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SPATIALLY STRUCTURED NETWORKS OF PULSE-COUPLED PHASE OSCILLATORS ON METRIC SPACES

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Abstract. The Winfree model describes finite networks of phase oscillators. Oscillators interact by broadcasting pulses that modulate the frequencies of connected oscillators. We study a generalization of the model and its fluid-dynamical limit for networks, where oscillators are distributed on some abstract σ-finite Borel measure space over a separable metric space. We give existence and uniqueness statements for solutions to the continuity equation for the oscillator phase densities. We further show that synchrony in networks of identical oscillators is locally asymptotically stable for finite, strictly positive measures and under suitable conditions on the oscillator response function and the coupling kernel of the network. The conditions on the latter are a generalization of the strong connectivity of finite graphs to abstract coupling kernels.

1. Introduction.

1.1. Historical background.

In the last 50 years, an ever increasing number of biological and physical systems displaying collective behavioural patterns has been modelled by networks of coupled oscillators. Examples include the synchronization of flashing fireflies, circadian rhythms in animals, the seemingly spontaneous activity coherence in the visual cortex, crowd synchronization on the Millennium Bridge, and large arrays of coupled Josephson junctions [39, 41, 36, 37, 38, 15]. To a large extent, these oscillators represent stable limit cycles, coupled so weakly that after any coupling-induced distortion they quickly return to their attractor, albeit with a certain phase shift.

One of the first models of such so-called phase oscillators was introduced by Winfree [40] and assumes the interaction to be driven by a mean field generated by pulses emitted by the network, with each oscillator's response depending on its current phase. In its typical form the model describes N coupled oscillators with

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time-dependent phases $\theta_1,\dots,\theta_N\in S^1$, constant intrinsic frequencies $\omega_1,\dots,\omega_N\in\mathbb{R}$ and dynamics

$$\dot\theta_i=\omega_i+\psi_i(\theta_i)\sum_{j=1}^{N}G(i,j)I_j(\theta_j),\qquad i\in\{1,\dots,N\}.\tag{1}$$

The functions $\psi_i: S^1\to\mathbb{R}$ and $I_i: S^1\to\mathbb{R}_+$ are called the *response function* and *pulse* of the i-th oscillator, respectively[1], $G\in\mathbb{R}^{N\times N}$ is the *coupling matrix*, and the weighted sum in (1) is the *stimulus* driving the i-th oscillator. A derivation of such dynamics is given by [26, §3.2 & §5.2] using a perturbation analysis around stable limit cycles, where the authors assume a short stimulus distorting the (multidimensional) oscillator dynamics as an additive term in the full equation of motion. Note that it is not always clear what form such a stimulus should have as a result of overall network activity. The mean field form used in (1), specifically the additive character of simultaneous perturbations, is justified by the assumption of weak interactions.

The Winfree model (1) has recently seen increased attention [4, 18, 32, 17, 7], though with a strong focus on the mathematically tractable case $G(i,j)\equiv G$ (*all-to-all* coupling), $\psi_i(\theta)=-\sin\theta$, and $I_j(\theta)=1+\cos\theta$, $\forall i,j$. Using this type of models to understand neural network activity, and in particular synchronization, currently seems to be a promising approach [35]. Note that the model describes the dynamics of oscillators on a finite graph. Given that the mammalian brain consists of billions of interacting neurons [14], large-network limits could be particularly fruitful, despite (or because of) their simplicity. As $N\to\infty$, an often considered limit of (1) for all-to-all coupled networks with identical oscillators ($\omega_i=\omega$, $I_i(\varphi)=I(\varphi)$, and $\psi_i(\varphi)=\psi(\varphi)$, $\forall i$) is the equation

$$\partial_t\rho(t,\vartheta)=-\partial_\vartheta\left\{\rho(t,\vartheta)\cdot\left[\omega+\psi(\vartheta)G\int_{S^1}I(\varphi)\rho(t,\varphi)\,d\varphi\right]\right\},\tag{2}$$

describing the evolution of the probability density $\rho(t,\vartheta)$ at time $t\in\mathbb{R}$ of an oscillator being at phase $\vartheta\in S^1$ [3]. This limit, analogous to the Boltzmann equation in kinetic gas theory, has already been considered in the celebrated Kuramoto model of coupled oscillators with great success [25, 26, 1].

1.2. **Models examined in this article.** In this article we introduce generalizations of the models (1) and (2) to the case where oscillators are distributed on some abstract σ-finite Borel measure space $(X,\mathscr{B}(X),\mu)$, over a separable metric space X with Borel σ-algebra $\mathscr{B}(X)$. The measure μ can be seen as a fixed *oscillator distribution* on X, similar to the continuous neuron distribution often assumed in neural field theory [9, 19]. The space X itself, which can be finite or infinite, may have virtually any topology of interest. Simple examples would be the n-dimensional sphere S^n or torus $\mathbb{T}^n$, but more complicated topologies comparable to the central nervous system are equally possible. In this context, the oscillator index $i\in\{1,\dots,N\}$ is replaced by a coordinate $x\in X$ and the coupling matrix becomes a coupling kernel $G: X\times X\to\mathbb{R}$. Furthermore, we do not a priori assume any particular shapes for ψ and I, which can even vary on X, though some of our results assume certain key features.

[1] Also known as the *sensitivity function* and the *influence function*, respectively.

In the most general case we consider the *field equation* (or *field model*)

$$\partial_t\theta(t,x) = u(t,x,\theta(t,x)) + \psi(x,\theta(t,x))\int_X G(x,y)I(y,\theta(t,y))\,d\mu(y) \tag{3}$$

in the time-dependent field $\theta : \mathbb{R}\times X \to S^1$ as a generalization of (1) and in analogy to field theories in continuum physics [21]. The function $u : \mathbb{R}\times X\times S^1 \to \mathbb{R}$ encapsulates any individual, possibly non-autonomous, dynamics of isolated oscillators. Similarly, we consider the *continuity equation* (or *fluid model*)

$$\partial_t\rho(t,x,\vartheta) = -\partial_\vartheta\left[\rho(t,x,\vartheta)\cdot \mathrm{v}(t,x,\vartheta,\rho(t))\right] \tag{4}$$

as a generalization of (2), with the *velocity field* v being given by

$$\mathrm{v}(t,x,\vartheta,\rho(t)) = u(t,x,\vartheta) + \psi(x,\vartheta)\int_X G(x,y)\int_{S^1} I(y,\varphi)\rho(t,y,\varphi)\,d\varphi\,d\mu(y). \tag{5}$$

Here, $\rho(t,x,\cdot)$ represents the phase-probability distribution on S^1 of oscillators located at the point $x\in X$ at time t. In the simple case where X is a finite set with counting measure μ and $u(t,x,\vartheta)=\omega(x)$, (3) takes the original form (1). Similarly, (4) can then be interpreted as a system of coupled continuity equations for finitely many networks, each one comprising infinitely many identical oscillators. Analogous models have been considered as limits of the Kuramoto model for infinitely many oscillators [34, 27, 29], for the cases when X is the real line, the two-dimensional plane, or the circle S^1.

In this article we show the existence and uniqueness of strong solutions to the initial value problem for the continuity equation (4), at least within a certain function class and for certain initial values. To the best of our knowledge, little has been published so far on this issue for continuity equations where the velocity field v explicitly depends on the current distribution $\rho(t)$. Furthermore, we study synchronization in certain subclasses of the field and fluid models, i.e., solutions where all oscillators share one common, time-dependent phase. In the field model (3) this corresponds to $\theta(t,x)$ not depending on x. In the fluid model (4), this formally corresponds to ρ being a rotating Dirac distribution. We give sufficient conditions for the existence and local stability of synchrony, which involve the structure of the oscillator response function as well as the connectivity properties of the coupling kernel. The latter is a generalization of strong connectivity of finite graphs. Our stability results show that prior findings for finite oscillator networks extend to the large-network limit in a wide range of connection topologies. Our approach removes the focus from the particular geometric structure of the underlying space X, placing it instead on the measure-transformation properties of the connection kernel G. Moreover, it is our impression that no similar results have been obtained for the fluid model, not even for all-to-all coupling.

1.3. **Article structure.** Section 2 introduces the basic notation and terminology, summarized in table 1. In section 2.3 we show how the field model connects to the fluid model, indicating that the latter can be interpreted as a statistical approximation of the former by looking at values of the field θ within small regions in the space X instead of single points $x\in X$. As will become clear in the proof of Theorem 4.16, a similar connection exists in the other direction as well.

In section 3 we study the existence and uniqueness of solutions to the continuity equation (4). We also point out the connection to the equivalent question for densities $\rho(t,x,\cdot)$ on $\mathbb{R}$, evolving within a velocity field $\mathrm{v}(t,x,\vartheta,\rho(t))$ that is periodic in

ϑ. The main results are given by Theorems 3.6 and 3.7 for quite abstract velocity fields. The velocity field (5) is only considered in Theorem 3.9 as a special case.

In section 4 we deal with the question of synchrony in both the field and fluid models for the case when $u(t,x,\vartheta)=\omega\ \forall(t,x,\vartheta)\in\mathbb{R}\times X\times S^1$ for some constant $\omega>0$, G is non-negative, and the response function $\psi(x,\cdot)$, pulse $I(x,\cdot)$, and the *total coupling* $\|G(x,\cdot)\|_{L^1(\mu)}$ do not depend on $x\in X$. In section 4.1 we introduce a generalization of strong graph connectivity to integral kernels on σ-finite measure spaces. The connectivity properties of the coupling kernel G play a key role in section 4.2. In particular, Theorem 4.10 gives sufficient conditions for the local stability of synchrony in the field model, thus generalizing the results of [18] who considered the special case of finitely many oscillators. In section 4.3 we study the stability of synchrony in the fluid model against distortions with an adequately narrow bandwidth. We show that, under certain assumptions on the model, the diameter $\operatorname{diam}\bigcup_{x\in X}\operatorname{supp}\rho(t,x,\cdot)$ decreases exponentially with time, provided it starts below a certain threshold. A key assumption of the theorem will be the stability of synchrony in the corresponding field model. For the stability analysis of synchrony in both the field and fluid model, we assume the measure μ to be finite.

2. Preliminaries.

2.1. Notation.

We denote $\mathbb{R}_+=[0,\infty)$ and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. The indicator function of a set A is denoted by 1_A. For an interval $J\subseteq\mathbb{R}$, we write $|J|=\sup J-\inf J$. The derivative of a function on the boundary of J is understood to mean the appropriate one-sided derivative. The Borel σ-algebra of a topological space Q is denoted by $\mathscr{B}(Q)$; we will always assume the space to be endowed with that σ-algebra. The space of bounded linear operators between two Banach spaces E,F is denoted by $\mathcal{L}(E,F)$, and $\mathcal{L}(E,E)$ is abbreviated to $\mathcal{L}(E)$. For a linear operator $\mathcal{H}$ on a normed space, we write $\sigma(\mathcal{H})$ for its spectrum, $\sigma_p(\mathcal{H})$ for its point spectrum, $r(\mathcal{H})$ for its spectral radius and $\|\mathcal{H}\|$ for its operator norm. The symbol Id denotes the identity operator. For a measure space $(K,\mathscr{K},\kappa)$ and $1\le q\le\infty$, we write $L^q(\kappa)$ for the L^q-space defined on $(K,\mathscr{K},\kappa)$ and $\|\cdot\|_{L^q(\kappa)}$ for the corresponding norm. The notation $B_\varepsilon(x)$ denotes the closed ball of radius $\varepsilon\ge 0$ centered at the point x in a metric space. The symbol 'd' denotes the metric in the given context. The *diameter* of a set A is given by $\operatorname{diam}A=\sup_{x,y\in A}d(x,y)$. If f maps to a metric space, the *support* of f is the set $\operatorname{supp}f:=\operatorname{cl}\{x\in X: f(x)\neq 0\}$, where "cl" denotes closure.

The Cartesian product $Q_1\times\cdots\times Q_n$ of topological spaces $Q_1,\dots,Q_n$ is assumed to be endowed with the product topology. The Cartesian product $X_1\times\cdots\times X_n$ of metric spaces $(X_1,d_1),\dots,(X_n,d_n)$ is assumed to be endowed with the product metric $d((x_1,\dots,x_n),(y_1,\dots,y_n))=\sum_{i=1}^n d_i(x_i,y_i)$. The direct product $V_1\times\cdots\times V_n$ of Banach spaces $(V_1,\|\cdot\|_1),\dots,(V_n,\|\cdot\|_n)$ is assumed to be endowed with the product norm $\|(v_1,\dots,v_n)\|=\sum_{i=1}^n\|v_i\|_i$. The product $K_1\times\cdots\times K_n$ of measurable spaces $(K_1,\mathscr{K}_1),\dots,(K_n,\mathscr{K}_n)$ is assumed to be endowed with the product σ-algebra $\mathscr{K}_1\otimes\cdots\otimes\mathscr{K}_n$. Finally, recall that $\mathscr{B}(X_1\times\cdots\times X_n)=\mathscr{B}(X_1)\otimes\cdots\otimes\mathscr{B}(X_n)$ for separable metric spaces $X_1,\dots,X_n$ [10, Theorem 6.4.2].

The circle S^1 is identified with the quotient group $\mathbb{R}/\mathbb{Z}$. For a set Y and $f: Y\times\Theta\to\mathbb{R}$ we shall denote by $\operatorname{supp}_{\mathrm{ph}}f$ the closure of $\bigcup_{y\in Y}\operatorname{supp}f(y,\cdot)$ in Θ and $\operatorname{diam}_{\mathrm{ph}}f:=\operatorname{diam}(\operatorname{supp}_{\mathrm{ph}}f)$. For $\vartheta^b,\vartheta^f\in S^1$ we denote by $[\vartheta^b,\vartheta^f]$ the circular arc between ϑ^b and ϑ^f (inclusive), covered as one traverses S^1 from ϑ^b to ϑ^f in the positive sense. By convention $\left[\vartheta^b,\vartheta^b\right]:=\left\{\vartheta^b\right\}$.

Let V be a Banach space or a smooth, complete manifold with tangent bundle TV and let $\mathcal{H} : \mathbb{R} \times V \to TV$ be a time-dependent vector field on V. Let $U(t, t_o)$ be the flow generated by $\mathcal{H}$, i.e., the unique solution to the initial value problem

$$\frac{d}{dt}U(t,t_o)(v) = \mathcal{H}(t, U(t,t_o)(v)), \quad U(t_o,t_o)(v) = v.$$

We also refer to $U(t, t_o)$ as as the *propagator* from time t_o to t_1 induced by $\mathcal{H}$. It is said to be autonomous if $U(t, t_o)$ only depends on the difference $(t - t_o)$.

The coordinates $t \in \mathbb{R}$, $x \in X$ and $\vartheta \in S^1$ (or occasionally $\vartheta \in \mathbb{R}$) introduced in the models (3) and (4) will be referred to as *time*, *space*, and *phase* coordinates, respectively.

2.2. **Special function spaces.** Let M, N be measurable spaces, X, Y, Z metric spaces, E, F Banach spaces and P, Q topological spaces. We denote by $\mathcal{M}(M, N)$ the set of measurable functions $f : M \to N$, and by $\mathcal{M}_{\mathrm{b}}(M, X)$ the metric space of measurable, bounded functions $f : M \to X$ endowed with the supremum metric. $\mathcal{C}(P, Q)$ denotes the class of continuous functions from P to Q and $\mathcal{C}_{\mathrm{b}}(P, X)$ denotes the metric space of bounded continuous functions from P to X endowed with the supremum metric. $\mathcal{C}_{\mathrm{u}}(X, Y)$ denotes the class of uniformly continuous functions mapping X to Y. Let $\mathcal{C}_{\mathrm{u}}(X, Y)$ be endowed with the supremum distance $d_\infty(f, g) = \sup_{x\in X} d(f(x), g(x))$ (whenever defined). Then $\mathcal{C}_{\mathrm{u,b}}(X, Y)$ denotes the subclass of uniformly continuous, bounded functions, which forms a metric space under d_∞.

$\mathcal{C}(Z, \mathcal{C}_{\mathrm{u}}(X, Y))$ denotes the class of functions $f : Z \times X \to Y$ satisfying $f(z, \cdot) \in \mathcal{C}_{\mathrm{u}}(X, Y)$ for all $z \in Z$ and $d_\infty(f(z_n, \cdot), f(z, \cdot)) \overset{n\to\infty}{\longrightarrow} 0$ whenever $z_n \to z$ in Z. The notation $\mathcal{C}(Z, \mathcal{C}_{\mathrm{u,b}}(X, Y))$ shall have similar meaning, and $\mathcal{C}_{\mathrm{b}}(Z, \mathcal{C}_{\mathrm{u,b}}(X, Y))$ denotes the subclass of bounded functions in that space.

$\mathcal{C}^1(\mathbb{R}, \mathcal{C}_{\mathrm{u}}(X, F))$ denotes the subclass of functions $f \in \mathcal{C}(\mathbb{R}, \mathcal{C}_{\mathrm{u}}(X, F))$ for which there exists a mapping $d_t f \in \mathcal{C}(\mathbb{R}, \mathcal{C}_{\mathrm{u,b}}(X, F))$, called the *time-derivative* of f, and a so-called *error term* $R : \mathbb{R} \times X \times \mathbb{R} \to F$, such that

$$f(t + \varepsilon, x) = f(t, x) + (d_t f)(t, x) \cdot \varepsilon + R(t, x, \varepsilon)$$

and $\sup_{x\in X} \|R(t, x, \varepsilon)\| \in o(\varepsilon)$ as $\varepsilon \to 0$ for every $t \in \mathbb{R}$. The notations $\mathcal{C}^1(\mathbb{R}, \mathcal{C}_{\mathrm{b}}(P, S^1))$, $\mathcal{C}^1(\mathbb{R}, \mathcal{C}_{\mathrm{u,b}}(X, S^1))$ and $\mathcal{C}^1(\mathbb{R}, \mathcal{M}_{\mathrm{b}}(M, S^1))$ are to be understood in a similar way.

Remarks 2.1.

1. The class $\mathcal{C}_{\mathrm{u,b}}(X, Y)$ is a closed subset of the metric space $\mathcal{C}_{\mathrm{b}}(X, Y)$.
2. If Z is a compact metric space, then any $f \in \mathcal{C}(Z, \mathcal{C}_{\mathrm{u,b}}(X, F))$ is also in $\mathcal{C}_{\mathrm{u,b}}(Z \times X, F)$.

Definition 2.2 (Wrapping)**.** We call a function $\rho : S^1 \to \mathbb{R}$ a *wrapping* [30] of $\tilde{\rho} : \mathbb{R} \to \mathbb{R}$ and $\tilde{\rho}$ an *unwrapping* of ρ if

$$\rho(\theta) = \sum_{n\in\mathbb{Z}} \tilde{\rho}(\theta + n) =: \Pi_{\mathrm{w}}(\tilde{\rho})(\theta)$$

for all $\theta \in S^1$, with the series converging absolutely. For any set Y we call $\rho : Y \times S^1 \to \mathbb{R}$ a *wrapping* of $\tilde{\rho} : Y \times \mathbb{R} \to \mathbb{R}$ if $\rho(y, \cdot)$ is a wrapping of $\tilde{\rho}(y, \cdot)$ for every $y \in Y$. We then write $\rho = \Pi_{\mathrm{w}}(\tilde{\rho})$.

Remarks 2.3.

1. For any $\rho : S^1 \to \mathbb{R}$ there exists an infinite number of unwrappings.

Symbol	Description
$\mathbb{R}_+$	$[0,\infty)$.
$\mathbb{N}_0$	$\mathbb{N}\cup\{0\}$.
1_A	Indicator function of set A.
$\mathscr{B}(X)$	Borel σ-algebra of X.
$\mathcal{L}(E,F)$	Space of bounded linear operators from E to F.
$\sigma(\mathcal{H})$	Spectrum of operator $\mathcal{H}$.
$\sigma_p(\mathcal{H})$	Point spectrum of operator $\mathcal{H}$.
$r(\mathcal{H})$	Spectral radius of operator $\mathcal{H}$.
$B_\varepsilon(x)$	Closed ball of radius ε and centre x.
$\operatorname{supp} f$	Support of function f.
$\operatorname{diam} A$	Diameter of set A.
$\operatorname{supp}_{\mathrm{ph}} f$	Closure of $\bigcup_{x\in X}\operatorname{supp} f(x,\cdot)$.
$\operatorname{diam}_{\mathrm{ph}} f$	Diameter of $\operatorname{supp}_{\mathrm{ph}} f$.
$\Pi_{\mathrm{w}}(\tilde\rho)$	Wrapping of function $\tilde\rho$ around S^1.
Π_{c}	Canonical projection of $\mathbb{R}$ to $S^1=\mathbb{R}/\mathbb{Z}$.
$\mathcal{M}(M,N)$	Set of measurable functions from M to N.
$\mathcal{M}_{\mathrm{b}}(M,X)$	Set of measurable, bounded functions from M to N.
$\mathcal{C}(X,Y)$	Continuous functions from X to Y.
$\mathcal{C}^1(X,Y)$	Continuously differentiable functions from X to Y.
$\mathcal{C}_{\mathrm{b}}(X,Y)$	Bounded continuous functions from X to Y.
$\mathcal{C}_{\mathrm{u}}(X,Y)$	Uniformly continuous functions from X to Y.
$\mathcal{C}_{\mathrm{u,b}}(X,Y)$	Uniformly continuous, bounded functions from X to Y.
$\mathcal{C}(Z,\mathcal{C}_{\mathrm{u}}(X,Y))$	Continuous functions from Z to $\mathcal{C}_{\mathrm{u}}(X,Y)$.
$\mathcal{C}^1(\mathbb{R},\mathcal{C}_{\mathrm{u}}(X,F))$	Continuously differentiable functions from $\mathbb{R}$ to $\mathcal{C}_{\mathrm{u}}(X,F)$.

TABLE 1. Summary of considered function spaces and non-standard notation used in our article, elaborated on in sections 2.1, 2.2 and definition 2.2.

2. Each $\rho\in\mathcal{C}^k(S^1,\mathbb{R}_+)$ (with $k\in\mathbb{N}_0$) has at least one unwrapping $\tilde\rho\in\mathcal{C}^k(\mathbb{R},\mathbb{R}_+)$ with support in $[-1,+1]$.
3. The symbol Π_{c} shall denote the canonical projection of reals to the quotient group $\mathbb{R}/\mathbb{Z}=S^1$. Note that Π_{c} is a covering map for S^1. For any function f defined on S^1 we call the composition $f\circ\Pi_{\mathrm{c}}$ its *pullback* on $\mathbb{R}$. We will often abbreviate $f(\Pi_{\mathrm{c}}(\vartheta))$ with $f(\vartheta)$; the distinction should be clear from the context.
4. A similar notion to wrappings exists in the algebraic topology literature for maps $f\in\mathcal{C}(Y,S^1)$ defined on some topological space Y, where a map $\tilde f\in\mathcal{C}(Y,\mathbb{R})$ is called a *lift* of f if $\Pi_{\mathrm{c}}\circ\tilde f=f$.

2.3. **The fluid model as an approximation of the field model.** The two models (3) and (4) studied in this article are connected in a certain sense that might not be apparent at first sight. In Theorem 4.16 we will show that the stability of network synchrony in the fluid model is implied by the stability of synchrony in the corresponding field model under appropriate assumptions. In fact, in the proof of that theorem we will see that the mean oscillator phase solves the field equation (3), within the context of the fluid model and up to an error of order

$o(\sup_{x\in X} \operatorname{diam}\operatorname{supp}\rho(t,x,\cdot))$. We can thus interpret the field model as a limit of the fluid model as oscillator densities $\rho(t,x,\cdot)$ approach a Dirac distribution at each point x. As we will show below, this link works in both ways. We present a heuristic interpretation of the fluid model as a certain statistical approximation of the field model.

For simplicity, we consider X in this section to be a compact metric space and μ a finite, strictly positive Borel measure on X (i.e. $\mu(A) > 0$ for all open nonempty sets A). We also assume in this section the following *homogeneity condition*: The limit

$$\lim_{\varepsilon\to 0^+} \frac{\mu(B_\varepsilon(x))}{\mu(B_\varepsilon(y))} =: \mathscr{N}(x,y)$$

exists in $(0,\infty)$ for all $x, y \in X$, is approached uniformly in x, y, and the mapping $\mathscr{N} : X \times X \to (0,\infty)$ is continuous. A simple example of such a space would be the n-sphere $X = S^n$. Let $u \in \mathcal{C}_{\mathrm{u}}(\mathbb{R}\times X\times S^1,\mathbb{R})$, $\psi, I \in \mathcal{C}(X\times S^1,\mathbb{R})$, and $G \in \mathcal{C}(X\times X,\mathbb{R})$. Define $\mathrm{v} : \mathbb{R}\times X\times S^1\times \mathcal{M}(X,S^1) \to \mathbb{R}$ by

$$\mathrm{v}(t,x,\vartheta,\theta) = u(t,x,\vartheta) + \psi(x,\vartheta)\int_X G(x,y)I(y,\theta(y))\,d\mu(y).$$

Note that $\|\mathrm{v}(\cdot,x,\cdot,\cdot) - \mathrm{v}(\cdot,\tilde{x},\cdot,\cdot)\|_\infty \to 0$ as $d(x,\tilde{x}) \to 0$. Let $\theta \in \mathcal{C}^1(\mathbb{R},\mathcal{M}_{\mathrm{b}}(X,S^1))$ satisfy the field equation

$$\partial_t\theta(t,x) = \mathrm{v}(t,x,\theta(t,x),\theta(t,\cdot)).$$

For any $\varepsilon > 0$ and $(t,x) \in \mathbb{R}\times X$, let $\rho_\varepsilon(t,x,\cdot) \in \mathcal{L}(\mathcal{C}_{\mathrm{b}}(S^1,\mathbb{R}),\mathbb{R})$ stand for the non-negative bounded linear functional defined by

$$\langle\rho_\varepsilon(t,x,\cdot), f\rangle := \int_{B_\varepsilon(x)} f(\theta(t,y))\frac{d\mu(y)}{\mu(B_\varepsilon(x))}$$

for $f \in \mathcal{C}_{\mathrm{b}}(S^1,\mathbb{R})$. Its operator norm is given by $\|\rho_\varepsilon(t,x,\cdot)\| = \langle\rho_\varepsilon(t,x,\cdot),1\rangle = 1$. Furthermore, for $f \in \mathcal{C}^\infty(S^1,\mathbb{R})$ one has

$$\begin{aligned}
\langle\partial_t\rho_\varepsilon(t,x,\cdot), f\rangle &\overset{\mathrm{def}}{=} \partial_t\,\langle\rho_\varepsilon(t,x,\cdot), f\rangle = \int_{B_\varepsilon(x)} \partial_t f(\theta(t,y))\frac{d\mu(y)}{\mu(B_\varepsilon(x))}\\
&= \int_{B_\varepsilon(x)} f'(\theta(t,y))\cdot\partial_t\theta(t,y)\frac{d\mu(y)}{\mu(B_\varepsilon(x))}\\
&= \int_{B_\varepsilon(x)} f'(\theta(t,y))\cdot \mathrm{v}(t,y,\theta(t,y),\theta(t,\cdot))\frac{d\mu(y)}{\mu(B_\varepsilon(x))}\\
&= o(\varepsilon^0) + \int_{B_\varepsilon(x)} f'(\theta(t,y))\cdot \mathrm{v}(t,x,\theta(t,y),\theta(t,\cdot))\frac{d\mu(y)}{\mu(B_\varepsilon(x))}\\
&= o(\varepsilon^0) + \langle\rho_\varepsilon(t,x,\cdot),\ f'\cdot \mathrm{v}(t,x,\cdot,\theta(t,\cdot))\rangle\\
&\overset{\mathrm{def}}{=} o(\varepsilon^0) - \langle\partial_\vartheta\left[\rho_\varepsilon(t,x,\cdot)\cdot \mathrm{v}(t,x,\cdot,\theta(t,\cdot))\right],\ f\rangle,
\end{aligned}$$

while we interpret $\rho_\varepsilon(t,x,\cdot)$ as a distribution on $\mathcal{C}^\infty(S^1,\mathbb{R})$. This translates to

$$\partial_t\rho_\varepsilon(t,x,\cdot) = o(\varepsilon^0) - \partial_\vartheta\left[\rho_\varepsilon(t,x,\cdot)\cdot \mathrm{v}(t,x,\cdot,\theta(t,\cdot))\right] \tag{6}$$

in the distributional sense. The error $o(\varepsilon^0)$ scales down with $\varepsilon \to 0$ uniformly in (t,x,θ), but pointwise in the distribution's argument $f \in \mathcal{C}^\infty(S^1,\mathbb{R})$. Note that the mappings $y \mapsto \mu(B_\varepsilon(y))$ and $y \mapsto \langle\rho_\varepsilon(t,y,\cdot), f\rangle$ (with $f \in \mathcal{C}_{\mathrm{b}}(S^1,\mathbb{R})$) are both measurable. This follows from the measurability of $(X\times X,\mathscr{B}(X\times X)) \to$

$(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, $(y,z) \mapsto 1_{B_\varepsilon(y)}(z)$, and the fact that $\mathscr{B}(X) \times \mathscr{B}(X) = \mathscr{B}(X \times X)$, the latter holding since X is separable. Furthermore, one has

$$\begin{aligned}
&\int_X G(x,y) \left\langle \rho_\varepsilon(t,y,\cdot), I(y,\cdot) \right\rangle \, d\mu(y) \\
=&\int_X G(x,y) \int_{B_\varepsilon(y)} I(y,\theta(t,z)) \frac{d\mu(z)}{\mu(B_\varepsilon(y))} \, d\mu(y) \\
=&o(\varepsilon^0) + \int_X \int_{B_\varepsilon(y)} \frac{\mu(B_\varepsilon(z))}{\mu(B_\varepsilon(y))} G(x,z) I(z,\theta(t,z)) \frac{d\mu(z)}{\mu(B_\varepsilon(z))} \, d\mu(y) \\
=&o(\varepsilon^0) + \int_X \int_{B_\varepsilon(y)} G(x,z) I(z,\theta(t,z)) \frac{d\mu(z)}{\mu(B_\varepsilon(z))} \, d\mu(y) \\
=&o(\varepsilon^0) + \int_X G(x,z) I(z,\theta(t,z)) \int_X 1_{B_\varepsilon(y)}(z) \, d\mu(y) \, \frac{d\mu(z)}{\mu(B_\varepsilon(z))} \\
=&o(\varepsilon^0) + \int_X G(x,z) I(z,\theta(t,z)) \, d\mu(z),
\end{aligned}$$

the error $o(\varepsilon^0)$ scaling down as $\varepsilon \to 0$ uniformly in (t,x,θ). Note that for the error estimate we used the uniform continuity of G and $\mathscr{N}$ on X^2, the uniform continuity of I on $X \times S^1$, the uniform convergence $\mu(B_\varepsilon(z))/\mu(B_\varepsilon(y)) \to \mathscr{N}(x,y)$ as $\varepsilon \to 0^+$, the fact that $\mathscr{N}(y,y) = 1$ and $\mu(X) < \infty$. Therefore $\mathrm{v}(t,x,\vartheta,\theta(t,\cdot)) = o(\varepsilon^0) + \tilde{\mathrm{v}}(t,x,\vartheta,\rho_\varepsilon(t,\cdot,\cdot))$, with

$$\tilde{\mathrm{v}}(t,x,\vartheta,\rho_\varepsilon(t,\cdot,\cdot)) := u(t,x,\vartheta) + \psi(x,\vartheta) \int_X G(x,y) \left\langle \rho_\varepsilon(t,y,\cdot), I(y,\cdot) \right\rangle \, d\mu(y).$$

Together with (6) this implies

$$\partial_t \rho_\varepsilon(t,x,\cdot) = o(\varepsilon^0) - \partial_\vartheta \left[\rho_\varepsilon(t,x,\cdot) \tilde{\mathrm{v}}(t,x,\cdot,\rho_\varepsilon(t,\cdot,\cdot)) \right] \tag{7}$$

in the distributional sense. The error $o(\varepsilon^0)$ scales down as $\varepsilon \to 0$ uniformly in (t,x,θ) but pointwise in the distribution's argument $f \in \mathcal{C}^\infty(S^1, \mathbb{R})$. The distribution $\rho_\varepsilon(t,x,\cdot)$ can be interpreted as a *probability density* for the random variable $\theta(t,\hat{Y})$ on S^1, where $\hat{Y}$ is a random variable on $B_\varepsilon(x)$ distributed by the law $\mu/\mu(B_\varepsilon(x))$. The continuity conditions on u, ψ, G, and I with respect to X ensure that oscillators in the ball $B_\varepsilon(x)$ have similar properties, for sufficiently small $\varepsilon > 0$. As $\varepsilon \to 0$, (7) formally takes the form of the continuity equation in the fluid model.

The advantage of the fluid model over the field model is the possibility to represent local phenomena (other than synchronization), like statistically stationary states, in which current oscillator phases are even on a spatially local level and are best described in a statistical way. In particular, the *local variation* of $\theta(t,\cdot)$ might tend to infinity, with $\rho_\varepsilon(t,x,\cdot)$ still (weakly*) continuous in $x \in X$ (see Fig. 1). In the limit of infinitely many oscillators at each spatial point, an alternative justification of the continuity equation might be attempted using a method equivalent to the BBGKY hierarchy known from kinetic gas theory. A similar approach has been taken by [13, Appendix B] for a generalization of the Kuramoto model.

3. **Existence and uniqueness results for the fluid model.** We study the existence and uniqueness of solutions to the initial value problem of the continuity equation (4). Unless stated otherwise, here X will be a separable metric space and

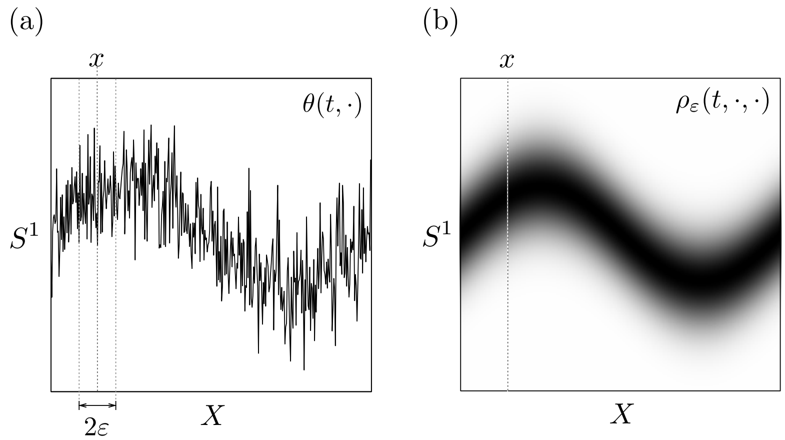

FIGURE 1. Motivation of the fluid model as a statistical approximation to the field model, as described in section 2.3. (a) Example field $\theta(t,\cdot) \in \mathcal{M}(X, S^1)$, plotted over the one-dimensional space X. (b) The corresponding density $\rho_\varepsilon(t,\cdot,\cdot)$ for some $\varepsilon > 0$, illustrated as a colour map over $X \times S^1$ with darker colours representing a higher density. The density $\rho_\varepsilon(t,x,\cdot)$ at x results from a statistical evaluation of the field values within the disc $B_\varepsilon(x)$.

μ a σ-finite Borel measure on X. For $\Theta \in \{\mathbb{R}, S^1\}$ we shall consider

$$\Omega_{o,\Theta} := \Big\{\rho_o \in \mathcal{C}_{\mathrm{u,b}}(X \times \Theta, \mathbb{R}_+) : \|\rho_o(x,\cdot)\|_{L^1(\Theta)} = 1 \ \forall x \in X\Big\} \tag{8}$$

to be endowed with the supremum metric. The aim of this section is to provide sufficient conditions for the existence of maximal solutions to initial value problems for the continuity equation in the function space

$$\begin{aligned}\Omega_\Theta := \{\rho \in \mathcal{C}(J, \Omega_{o,\Theta}) : 0 \in J \subseteq \mathbb{R} \text{ interval } \wedge\ \partial_\vartheta \rho, \partial_t \rho \text{ exist}\\ \wedge\ (\partial_t \rho)(\cdot, x, \cdot) \in \mathcal{C}(J \times \Theta, \mathbb{R})\ \forall x \in X\}.\end{aligned} \tag{9}$$

For this purpose, we write the initial value problem as an integral fixed-point equation and apply Banach's fixed point theorem. The integral operator is constructed using the flow on Θ, which is induced by the velocity field corresponding to a given $\rho \in \Omega_\Theta$. This connection between flows and densities satisfying the continuity equation has its roots in the observation that the continuity equation is an ensemble description of point orbits in Θ. The technical auxiliary statements 3.2 and 3.3 give a synopsis of the smoothness properties of flows generated by velocity fields and densities corresponding to flows. The auxiliary statement 3.4 gives an existence and uniqueness statement for the initial value problem for continuity equations with velocity fields not explicitly depending on the density itself, which is based on the ideas of [2, Lemma 8.1.6]. We make used of these auxiliary statements in Theorem 3.6, where we prove the existence and uniqueness of solutions for the case when the velocity field depends explicitly on the density ρ. For convenience, we occasionally identify the continuity equation on S^1 with its equivalent counterpart on $\mathbb{R}$ for phase-periodic densities and phase-periodic velocity fields. In Theorem 3.7 we provide a connection between solutions $\rho \in \Omega_{S^1}$ of the continuity equation and the evolution of unwrappings of ρ. Theorem 3.9 concretizes these results to our model at hand, that is the continuity equation for pulse-coupled phase oscillators with the velocity field (5).

Remark 3.1. As already mentioned, there is a relationship between the phase density and the flow generated by the velocity field that we would like to note here for future reference. Let $J \subseteq \mathbb{R}$ be some time interval (finite or infinite) and $\Theta \in \{\mathbb{R}, S^1\}$. Let $\mathrm{v} : J \times \Theta \to \mathbb{R}$ be such that both v and $\partial_\vartheta \mathrm{v}$ are continuous and bounded on $J \times \Theta$. Suppose that $\eta : J \times \Theta \to \mathbb{R}$ is differentiable in t and ϑ, and $\partial_t \eta$ is continuous on $J \times \Theta$. Let $\Phi : J^2 \times \Theta \to \Theta$ be the flow induced on Θ by the velocity field v. Let η satisfy the continuity equation

$$\partial_t \eta(t, \vartheta) = -\partial_\vartheta \left[\eta(t, \vartheta) \cdot \mathrm{v}(t, \vartheta)\right].$$

Then η is of the form

$$\eta(t, \vartheta) = \eta(t_o, \Phi(t_o, t, \vartheta)) \cdot \exp\left[\int_t^{t_o} (\partial_\vartheta \mathrm{v})\,(\tau, \Phi(\tau, t, \vartheta))\; d\tau\right] \tag{10}$$

for $(t, \vartheta) \in J \times \Theta$ and any $t_o \in J$ (note the reversed role of t_o and t in the flow). Furthermore,

$$\operatorname{supp} \eta(t, \cdot) \subseteq \left\{\vartheta \in \Theta : d(\vartheta, \operatorname{supp} \eta(t_o, \cdot)) \le |t - t_o|\, \|\mathrm{v}\|_\infty\right\}. \tag{11}$$

Representation (10) has already been pointed out by [2, Lemma 8.1.6] in the context of gradient flows in spaces of probability measures. The estimate (11) follows from $d(\Phi(t, t_o, \vartheta), \vartheta) \le |t - t_o|\, \|\mathrm{v}\|_\infty$.

Auxiliary statement 3.2 (Smoothness of flows)**.** *Let $J \subseteq \mathbb{R}$ be a time interval. Let $\mathrm{v} : J \times X \times \mathbb{R} \to \mathbb{R}$ be such that the partial derivatives $\partial_\vartheta \mathrm{v}, \partial_\vartheta^2 \mathrm{v}$ exist and $\mathrm{v}, \partial_\vartheta \mathrm{v}, \partial_\vartheta^2 \mathrm{v}$ are uniformly continuous and bounded. Let $\Phi : J^2 \times X \times \mathbb{R} \to \mathbb{R}$ be the flow induced by v, defined as the solution to the initial value problem*

$$\begin{aligned} &\Phi(t_o, t_o, x, \vartheta) = \vartheta, \\ &\frac{d}{dt} \Phi(t, t_o, x, \vartheta) = \mathrm{v}(t, x, \Phi(t, t_o, x, \vartheta)) \end{aligned}$$

for every $(t, t_o, x, \vartheta) \in J^2 \times X \times \mathbb{R}$. Then the following hold:

1. *$\Phi(\cdot, t_o, \cdot, \cdot) \in \mathcal{C}^1(J, \mathcal{C}_\mathrm{u}(X \times \mathbb{R}, \mathbb{R}))$ for each $t_o \in J$. Furthermore, $\Phi(\cdot, \cdot, \cdot, \cdot)$ is uniformly continuous on $K^2 \times X \times \mathbb{R}$ for any compact sub-interval $K \subseteq J$.*
2. *$\Phi(t, \cdot, \cdot, \cdot) \in \mathcal{C}^1(J, \mathcal{C}_\mathrm{u}(X \times \mathbb{R}, \mathbb{R}))$ for each $t \in J$. The derivative $(d_{t_o}\Phi)(t, t_o, x, \vartheta)$ is given as the solution to the initial value problem*

$$\begin{aligned} &(d_{t_o}\Phi)(t_o, t_o, x, \vartheta) = -\mathrm{v}(t_o, x, \vartheta), \\ &\frac{d}{dt}(d_{t_o}\Phi)(t, t_o, x, \vartheta) = (\partial_\vartheta \mathrm{v})(t, x, \Phi(t, t_o, x, \vartheta)) \cdot (d_{t_o}\Phi)(t, t_o, x, \vartheta) \end{aligned}$$

   in $\mathbb{R}$. Moreover, $(d_{t_o}\Phi)$ is uniformly continuous and bounded on $K^2 \times X \times \mathbb{R}$ for any compact time interval $K \subseteq J$.
3. *$\Phi(t, t_o, x, \vartheta)$ is differentiable in ϑ with the derivative $\partial_\vartheta \Phi$ given as the solution to the initial value problem*

$$\begin{aligned} &(\partial_\vartheta \Phi)(t_o, t_o, x, \vartheta) = \mathrm{Id}_\mathbb{R}, \\ &\frac{d}{dt}(\partial_\vartheta \Phi)(t, t_o, x, \vartheta) = (\partial_\vartheta \mathrm{v})(t, x, \Phi(t, t_o, x, \vartheta)) \cdot (\partial_\vartheta \Phi)(t, t_o, x, \vartheta) \end{aligned}$$

   in $\mathbb{R}$. Moreover, $(\partial_\vartheta \Phi)$ is uniformly continuous and bounded on $K^2 \times X \times \mathbb{R}$ for any compact time interval $K \subseteq J$.

4. *$(\partial_\vartheta\Phi)(t,\cdot,\cdot,\cdot) \in \mathcal{C}^1(J,\mathcal{C}_{\mathrm{u,b}}(X\times\mathbb{R},\mathbb{R}))$ for any fixed $t\in J$. The derivative $(d_{t_o}\partial_\vartheta\Phi)(t,t_o,x,\vartheta)$ is given as the solution to the initial value problem*

$$\begin{aligned}(d_{t_o}\partial_\vartheta\Phi)(t_o,t_o,x,\vartheta) =& -(\partial_\vartheta \mathrm{v})(t_o,x,\vartheta),\\ \frac{d}{dt}(d_{t_o}\partial_\vartheta\Phi)(t,t_o,x,\vartheta) =& (\partial_\vartheta^2\mathrm{v})(t,x,\Phi(t,t_o,x,\vartheta))\\ &\times(\partial_\vartheta\Phi)(t,t_o,x,\vartheta)\cdot(d_{t_o}\Phi)(t,t_o,x,\vartheta)\\ &+(\partial_\vartheta\mathrm{v})(t,x,\Phi(t,t_o,x,\vartheta))\cdot(d_{t_o}\partial_\vartheta\Phi)(t,t_o,x,\vartheta)\end{aligned}$$

in $\mathbb{R}$. Furthermore, $d_{t_o}\partial_\vartheta\Phi$ is uniformly continuous and bounded on $K^2\times X\times\mathbb{R}$ for every compact time interval $K\subseteq J$.

5. *The partial derivative $\partial_\vartheta^2\Phi(t,t_o,x,\vartheta)$ exists and is given as the solution to the initial value problem*

$$\begin{aligned}(\partial_\vartheta^2\Phi)(t_o,t_o,x,\vartheta) =& 0,\\ \frac{d}{dt}(\partial_\vartheta^2\Phi)(t,t_o,x,\vartheta) =& (\partial_\vartheta^2\mathrm{v})(t,x,\Phi(t,t_o,x,\vartheta))\cdot[\partial_\vartheta\Phi(t,t_o,x,\vartheta)]^2\\ &+(\partial_\vartheta\mathrm{v})(t,x,\Phi(t,t_o,x,\vartheta))\cdot(\partial_\vartheta^2\Phi)(t,t_o,x,\vartheta)\end{aligned}$$

in $\mathbb{R}$. Furthermore, $\partial_\vartheta^2\Phi$ is uniformly continuous and bounded on $K^2\times X\times\mathbb{R}$ for every compact time interval $K\subseteq J$.

The assertion is merely an extension of known facts about flows induced by ODEs and can be proven using standard techniques [5, 28].

Auxiliary statement 3.3 (Smoothness of densities corresponding to flows)**.** *Let $J\subseteq\mathbb{R}$ be a time interval and let $\Phi: J^2\times X\times\mathbb{R}\to\mathbb{R}$ satisfy the following:*

F1. $\Phi(t,\cdot,\cdot,\cdot)\in\mathcal{C}^1(J,\mathcal{C}_{\mathrm{u}}(X\times\mathbb{R},\mathbb{R}))$ *for all $t\in J$,*
F2. $(\partial_\vartheta\Phi)(t,\cdot,\cdot,\cdot)\in\mathcal{C}^1(J,\mathcal{C}_{\mathrm{u,b}}(X\times\mathbb{R},\mathbb{R}))$ *for all $t\in J$,*
F3. $(\partial_\vartheta^2\Phi)(t,t_o,\cdot,\cdot)$ *is uniformly continuous and bounded for all $t,t_o\in J$, and $(\partial_\vartheta^2\Phi)(t,\cdot,\cdot,\cdot)$ is continuous for all $t\in J$.*

Fix some $t_o\in J$ and let the initial state $\rho_o: X\times\mathbb{R}\to\mathbb{R}$ be such that $\rho_o,\partial_\vartheta\rho_o$ are uniformly continuous and bounded. Then the mapping $\rho: J\times X\times\mathbb{R}\to\mathbb{R}$ defined by

$$\rho(t,x,\vartheta) := \rho_o\left(x,\Phi(t_o,t,x,\vartheta)\right)\cdot(\partial_\vartheta\Phi)(t_o,t,x,\vartheta)$$

(note the reversed role of t_o and t) satisfies the following:

1. $\rho\in\mathcal{C}^1(J,\mathcal{C}_{\mathrm{u,b}}(X\times\mathbb{R},\mathbb{R}))$,
2. *$\partial_\vartheta\rho$ is continuous and $(\partial_\vartheta\rho)(t,\cdot,\cdot)$ is continuous and bounded for every $t\in J$.*

The assertion follows directly from the assumptions on Φ and ρ_o using standard calculus. We will make use of it in the auxiliary statement 3.4 below.

Auxiliary statement 3.4. *Let $J\subseteq\mathbb{R}$ be a time-interval. Suppose that the initial state $\rho_o: X\times\mathbb{R}\to\mathbb{R}$ and the velocity field $\mathrm{v}: J\times X\times\mathbb{R}\to\mathbb{R}$ are such that $\rho_o,\partial_\vartheta\rho_o,\mathrm{v},\partial_\vartheta\mathrm{v}$ and $\partial_\vartheta^2\mathrm{v}$ are uniformly continuous and bounded. Define the flow $\Phi: J^2\times X\times\mathbb{R}\to\mathbb{R}$ as the solution to the initial value problem*

$$\Phi(t_o,t_o,x,\vartheta)=\vartheta,\qquad \frac{d\Phi}{dt}(t,t_o,x,\vartheta)=\mathrm{v}\left(t,x,\Phi(t,t_o,x,\vartheta)\right)$$

for each $(t,t_o,x,\vartheta)\in J^2\times X\times\mathbb{R}$. Then the following hold:

1. *For each $s \in J$, the mapping $\rho : J \times X \times \mathbb{R} \to \mathbb{R}$ defined by*

$$\rho(t,x,\vartheta) := \rho_o\left(x, \Phi(s,t,x,\vartheta)\right) \cdot (\partial_\vartheta \Phi)(s,t,x,\vartheta) \tag{12}$$

is in $\mathcal{C}^1(J, \mathcal{C}_{\mathrm{u,b}}(X \times \mathbb{R}, \mathbb{R}))$. Furthermore $\partial_\vartheta \rho$ is continuous and $(\partial_\vartheta \rho)(t,\cdot,\cdot)$ is uniformly continuous and bounded for every $t \in J$.

2. *The continuity equation*

$$\partial_t \rho(t,x,\vartheta) = -\partial_\vartheta \left[\rho(t,x,\vartheta) \cdot \mathrm{v}(t,x,\vartheta)\right] \tag{13}$$

and the initial condition $\rho(s,\cdot,\cdot) = \rho_o$ are satisfied.

3. *Within the class of functions $\tilde{\rho} : J \times X \times \mathbb{R} \to \mathbb{R}$ with partial derivatives $\partial_t \tilde{\rho}$ and $\partial_\vartheta \tilde{\rho}$ such that $\partial_t \tilde{\rho}(\cdot, x, \cdot)$ is continuous in $J \times \mathbb{R}$ for every $x \in X$, there exists a maximal solution on J for the continuity equation (13) with initial value $\tilde{\rho}(s,\cdot,\cdot) = \rho_o$. That solution is given by (12).*

Proof. To avoid any ambiguity, we shall denote where needed the partial derivatives of Φ by $\partial_1 \Phi, \dots, \partial_4 \Phi$ (similarly for v and ρ_o), indexed by the argument with respect to which we differentiate. Observe that by the conditions on the velocity field v, the flow Φ is indeed well-defined.

1. By the auxiliary statement 3.2 the flow Φ is differentiable in ϑ, so that ρ is well defined. In fact, by the same proposition it satisfies the assumptions of the auxiliary statement 3.3, which in turn implies all of the assertions.

2. By the auxiliary statement 3.2 the partial derivatives $\partial_4 \Phi$, $\partial_2 \Phi$ take the form

$$\begin{aligned}
(\partial_4 \Phi)(s,t,x,\vartheta) &= \exp\left[\int_t^s (\partial_3 \mathrm{v})(\tau, x, \Phi(\tau,t,x,\vartheta))\, d\tau\right], \\
(\partial_2 \Phi)(s,t,x,\vartheta) &= -\exp\left[\int_t^s (\partial_3 \mathrm{v})(\tau, x, \Phi(\tau,t,x,\vartheta))\, d\tau\right] \cdot \mathrm{v}(t,x,\vartheta) \\
&= -(\partial_4 \Phi)(s,t,x,\vartheta) \cdot \mathrm{v}(t,x,\vartheta).
\end{aligned} \tag{14}$$

Note that by 3.2(1) and 3.2(3), Φ and $\partial_\vartheta \Phi : J^2 \times X \times \Theta \to \Theta$ are both continuous. Consequently, the compositions $(\tau,t,x,\vartheta) \mapsto (\partial_3 \mathrm{v})(\tau,x,\Phi(\tau,t,x,\vartheta))$ and $(\tau,t,x,\vartheta) \mapsto (\partial_3^2 \mathrm{v})(\tau,x,\Phi(\tau,t,x,\vartheta))$ are also continuous. We can therefore use (14) to find that $\partial_2 \partial_4 \Phi$ exists and takes the form

$$\begin{aligned}
\partial_2 \partial_4 \Phi(s,t,x,\vartheta) =& (\partial_4 \Phi)(s,t,x,\vartheta) \\
&\times \int_t^s (\partial_3^2 \mathrm{v})(\tau,x,\Phi(\tau,t,x,\vartheta)) \cdot \partial_2 \Phi(\tau,t,x,\vartheta)\, d\tau \\
&- (\partial_4 \Phi)(s,t,x,\vartheta) \cdot (\partial_3 \mathrm{v})(t,x,\vartheta).
\end{aligned} \tag{15}$$

Similarly,

$$\begin{aligned}
\partial_4^2 \Phi(s,t,x,\vartheta)) =& \partial_4 \Phi(s,t,x,\vartheta) \\
&\times \int_t^s (\partial_3^2 \mathrm{v})(\tau,x,\Phi(\tau,t,x,\vartheta)) \cdot \partial_4 \Phi(\tau,t,x,\vartheta)\, d\tau.
\end{aligned} \tag{16}$$

Starting from (12) and using (14), (15) and (16), we find that indeed

$$\partial_t \rho(t,x,\vartheta) + \partial_\vartheta \left[\rho(t,x,\vartheta) \cdot \mathrm{v}(t,x,\vartheta)\right] = 0.$$

3. The existence of a solution is ensured by assertion (2). Its uniqueness is given by Remark 3.1. $\square$

Representation (12) is essentially a special case of the known transformation of Lebesgue densities under diffeomorphic coordinate transformations [11], the transformation in our case being the flow Φ defined by the velocity field v.

Remarks 3.5. Let ρ_o and ρ be as described in the auxiliary statement 3.4. Then the following are easily verified:

(i) If $\rho_o \geq 0$ (resp., $\rho_o > 0$), then $\rho \geq 0$ (resp., $\rho > 0$).

(ii) If $\rho_o(x, \vartheta)$ and $\mathrm{v}(t, x, \vartheta)$ are 1-periodic in $\vartheta \in \mathbb{R}$, then so is $\rho(t, x, \vartheta)$. In that case, the value $\int_{S^1} \rho(t, x, \varphi)\, d\varphi$ is a constant in time for all $x \in X$. The latter is in fact a common property of the continuity equation.

(iii) If $\bigcup_{x\in X} \operatorname{supp} \rho_o(x, \cdot)$ is relatively compact in $\mathbb{R}$ (i.e. has compact closure), then for any compact interval $J_o \subseteq J$ the union $\bigcup_{t\in J_o} \bigcup_{x\in X} \operatorname{supp} \rho(t, x, \cdot)$ is also relatively compact. In that case, the value $\int_\Theta \rho(t, x, \varphi)\ d\varphi$ is a constant in time for all $x \in X$.

Theorem 3.6 (Periodic solutions to the continuity equation)**.** *Let Ω_{o,S^1} and Ω_{S^1} be the function spaces defined in (8) and (9) respectively. Suppose the velocity field $\mathrm{v} : \mathbb{R} \times X \times S^1 \times \Omega_{o,S^1} \to \mathbb{R}$ satisfies the following conditions:*

V1. The mappings given by $\rho_1 \mapsto \mathrm{v}(\cdot, \cdot, \cdot, \rho_1)$, $\rho_1 \mapsto \partial_\vartheta \mathrm{v}(\cdot, \cdot, \cdot, \rho_1)$, and $\rho_1 \mapsto \partial^2_\vartheta \mathrm{v}(\cdot, \cdot, \cdot, \rho_1)$ are well-defined, bounded, and continuous, as mappings from Ω_{o,S^1} to $\mathcal{C}_{\mathrm{u,b}}(\mathbb{R} \times X \times S^1, \mathbb{R})$.

V2. Both v and $\partial_\vartheta \mathrm{v}$ are Lipschitz continuous in Ω_{o,S^1} with a uniform Lipschitz constant.

Let the initial state $\rho_o \in \Omega_{o,S^1}$ be such that $\partial_\vartheta \rho_o$ is uniformly continuous and bounded. Then in Ω_{S^1} there exists a maximal solution to the initial value problem

$$\rho(0, \cdot, \cdot) = \rho_o, \qquad \partial_t \rho(t, x, \vartheta) = -\partial_\vartheta \left[\rho(t, x, \vartheta) \cdot \mathrm{v}(t, x, \vartheta, \rho(t, \cdot, \cdot))\right], \tag{17}$$

defined for all times $t \in \mathbb{R}$. This solution ρ is of class $\mathcal{C}^1\big(\mathbb{R}, \mathcal{C}_{\mathrm{u,b}}(X \times S^1, \mathbb{R})\big)$, $\partial_\vartheta \rho$ is continuous and $(\partial_\vartheta \rho)(t, \cdot, \cdot)$ is uniformly continuous and bounded for every $t \in \mathbb{R}$.

Proof. We identify v with its pullback on $\mathbb{R}$ (with respect to the phase ϑ) and identify

$$\begin{aligned}\Omega_{o,S^1} &= \{\rho \in \mathcal{C}_{\mathrm{u,b}}(X \times \mathbb{R}, \mathbb{R}_+) :\ \rho(x, \cdot) \text{ 1-periodic} \\ &\qquad\qquad \wedge\ \|\rho(x, \cdot)\|_{L^1(S^1)} = 1\ \forall x \in X\}, \\ \Omega_{S^1} &= \{\rho \in \mathcal{C}(J, \Omega_{o,S^1}) :\ 0 \in J \subseteq \mathbb{R} \text{ interval } \wedge\ \exists \partial_\vartheta \rho, \partial_t \rho \\ &\qquad\qquad \wedge\ (\partial_t \rho)(\cdot, x, \cdot) \text{ is continuous } \forall x \in X\}.\end{aligned} \tag{18}$$

We prove the theorem without loss of generality for the spaces (18) instead. Fix some compact time interval $J = [t_o, t_1] \subseteq \mathbb{R}$ ($t_1 > t_o$). Note that ρ_o satisfies the axioms of the auxiliary statement 3.4. For any $\rho \in \mathcal{C}(J, \Omega_{o,S^1})$ define $\mathrm{v}[\rho] : J \times X \times \mathbb{R} \to \mathbb{R}$ by $\mathrm{v}[\rho](t, x, \vartheta) = \mathrm{v}(t, x, \vartheta, \rho(t, \cdot, \cdot))$. Then $\mathrm{v}[\rho]$ satisfies the assumptions of the auxiliary statement 3.4. To see this, observe that the image $\rho(J) \subseteq \Omega_{o,S^1}$ is compact so that v is by Remark 2.1(2) uniformly continuous and bounded on $\mathbb{R} \times X \times \mathbb{R} \times \rho(J)$. Since $\rho(t, \cdot, \cdot)$ is uniformly continuous in t, this shows that the mapping $(t, x, \vartheta) \mapsto \mathrm{v}(t, x, \vartheta, \rho(t, \cdot, \cdot))$ is uniformly continuous and bounded. Similar arguments hold for $\partial_\vartheta \mathrm{v}$ and $\partial^2_\vartheta \mathrm{v}$. Let $\Phi[\rho] : J^2 \times X \times \mathbb{R} \to \mathbb{R}$ be the flow generated by $\mathrm{v}[\rho]$ as in the auxiliary statement 3.4. Consider the metric space

$$\Omega_J = \left\{\rho \in \mathcal{C}_{\mathrm{b}}(J, \Omega_{o,S^1}) : \rho(t_o, \cdot, \cdot) = \rho_o\right\},$$

endowed with the supremum metric. Note that Ω_{o,S^1} is a closed subset of the complete metric space $\mathcal{C}_{\mathrm{u,b}}(X\times\mathbb{R},\mathbb{R})$, so that Ω_J is complete. Consider the function $A:\Omega_J\to\Omega_J$ defined by

$$A[\rho](t,x,\vartheta)=\rho_o\left(x,\Phi[\rho](t_o,t,x,\vartheta)\right)\cdot(\partial_\vartheta\Phi[\rho])(t_o,t,x,\vartheta), \tag{19}$$

for $(t,x,\vartheta)\in J\times X\times\mathbb{R}$. Note that by 3.4(1) and Remarks 3.5(i) and 3.5(ii), A indeed maps Ω_J into Ω_J. We will show that $A[\rho]$ is a contraction on the complete metric space Ω_J for sufficiently small $|J|$.

Let L_{v} be a Lipschitz constant for v and ∂_ϑv in Ω_{o,S^1}, as predicted by (V2). Since ∂_ϑv and ∂_ϑ^2v are bounded, v and ∂_ϑv are Lipschitz continuous in ϑ with Lipschitz constant, say without loss of generality, L_{v}. For any $(t,s,x,\vartheta)\in J^2\times X\times\mathbb{R}$ and $\rho_1,\rho_2\in\Omega_J$, one can therefore estimate

$$\begin{aligned}
&|\Phi[\rho_1](t,s,x,\vartheta)-\Phi[\rho_2](t,s,x,\vartheta)|\\
&\leq\int_s^t\big|\mathrm{v}[\rho_1]\left(\tau,x,\Phi[\rho_1](\tau,s,x,\vartheta)\right)-\mathrm{v}[\rho_2]\left(\tau,x,\Phi[\rho_2](\tau,s,x,\vartheta)\right)\big|\,d\tau\\
&\leq\int_s^t\big|\mathrm{v}[\rho_1]\left(\tau,x,\Phi[\rho_1](\tau,s,x,\vartheta)\right)-\mathrm{v}[\rho_1]\left(\tau,x,\Phi[\rho_2](\tau,s,x,\vartheta)\right)\big|\,d\tau\\
&\quad+\int_s^t\big|\mathrm{v}[\rho_1]\left(\tau,x,\Phi[\rho_2](\tau,s,x,\vartheta)\right)-\mathrm{v}[\rho_2]\left(\tau,x,\Phi[\rho_2](\tau,s,x,\vartheta)\right)\big|\,d\tau\\
&\leq\int_s^t L_{\mathrm{v}}\big|\Phi[\rho_1](\tau,s,x,\vartheta)-\Phi[\rho_2](\tau,s,x,\vartheta)\big|\,d\tau+|J|\,L_{\mathrm{v}}\,\|\rho_1-\rho_2\|\,.
\end{aligned}$$

By the integral Grönwall inequality [8], this implies

$$\begin{aligned}
|\Phi[\rho_1](t,s,x,\vartheta)-\Phi[\rho_2](t,s,x,\vartheta)|\leq&e^{|J|L_{\mathrm{v}}}\,|J|\,L_{\mathrm{v}}\,\|\rho_1-\rho_2\|\\
=:&L_\Phi(|J|)\,\|\rho_1-\rho_2\|\,.
\end{aligned}\tag{20}$$

From the auxiliary statement 3.2(3) we know that

$$\partial_\vartheta\Phi[\rho](t,s,x,\vartheta)=\exp\left[\int_s^t(\partial_\vartheta\mathrm{v}[\rho])(\tau,x,\Phi[\rho](\tau,s,x,\vartheta)\,d\tau\right],$$

so we can estimate

$$\begin{aligned}
&|\partial_\vartheta\Phi[\rho_1](t,s,x,\vartheta)-\partial_\vartheta\Phi[\rho_2](t,s,x,\vartheta)|\\
\leq&e^{|J|\|\partial_\vartheta\mathrm{v}\|_\infty}\int_s^t|(\partial_\vartheta\mathrm{v}[\rho_1])(\tau,x,\Phi[\rho_1](\tau,s,x,\vartheta))\\
&\qquad\qquad-(\partial_\vartheta\mathrm{v}[\rho_2])(\tau,x,\Phi[\rho_2](\tau,s,x,\vartheta))|\,d\tau\\
\leq&e^{|J|\|\partial_\vartheta\mathrm{v}\|_\infty}\int_s^t|(\partial_\vartheta\mathrm{v}[\rho_1])(\tau,x,\Phi[\rho_1](\tau,s,x,\vartheta))\\
&\qquad\qquad-(\partial_\vartheta\mathrm{v}[\rho_1])(\tau,x,\Phi[\rho_2](\tau,s,x,\vartheta))|\,d\tau\\
&+e^{|J|\|\partial_\vartheta\mathrm{v}\|_\infty}\int_s^t|(\partial_\vartheta\mathrm{v}[\rho_1])(\tau,x,\Phi[\rho_2](\tau,s,x,\vartheta))\\
&\qquad\qquad-(\partial_\vartheta\mathrm{v}[\rho_2])(\tau,x,\Phi[\rho_2](\tau,s,x,\vartheta))|\,d\tau\\
\leq&e^{|J|\|\partial_\vartheta\mathrm{v}\|_\infty}\cdot L_{\mathrm{v}}\int_s^t\big|\Phi[\rho_1](\tau,s,x,\vartheta)-\Phi[\rho_2](\tau,s,x,\vartheta)\big|\,d\tau\\
&+e^{|J|\|\partial_\vartheta\mathrm{v}\|_\infty}L_{\mathrm{v}}\,|J|\,\|\rho_1-\rho_2\|\\
\leq&e^{|J|\|\partial_\vartheta\mathrm{v}\|_\infty}L_{\mathrm{v}}\,|J|\,\big[L_\Phi(|J|)+1\big]\,\|\rho_1-\rho_2\|=:L_{\partial_\vartheta\Phi}(|J|)\,\|\rho_1-\rho_2\|\,.
\end{aligned}\tag{21}$$

For the last inequality we used (20). Since $\partial_\vartheta \rho_o$ is bounded, ρ_o is Lipschitz continuous in ϑ with Lipschitz constant $\|\partial_\theta \rho_o\|_\infty$. Using (20) and (21), as well as the bound

$$\|\partial_\vartheta \Phi[\rho](t_o, t, x, \vartheta)\| \le e^{|t-t_o|\|\partial_\vartheta \mathrm{v}\|_\infty}, \tag{22}$$

we can estimate, for $(t, x, \vartheta) \in J \times X \times \mathbb{R}$ and $\rho_1, \rho_2 \in \Omega_J$,

$$\begin{aligned}
&|A[\rho_1](t,x,\vartheta) - A[\rho_2](t,x,\vartheta)| \\
\le{}& \|\rho_o\|_\infty \|\partial_\vartheta \Phi[\rho_1](t_o,t,x,\vartheta) - \partial_\vartheta \Phi[\rho_2](t_o,t,x,\vartheta)\| \\
&+ e^{|J|\|\partial_\vartheta \mathrm{v}\|_\infty} \|\partial_\theta \rho_o\|_\infty \|\Phi[\rho_1](t_o,t,x,\vartheta) - \Phi[\rho_2](t_o,t,x,\vartheta)\| \\
\le{}& \Big[\|\rho_o\|_\infty L_{\partial_\vartheta \Phi}(|J|) + e^{|J|\|\partial_\vartheta \mathrm{v}\|_\infty} \|\partial_\theta \rho_o\|_\infty L_\Phi(|J|)\Big] \|\rho_1 - \rho_2\| \\
=:{}& L_A\left[|J|, \|\rho_o\|_\infty, \|\partial_\theta \rho_o\|_\infty\right] \|\rho_1 - \rho_2\|,
\end{aligned}$$

with L_Φ and $L_{\partial_\vartheta \Phi}$ as defined in (20) and (21) respectively. This shows that A is Lipschitz continuous on Ω_J with Lipschitz constant $L_A\left[|t_1 - t_o|, \|\rho_o\|_\infty, \|\partial_\theta \rho_o\|_\infty\right]$, the latter tending to 0 as $|t_1 - t_o| \to 0$. Now choose $|J| = |t_1 - t_o|$ small enough so that $L_A\left[|t_1 - t_o|, \|\rho_o\|_\infty, \|\partial_\theta \rho_o\|_\infty\right] < 1$. Then by Banach's fixed point theorem, A has a unique fixed point ρ_J in Ω_J. By the auxiliary statement 3.4, ρ_J is in $\mathcal{C}^1(J, \Omega_{o,S^1})$ and satisfies $\partial_\vartheta \rho_J(t, \cdot, \cdot) \in \mathcal{C}_{\mathrm{u,b}}(X \times \mathbb{R}, \mathbb{R})$ for every $t \in J$ as well as $\partial_\vartheta \rho_J \in \mathcal{C}(J \times X \times \mathbb{R}, \mathbb{R})$. By 3.4(2) it satisfies the continuity equation $\partial_t \rho_J = -\partial_\vartheta(\rho_J \cdot \mathrm{v}[\,\rho_J])$ and by construction the initial condition $\rho_J(t_o, \cdot, \cdot) = \rho_o$.

Note that $\rho_J(t_1, \cdot, \cdot)$ satisfies again the conditions on the initial state imposed in this theorem. Therefore, ρ_J can be extended to $[t_o, t_2]$ for some $t_2 > t_1$ sufficiently small so that $L_A\left[|t_2 - t_1|, \|\rho_J(t_1)\|_\infty, \|\partial_\theta \rho_J(t_1)\|_\infty\right] < 1$. That extension is in $\mathcal{C}([t_o, t_2], \Omega_{o,S^1})$. Since by the continuity equation $\partial_t \rho_J(t_1, \cdot, \cdot)$ is completely determined by $\rho_J(t_1, \cdot, \cdot)$, this extension is in fact in $\mathcal{C}^1([t_o, t_2], \Omega_{o,S^1})$. Similarly, $\partial_\vartheta \rho_J \in \mathcal{C}([t_o, t_2] \times X \times \mathbb{R}, \mathbb{R})$. In order to apply an induction argument to extend ρ_J to the whole real line, we need to make sure that the size of possible time steps $|t_{n+1} - t_n|$ does not decrease too fast with increasing n. By the auxiliary statement 3.2(5) and estimate (22), one can either using Grönwall's inequality or direct calculation to estimate

$$\|\partial_\vartheta^2 \Phi[\rho](t_o, t, \cdot, \cdot)\|_\infty \le e^{3|t-t_o|\|\partial_\vartheta \mathrm{v}\|_\infty} |t - t_o| \, \|\partial_\vartheta^2 \mathrm{v}\|_\infty.$$

By (19) this leads to the estimates

$$\begin{aligned}
&\|\rho_J(t, \cdot, \cdot)\|_\infty \le \|\rho_o\|_\infty e^{|t-t_o|\|\partial_\vartheta \mathrm{v}\|_\infty}, \\
&\|\partial_\vartheta \rho_J(t, \cdot, \cdot)\|_\infty \le e^{3|t-t_o|\|\partial_\vartheta \mathrm{v}\|_\infty} \left[\|\partial_\vartheta \rho_o\|_\infty + \|\rho_o\|_\infty |t - t_o| \, \|\partial_\vartheta^2 \mathrm{v}\|_\infty\right]
\end{aligned} \tag{23}$$

for all t in the domain of ρ_J. This implies that we can always extend ρ_J to any arbitrary finite time interval, say for example $[t_o, t_o + 1]$. Indeed, at any time $t_n \in [t_o, t_o + 1]$ in the domain of ρ_J and any $t_{n+1} \in [t_n, t_o + 1]$ one can by (23) estimate

$$\begin{aligned}
&L_A\left[|t_{n+1} - t_n|, \|\rho_J(t_n)\|_\infty, \|\partial_\vartheta \rho_J(t_n)\|_\infty\right] \\
\le{}& e^{4\|\partial_\vartheta \mathrm{v}\|_\infty} \big[\, \|\rho_o\|_\infty L_{\partial_\vartheta \Phi}(|t_{n+1} - t_n|) \\
&+ \left[\|\partial_\vartheta \rho_o\|_\infty + \|\rho_o\|_\infty \|\partial_\vartheta^2 \mathrm{v}\|_\infty\right] L_\Phi(|t_{n+1} - t_n|)\big].
\end{aligned} \tag{24}$$

The length $|t_{n+1} - t_n|$ can thus be chosen small enough for the Lipschitz constant on the left hand side of (24) to be less than 1, uniformly in $t_n \in [t_o, t_o + 1]$. Consequently, a solution $\rho \in \mathcal{C}^1(\mathbb{R}, \Omega_{o,S^1})$ can by induction be constructed for the initial value problem (17), satisfying $\partial_\vartheta \rho(t,\cdot,\cdot) \in \mathcal{C}_{\mathrm{u,b}}(X\times\mathbb{R},\mathbb{R})$ for every $t\in\mathbb{R}$ as well as $\partial_\vartheta \rho_J \in \mathcal{C}(\mathbb{R}\times X\times\mathbb{R},\mathbb{R})$.

It remains to show uniqueness. By Remark 3.1, any local solution $\tilde\rho\in\Omega_{S^1}$ to the initial value problem defined on some interval $0\in\tilde J\subseteq\mathbb{R}$ is a fixed point of $A:\Omega_{\tilde J}\to\Omega_{\tilde J}$. If $L_A(|\tilde J|, \|\rho_o\|_\infty, \|\partial_\vartheta\rho_o\|_\infty)<1$, then the uniqueness of the fixed point implies $\tilde\rho=\rho|_{\tilde J}$. In any other case a similar induction by interval-splitting can be used to show that $\tilde\rho=\rho|_{\tilde J}$, noting that $\rho(t,\cdot,\cdot)$ satisfies for all $t\in\mathbb{R}$ the initial state conditions of this theorem and an estimate similar to (23). ☐

Theorem 3.7 (Evolution of unwrappings)**.** *Let the velocity field* $\mathrm{v}:\mathbb{R}\times X\times S^1\times\Omega_{o,S^1}\to\mathbb{R}$ *satisfy the assumptions (V1) and (V2) of Theorem 3.6. Let* $\nu:\mathbb{R}\times X\times\mathbb{R}\times\Omega_{o,\mathbb{R}}\to\mathbb{R}$ *satisfy the following conditions:*

N1. ν *is bounded.*
N2. $\nu(\cdot,x,\cdot,\cdot)$ *and* $\partial_\vartheta\nu(\cdot,x,\cdot,\cdot)$ *are continuous and bounded for every* $x\in X$*.*
N3. ν *is Lipschitz continuous in* ϑ *with a uniform Lipschitz constant.*
N4. For all $\rho_1\in\Omega_{o,\mathbb{R}}$ *with* $\operatorname{diam}_{\mathrm{ph}}\rho_1<\infty$ *one has* $\nu(t,x,\vartheta,\rho_1)=\mathrm{v}(t,x,\vartheta,\Pi_{\mathrm{w}}(\rho_1))$*, with the wrapping operator* Π_{w} *defined in definition 2.2.*

Let the initial state $\rho_o\in\Omega_{o,S^1}$ *be such that* $\partial_\vartheta\rho_o$ *is uniformly continuous and bounded. Let* $\rho\in\mathcal{C}^1(\mathbb{R},\mathcal{C}_{\mathrm{u,b}}(X\times S^1,\mathbb{R}))$ *be the maximal solution to the initial value problem*

$$\rho(0,\cdot,\cdot)=\rho_o,\qquad \partial_t\rho(t,x,\vartheta)=-\partial_\vartheta\left[\rho(t,x,\vartheta)\cdot\mathrm{v}(t,x,\vartheta,\rho(t,\cdot,\cdot))\right]$$

within Ω_{S^1}*, as predicted by Theorem 3.6. Let* $\zeta_o\in\Omega_{o,\mathbb{R}}$ *be a an unwrapping of* ρ_o *with* $\operatorname{diam}_{\mathrm{ph}}\zeta_o<\infty$ *and such that* $\partial_\vartheta\zeta_o$ *is uniformly continuous and bounded. Then there exists in* $\Omega_{\mathbb{R}}$ *a maximal solution* ζ *to the continuity equation*

$$\partial_t\zeta(t,x,\vartheta)=-\partial_\vartheta\left[\zeta(t,x,\vartheta)\cdot\nu(t,x,\vartheta,\zeta(t,\cdot,\cdot))\right],\quad \zeta(0,\cdot,\cdot)=\zeta_o. \tag{25}$$

The solution $\zeta(t,\cdot,\cdot)$ *is an unwrapping of* $\rho(t,\cdot,\cdot)$ *for all* $t\in\mathbb{R}$ *and satisfies* $\zeta\in\mathcal{C}^1(\mathbb{R},\mathcal{C}_{\mathrm{u,b}}(X\times\mathbb{R},\mathbb{R}))$*, while* $\partial_\vartheta\zeta$ *is continuous. Furthermore,* $\partial_\vartheta\zeta(t,\cdot,\cdot)\in\mathcal{C}_{\mathrm{u,b}}(X\times\mathbb{R},\mathbb{R})$ *and*

$$\operatorname{supp}_{\mathrm{ph}}\zeta(t,\cdot,\cdot)\subseteq\left\{\vartheta\in\mathbb{R}: d\left(\vartheta,\ \operatorname{supp}_{\mathrm{ph}}\zeta_o\right)\le\|\nu\|_\infty|t|\right\}$$

for every $t\in\mathbb{R}$*.*

Proof. We start with uniqueness. Suppose $\zeta:J\times X\times\mathbb{R}\to\mathbb{R}_+$ is of class $\Omega_{\mathbb{R}}$, defined on some time interval $J\subseteq\mathbb{R}$ and solving the initial value problem (25). Fix $x\in X$. Then the mappings $(t,\vartheta)\mapsto\nu(t,x,\vartheta,\zeta(t,\cdot,\cdot))$ and $(t,\vartheta)\mapsto\partial_\vartheta\nu(t,x,\vartheta,\zeta(t,\cdot,\cdot))$ are bounded and continuous by assumption (N2). The velocity field $\nu(t,x,\vartheta,\zeta(t,\cdot,\cdot))$ therefore satisfies the assumptions in Remark 3.1. By that remark, ζ is of the form

$$\zeta(t,x,\vartheta)=\zeta_o(x,\Phi(0,t,x,\vartheta))\cdot\exp\left[\int_t^0\left(\partial_\vartheta\nu\right)(\tau,x,\Phi(\tau,t,x,\vartheta),\zeta(\tau,\cdot,\cdot))\,d\tau\right], \tag{26}$$

with $\Phi:J^2\times X\times\mathbb{R}\to\mathbb{R}$ being the flow generated by the velocity field $\nu:(t,\vartheta)\mapsto\nu(t,x,\vartheta,\zeta(t,\cdot,\cdot))$. Furthermore, by that remark $\zeta(t,\cdot,\cdot)$ satisfies

$$\operatorname{supp}_{\mathrm{ph}}\zeta(t,\cdot,\cdot)\subseteq\left\{\vartheta\in\mathbb{R}: d\left(\vartheta,\ \operatorname{supp}_{\mathrm{ph}}\zeta_o\right)\le\|\nu\|_\infty|t|\right\}$$

for all $t \in J$. Therefore by assumption (N4) $\nu(t,x,\vartheta,\zeta(t,\cdot,\cdot)) = \mathrm{v}(t,x,\vartheta,\Pi_{\mathrm{w}}(\zeta(t,\cdot,\cdot)))$ and by the continuity equation,

$$\begin{aligned}\partial_t\Pi_{\mathrm{w}}(\zeta(t,\cdot,\cdot))(x,\vartheta) &= \sum_{n\in\mathbb{Z}}\partial_t\zeta(t,x,\vartheta+n)\\ &= -\partial_\vartheta\sum_{n\in\mathbb{Z}}\nu(t,x,\vartheta+n,\zeta(t,\cdot,\cdot))\cdot\zeta(t,x,\vartheta+n)\\ &= -\partial_\vartheta\Big[\mathrm{v}(t,x,\vartheta,\Pi_{\mathrm{w}}(\zeta(t,\cdot,\cdot)))\sum_{n\in\mathbb{Z}}\zeta(t,x,\vartheta+n)\Big]\\ &= -\partial_\vartheta\left[\mathrm{v}(t,x,\vartheta,\Pi_{\mathrm{w}}(\zeta(t,\cdot,\cdot)))\cdot\Pi_{\mathrm{w}}(\zeta(t,\cdot,\cdot))(x,\vartheta)\right].\end{aligned} \tag{27}$$

Note that the sum in (27) a finite one for all $t \in J$, whose order can be chosen to be constant for any compact time interval. Therefore $\Pi_{\mathrm{w}}(\zeta) \in \Omega_{S^1}$ and $\Pi_{\mathrm{w}}(\zeta)$ satisfies the continuity equation with initial value $\Pi_{\mathrm{w}}(\zeta(0,\cdot,\cdot)) = \rho_o$. By maximality of ρ, this implies $\Pi_{\mathrm{w}}(\zeta) = \rho|_J$. Therefore $\nu(t,x,\vartheta,\zeta(t,\cdot,\cdot)) = \mathrm{v}(t,x,\vartheta,\rho(t,\cdot,\cdot))$ for all $t \in J$, which by (26) implies the uniqueness of ζ on J.

It remains to show existence. Define the velocity field $\kappa : \mathbb{R}\times X\times\mathbb{R}\to\mathbb{R}$ by $\kappa(t,x,\vartheta) = \mathrm{v}(t,x,\vartheta,\rho(t,\cdot,\cdot))$. Then for any compact time interval $J := [0,t_1] \subseteq \mathbb{R}$, the mapping $t \mapsto \rho(t,\cdot,\cdot)$ is uniformly continuous as a function $J \to \Omega_{o,S^1}$, and its range over J is compact. By assumption (V1) and Remark 2.1(2), the restriction $\mathrm{v} : J\times X\times\mathbb{R}\times\rho(J)\to\mathbb{R}$ is uniformly continuous and bounded, hence so is $\kappa : J\times X\times\mathbb{R}\to\mathbb{R}$. Similar conclusions hold for $\partial_\vartheta\kappa$ and $\partial_\vartheta^2\kappa$. Thus κ satisfies the axioms of the auxiliary statement 3.4, by which the initial value problem $\partial_t\zeta = -\partial_\vartheta(\zeta\cdot\kappa)$ with initial value $\zeta(0,\cdot,\cdot) = \zeta_o$ has a solution $\zeta \in \mathcal{C}^1(J,\mathcal{C}_{\mathrm{u,b}}(X\times\mathbb{R},\mathbb{R}))$. This solution satisfies $\partial_\vartheta\zeta(t,\cdot,\cdot) \in \mathcal{C}_{\mathrm{u,b}}(X\times\mathbb{R},\mathbb{R})$ for every $t \in J$ and $\partial_\vartheta\zeta \in \mathcal{C}(J\times X\times\mathbb{R},\mathbb{R})$. By Remarks 3.5 (i) and (iii), one has $\zeta(t,\cdot,\cdot) \in \Omega_{o,\mathbb{R}}$, $\forall t \in J$. Furthermore, the set $\bigcup_{t\in J}\bigcup_{x\in X}\operatorname{supp}\zeta(t,x,\cdot)$ is bounded in $\mathbb{R}$ by Remark 3.5(iii). Since $\zeta(t_1,\cdot,\cdot)$ again satisfies the initial state conditions of the auxiliary statement 3.4, the same reasoning can be used to extend ζ to $[0,2t_1]$, resulting in $\zeta \in \mathcal{C}([0,2t_1],\Omega_{o,\mathbb{R}})$. Since $\partial_t\zeta(t_1,\cdot,\cdot)$ depends by the continuity equation only on $\zeta(t_1,\cdot,\cdot)$, one has in fact $\zeta \in \mathcal{C}^1([0,2t_1],\mathcal{C}_{\mathrm{u,b}}(X\times\mathbb{R},\mathbb{R}))$. Similarly, $\partial_\vartheta\zeta \in \mathcal{C}([0,2t_1]\times X\times\mathbb{R},\mathbb{R})$. Inductively, one arrives at constructing $\zeta \in \mathcal{C}^1(\mathbb{R},\mathcal{C}_{\mathrm{u,b}}(X\times\mathbb{R},\mathbb{R}))\cap\Omega_{\mathbb{R}}$ with the smoothness properties postulated by the theorem. Just as in (27), we find that

$$\partial_t\Pi_{\mathrm{w}}(\zeta)(t,x,\vartheta) = -\partial_\vartheta\left[\Pi_{\mathrm{w}}(\zeta)(t,x,\vartheta)\cdot\mathrm{v}(t,x,\vartheta,\rho(t,\cdot,\cdot))\right],$$

with $\Pi_{\mathrm{w}}(\zeta) \in \Omega_{o,S^1}$ and $\Pi_{\mathrm{w}}(\zeta)(0,\cdot,\cdot) = \rho_o$. By uniqueness of solutions given in the auxiliary statement 3.4(3), one has $\Pi(\zeta) = \rho$, so that $\partial_t\zeta(t,x,\vartheta) = -\partial_\vartheta\left[\zeta(t,x,\vartheta)\cdot\nu(t,x,\vartheta,\zeta(t,\cdot,\cdot))\right]$. ☐

Theorem 3.6 is our main result on the existence and uniqueness of solutions to the continuity equation on the circle S^1. Theorem 3.7 connects this result to similar statements on the continuity equation for densities on the real line. The theorem shows what should be intuitively clear: For velocity fields $\nu(t,x,\vartheta,\rho)$ depending only on the projection of ϑ to S^1 and the wrapping of ρ, the wrapping and evolution of densities should be commuting operations. In the next Theorem 3.9, we apply the above results to our fluid model of pulse-coupled phase oscillators, defined by the velocity field (5). For the proof of the theorem we require the following technical auxiliary statement.

Auxiliary statement 3.8 (Continuity of stimulus in oscillator networks)**.** *Let $(K, \mathscr{K}, \kappa)$ be a σ-finite measure space. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$ and consider the function class*

$$\Omega_o = \left\{ \rho \in \mathcal{M}_{\mathrm{b}}(X \times K, \mathbb{K}) : \|\rho(x,\cdot)\|_{L^1(\kappa)} = 1 \ \forall x \in X \right\},$$

endowed with the supremum metric. Let $I : X \times K \to \mathbb{K}$ be measurable and bounded. Let $G : X \times X \to \mathbb{K}$ be measurable, such that the mapping $x \mapsto G(x,\cdot) \in L^1(\mu)$ is uniformly continuous and bounded. Then the stimulus

$$S(x,\rho) := \int_X G(x,y) \int_K \rho(y,\varphi) I(y,\varphi) \ d\kappa(\varphi) \ d\mu(y)$$

is well-defined for any $x \in X$ and $\rho \in \Omega_o$ and satisfies the following:

1. *$S(\cdot,\rho)$ is uniformly continuous and bounded for every $\rho \in \Omega_o$, and $S(x,\cdot)$ is continuous and bounded for every $x \in X$. Furthermore, S is bounded on $X \times \Omega_o$.*
2. *If $\kappa(K)$ is finite, then S is uniformly continuous and bounded on $X \times \Omega_o$ and Lipschitz continuous in Ω_o with a Lipschitz constant uniform in X.*

For a proof see appendix A.1.

Theorem 3.9 (Solutions for oscillator networks)**.** *Let $G : X \times X \to \mathbb{R}$ be measurable and such that the mapping $x \mapsto G(x,\cdot) \in L^1(\mu)$ is bounded and uniformly continuous. Let $I : X \times S^1 \to \mathbb{R}$ be measurable and bounded. Let $u, \psi : \mathbb{R} \times X \times S^1 \to \mathbb{R}$ be such that $\psi, \partial_\vartheta \psi, \partial_\vartheta^2 \psi, u, \partial_\vartheta u$ and $\partial_\vartheta^2 u$ are uniformly continuous and bounded. Let Ω_{o,S^1}, $\Omega_{o,\mathbb{R}}$, Ω_{S^1} and $\Omega_{\mathbb{R}}$ be the function spaces defined in (8) and (9). Define the velocity field $\mathrm{v} : \mathbb{R} \times X \times S^1 \times \Omega_{o,S^1} \to \mathbb{R}$ as*

$$\mathrm{v}(t,x,\vartheta,\rho_o) := u(t,x,\vartheta) + \psi(t,x,\vartheta) \cdot \int_X G(x,y) \int_{S^1} \rho_o(y,\varphi) I(y,\varphi) \ d\varphi \ d\mu(y).$$

Identify u, ψ and I with their pullbacks on $\mathbb{R}$ (with respect to ϑ) and define the velocity field $\nu : \mathbb{R} \times X \times \mathbb{R} \times \Omega_{o,\mathbb{R}} \to \mathbb{R}$ as

$$\nu(t,x,\vartheta,\zeta_o) := u(t,x,\vartheta) + \psi(t,x,\vartheta) \cdot \int_X G(x,y) \int_{\mathbb{R}} \zeta_o(y,\varphi) I(y,\varphi) \ d\varphi \ d\mu(y).$$

Let $\rho_o \in \Omega_{o,S^1}$ be an initial state and $\zeta_o \in \Omega_{o,\mathbb{R}}$ an unwrapping of ρ_o, such that $\partial_\vartheta \rho_o, \partial_\vartheta \zeta_o$ are uniformly continuous and bounded and $\mathrm{diam}_{\mathrm{ph}}\, \zeta_o < \infty$. Then there exists in Ω_{S^1} a maximal, global solution to the initial value problem

$$\rho(0,\cdot,\cdot) = \rho_o, \qquad \partial_t \rho(t,x,\vartheta) = -\partial_\vartheta \left[\rho(t,x,\vartheta) \cdot \mathrm{v}(t,x,\vartheta,\rho(t,\cdot,\cdot)) \right].$$

The solution ρ is in $\mathcal{C}^1(\mathbb{R}, \mathcal{C}_{\mathrm{u,b}}(X \times S^1, \mathbb{R}))$, $\partial_\vartheta \rho(t,\cdot,\cdot)$ is uniformly continuous and bounded for all $t \in \mathbb{R}$, and $\partial_\vartheta \rho$ is continuous. Similarly, there exists in $\Omega_{\mathbb{R}}$ a maximal, global solution to the initial value problem

$$\zeta(0,\cdot,\cdot) = \zeta_o, \qquad \partial_t \zeta(t,x,\vartheta) = -\partial_\vartheta \left[\zeta(t,x,\vartheta) \cdot \nu(t,x,\vartheta,\zeta(t,\cdot,\cdot)) \right].$$

The solution ζ is in $\mathcal{C}^1(\mathbb{R}, \mathcal{C}_{\mathrm{u,b}}(X \times \mathbb{R}, \mathbb{R}))$, $\partial_\vartheta \zeta(t,\cdot,\cdot)$ is uniformly continuous and bounded for all $t \in \mathbb{R}$, and $\partial_\vartheta \zeta$ is continuous. Furthermore, for all $t \in \mathbb{R}$ one has

$$\mathrm{supp}_{\mathrm{ph}}\, \zeta(t,\cdot,\cdot) \subseteq \left\{ \vartheta \in \mathbb{R} : d\left(\vartheta, \ \mathrm{supp}_{\mathrm{ph}}\, \zeta_o\right) \le \|\nu\|_\infty |t| \right\}.$$

Finally, $\rho(t,\cdot,\cdot)$ is a wrapping of $\zeta(t,\cdot,\cdot)$, $\forall t \in \mathbb{R}$.

The last statement underlines the equivalence of solutions obtained by either regarding the oscillator phase on the real line, or on S^1.

Proof. We will show that the velocity fields v and ν satisfy the assumptions of Theorem 3.7. For $\rho_1 \in \Omega_{o,S^1}, \zeta_1 \in \Omega_{o,\mathbb{R}}$, and $x \in X$ denote

$$S_{S^1}(x,\rho_1) := \int_X G(x,y) \int_{S^1} \rho_1(y,\varphi) I(y,\varphi)\ d\varphi\, d\mu(y),$$
$$S_{\mathbb{R}}(x,\zeta_1) := \int_X G(x,y) \int_{\mathbb{R}} \zeta_1(y,\varphi) I(y,\varphi)\ d\varphi\, d\mu(y).$$

By the auxiliary statement 3.8(2) we know that $S_{S^1} \in \mathcal{C}_{\mathrm{u,b}}(X \times \Omega_{o,S^1}, \mathbb{R})$ and S_{S^1} is Lipschitz continuous in Ω_{o,S^1} uniformly in X. By the assumptions on u and v, the velocity field $\mathrm{v} = u + \psi S_{S^1}$ as well as its derivatives $\partial_\vartheta \mathrm{v}$, $\partial^2_\vartheta \mathrm{v}$ are thus in $\mathcal{C}_{\mathrm{u,b}}(\mathbb{R} \times X \times S^1 \times \Omega_{o,S^1}, \mathbb{R})$. Similarly, v and $\partial_\vartheta \mathrm{v}$ are Lipschitz continuous in Ω_{o,S^1} uniformly in $\mathbb{R} \times X \times S^1$. Therefore v and ρ_o satisfy the assumptions of Theorem 3.7.

By 3.8(1), $S_{\mathbb{R}}$ is bounded and one has $S_{\mathbb{R}}(x,\cdot) \in \mathcal{C}_{\mathrm{b}}(\Omega_{o,\mathbb{R}}, \mathbb{R})$ for every $x \in X$. Consequently, ν satisfies conditions (N1) and (N2) of Theorem 3.7. Since $\partial_\vartheta \nu$ is bounded, ν is Lipschitz continuous in ϑ, uniformly in $\mathbb{R} \times X \times \Omega_{o,\mathbb{R}}$. Condition 3.7(N4) is also clearly satisfied. Applying Theorems 3.6 and 3.7 completes the proof. $\square$

4. **Synchrony.** In this section we consider the field and fluid model for networks composed of identical oscillators, that is with $u(t,x,\vartheta) \equiv \omega$, for some *intrinsic frequency* $\omega > 0$, $\psi(x,\vartheta) = \psi(\vartheta)$ and $I(x,\vartheta) = I(\vartheta)$ for all $x \in X$, $\vartheta \in S^1$, and $t \in \mathbb{R}$. We give sufficient conditions on the coupling kernel G, the response function ψ, and the pulse I for the existence and local stability of synchrony in both models. In the field model defined by the field equation

$$\partial_t \theta(t,x) = \omega + \psi(\theta(t,x)) \int_X G(x,y) I(\theta(t,y))\, d\mu(y), \tag{28}$$

synchrony takes the form $\theta(t,x) = \phi(t)$ for some function $\phi : \mathbb{R} \to S^1$. In the fluid model described by the continuity equation

$$\partial_t \rho(t,x,\vartheta) = -\partial_\vartheta \left[\rho(t,x,\vartheta) \left[\omega + \psi(\vartheta) \int_X G(x,y) \int_{S^1} I(\varphi)\rho(t,y,\varphi)\, d\varphi\, d\mu(y) \right] \right],$$

it takes the formal form of a Dirac distribution $\rho(t,x,\vartheta) = \delta(\vartheta - \phi(t))$ with time-dependent support $\phi(t)$. Since Dirac distributions are somewhat pathological when it comes to stability analysis, the methods we use and the results we obtain are of a special character, the measure of deviation from synchrony being the distribution's bandwidth. Throughout this section X will be a separable metric space and μ a σ-finite Borel measure on it.

In section 4.1 we introduce the property of *strong connectivity* for integral kernels as a generalization of irreducibility of non-negative matrices. This property will allow us in section 4.2 to use the generalized Jentzsch-Perron theorem, a generalization of the Perron-Frobenius theorem on irreducible matrices to operators on Banach lattices. Applying it to the coupling kernel G will give us the necessary insight to the spectrum of the linearized dynamics around the limit cycle of synchrony. With Theorem 4.10 we give a sufficient condition for the stability of synchrony in the field model. In section 4.3, and in particular Theorem 4.16, we prove the stability of synchrony for the fluid model under the assumption of its stability in the corresponding field model. This will be done by looking at the dynamics of the mean phase and any deviations about it. The proof of the theorem will underline the relationship between the two models, supplementing the interpretation provided in section 2.3.

4.1. **Strongly connecting integral kernels.** The connectivity of the oscillator network inevitably plays a crucial role in the emergence and stability of synchronization. For finite networks of a structure similar to the Winfree model, it has been shown that irreducibility of the coupling matrix is key to the stability of synchronization [18, 23]. In both of the cited articles, the Perron-Frobenius theorem was used to show the existence of a maximal positive eigenvalue of the coupling matrix with simple algebraic multiplicity. Irreducibility of the (non-negative) coupling matrix is equivalent to the strong connectivity of the graph it defines. We generalize the notion of irreducibility to so-called strong connectivity of integral kernels on σ-finite measure spaces. The idea is that, just as any two vertices in a strongly connected graph are connected by a directed path, so a strongly connecting kernel should *spread* any arbitrarily small *bump* to the whole space by means of iterated convolutions. We are then able to use a generalization of the Perron-Frobenius theorem for band-irreducible, positive operators on Banach lattices.

We refer the reader to [33] and [42] for extensive information on Banach lattices and the terminology used in this article. As an overview, a *lattice vector space* is a partially ordered vector space where the partial order is a lattice. A *Banach lattice* is a Banach space with a vector lattice structure that is compatible with the norm, that is $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. A sub-lattice vector space $\mathcal{B}_o$ of some lattice $\mathbb{R}$-vector space V_o is called a *band* in V_o if:

B1. For each $v \in V_o$ and $b \in \mathcal{B}_o$ with $|v| \leq |b|$, also $v \in \mathcal{B}_o$.

B2. For each $\emptyset \neq A \subseteq \mathcal{B}_o$ for which the supremum $\sup A$ exists in V_o, one has $\sup A \in \mathcal{B}_o$.

If V is the lattice $\mathbb{C}$-vector space given by complexification of V_o, then a *band* $\mathcal{B}$ in V is the complexification $\mathcal{B}_o + i\mathcal{B}_o$ of a band $\mathcal{B}_o$ in V_o. It still satisfies axiom (B1) with respect to the complex modulus $|x+iy| := \sup_{\vartheta \in S^1} |x\cos\vartheta + y\sin\vartheta|$ $(x, y \in V_o)$. A band $\mathcal{B}$ is called *trivial* if either $\mathcal{B} = \{0\}$ or $\mathcal{B} = V$. For any sequence $(h_n)_{n\in\mathbb{N}} \subseteq V_o$ and $h \in V_o$, we write $h_n \downarrow h$ if the sequence $(h_n)_n$ is decreasing and h is its infimum. A linear operator K on V is called *band irreducible* if there exists no non-trivial, K-invariant band in V. We say K is *σ-order continuous* if from $V_o \ni h_n \downarrow 0$ follows $Kh_n \downarrow 0$.

Definition 4.1 (Strongly connecting integral kernel)**.** Let $(K, \mathcal{K}, \kappa)$ be a σ-finite measure space and $G : K \times K \to \mathbb{R}_+$ be measurable. Let $\mathcal{F}$ be the set of measurable functions $h : K \to \mathbb{R}_+ \cup \{\infty\}$ and $\hat{G} : \mathcal{F} \to \mathcal{F}$ be the *integral operator corresponding to G*, defined as

$$(\hat{G}h)(x) = \int_K G(x,y)h(y)\, d\kappa(y), \quad h \in \mathcal{F},\, x \in K.$$

Note that $\hat{G}h : K \to \mathbb{R}_+$ is measurable by Tonelli's theorem [6, Theorem 5.2.1]. We say that G *strongly connects* K if for every $h \in \mathcal{F}$ with $\|h\|_{L^1(\kappa)} > 0$, the set

$$\bigcup_{n\in\mathbb{N}_0} \left(\hat{G}^n h\right)^{-1}((0,\infty])$$

has full measure, that is, its complement has measure zero. Note that for the definition it suffices to consider only indicator functions $h = 1_A$ of sets $A \in \mathcal{K}$ of non-zero measure.

Examples 4.2.

(i) If there exists a set $A \in \mathcal{K}$ such that $\kappa(A) > 0$, $\kappa(K \setminus A) > 0$, and $G(x, y) = 0$ for all $x \in A$, $y \in K \setminus A$, then G does not strongly connect K. See figure 2 for an example.

(ii) Let $G \in \mathbb{R}_+^{N \times N}$ be a non-negative matrix and let $K = \{1, .., N\}$ be endowed with the counting measure. Interpret $G : K \times K \to \mathbb{R}_+$ as an integral kernel. Then G strongly connects K if and only if the matrix G is irreducible, or equivalently, if and only if the underlying directed graph (which has a directed edge $m \to n$ whenever $G_{nm} > 0$) is strongly connected.

(iii) If the convolution $G * \cdots * G$ is strictly positive everywhere for some sufficiently large convolution order, then G strongly connects K.

(iv) Let K be a path-connected separable metric space and κ a strictly positive σ-finite Borel measure on K. Let G satisfy the following *blurring* property: For every $x \in K$ there exists an open neighbourhood U_x of x such that whenever $\kappa(U_x \cap A) > 0$ for some $A \in \mathscr{B}(K)$, one has $(\hat{G}1_A)|_{U_x} > 0$. Then G strongly connects K.

Proof. Note that the blurring property is equivalent to the following statement: For every $x \in K$ there exists an open neighbourhood U_x of x such that whenever $\|h \cdot 1_{U_x}\|_{L^1(\kappa)} > 0$ for some $h \in \mathcal{F}$, one has $(\hat{G}h)|_{U_x} > 0$. Now let $h \in \mathcal{F}$ be arbitrary with $\|h\|_{L^1(\kappa)} > 0$. For each $x \in K$ let U_x be a neighbourhood as described above. Since K is a Lindelöf space, it is the union of countably many of those neighbourhoods $\{U_x\}_{x \in K}$. Thus $\|h \cdot 1_{U_x}\|_{L^1(\kappa)} > 0$ for at least one $x \in K$ and by the blurring property $(\hat{G}h)|_{U_x} > 0$. Now let $y \in K$ be any other point. We shall show that $(\hat{G}^n h)(y) > 0$ for some $n \in \mathbb{N}$. Since K is path connected, there exists a finite sequence of points $x = x_1, x_2, \dots, x_n = y$ in K such that $U_{x_i} \cap U_{x_{i+1}} \neq \emptyset$ for $1 \le i \le n-1$. We have seen that $(\hat{G}h)|_{U_{x_1}} > 0$. Since κ is strictly positive we know that $\kappa(U_{x_1} \cap U_{x_2}) > 0$, so that $\|(\hat{G}h) \cdot 1_{U_{x_2}}\|_{L^1(\kappa)} > 0$, and thus by the blurring property $((\hat{G})^2 h)|_{U_{x_2}} > 0$. By induction we conclude that $((\hat{G})^n h)|_{U_{x_n}} > 0$. □

(v) Let K be a separable metric space and κ a σ-finite Borel measure on it. Let $G : K \times K \to \mathbb{R}_+$ be continuous and satisfy $G(x, x) > 0$ for every $x \in K$. Then G satisfies the blurring property from example (iv). See figure 2 for an example.

Proof. For $x \in K$ choose the open neighbourhood U_x so that $G|_{U_x \times U_x} > 0$. Let $A \in \mathscr{B}(K)$ be such that $\kappa(U_x \cap A) > 0$. Then for any $z \in U_x$ one can estimate

$$(\hat{G}1_A)(z) = \int_A G(z, y)\, d\kappa(y) \ge \int_{U_x \cap A} G(z, y)\, d\kappa(y) > 0.$$

□

(vi) Let $\mathbb{T}^m = \mathbb{R}^m / \mathbb{Z}^m$ be the m-dimensional torus with the metric inherited from $\mathbb{R}^m$ and κ be the normalized Haar measure on it. Let $G : \mathbb{T}^m \times \mathbb{T}^m \to \mathbb{R}_+$ be of the form $G(x, y) = g(x - y)$ for some continuous $g : \mathbb{T}^m \to \mathbb{R}_+$. If g is not identically zero, then G strongly connects $\mathbb{T}^m$.

Proof. By continuity of g there exists a ball $B_{2r}(z) \subseteq \mathbb{T}^m$ of radius $2r > 0$ and centre $z \in \mathbb{T}^m$ such that $g|_{B_{2r}(z)} > 0$. Let $A \subseteq \mathbb{T}^m$ be of strictly positive measure, then there exists some point $x_o \in \mathbb{T}^m$ such that $\kappa(A \cap B_r(x_o)) > 0$. Let $x_1 = x_o + z$, then for any $x \in B_r(x_1)$ one has

$$(\hat{G}1_A)(x) = \int_A g(x-y)\,dy \geq \int_{A \cap B_r(x_o)} g(x-y)\,dy > 0,$$

that is, $(\hat{G}1_A)|_{B_r(x_1)} > 0$. Note that in the last step we used the fact that the difference $(x-y)$ appearing in the integral is in $B_{2r}(z)$. Now suppose $h \in \mathcal{F}$ is strictly positive on some ball $B_{nr}(x_n)$. Set $x_{n+1} = x_n + z$, then for any $x = (x_{n+1} + \delta) \in B_{nr+r}(x_{n+1})$ one has

$$\begin{aligned} &\{y \in K : g(x-y)h(y) > 0\} \supseteq \{y \in K : y \in B_{nr}(x_n) \wedge (x-y) \in B_{2r}(z)\} \\ =&B_{nr}(x_n) \cap B_{2r}(x-z) = B_{nr}(x_n) \cap B_{2r}(x_n + \delta)\,. \end{aligned}$$

As the intersection $B_{nr}(x_n) \cap B_{2r}(x_n + \delta)$ is non-empty, one has $(\hat{G}h)(x) > 0$ for all $x \in B_{nr+r}(x_{n+1})$. By induction we conclude that $((\hat{G})^n 1_A)|_{B_{nr}(x_n)} > 0$ $\forall n \in \mathbb{N}$ for appropriate centres $x_n \in \mathbb{T}^m$. Since $\mathbb{T}^m$ has finite diameter, $((\hat{G})^n h)|_{\mathbb{T}^m} > 0$ for $n \in \mathbb{N}$ large enough. Thus G strongly connects $\mathbb{T}^m$. $\square$

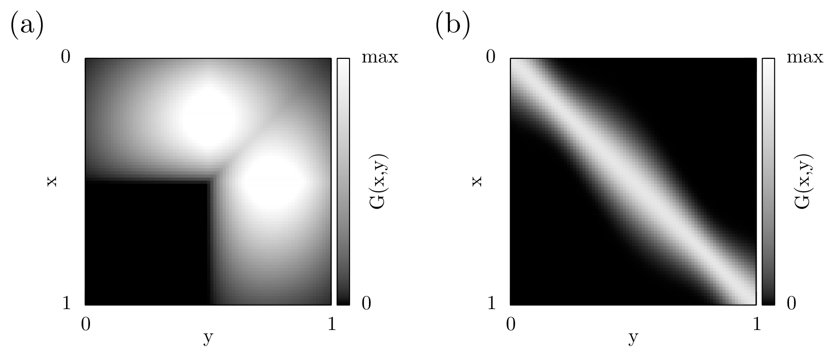

FIGURE 2. Illustration of two non-negative integral kernels on the interval $K = [0,1]$ endowed with the Lebesgue measure. Brighter colours correspond to larger values. The kernel in (a) does not strongly connect K, due to the invariance of the subset $\{h \in \mathcal{F} : \operatorname{supp} h \subseteq [0, 0.5]\}$ under the action of $\hat{G}$. The kernel in (b) strongly connects K. Note the apparent similarity to the notion of irreducibility of square matrices.

Lemma 4.3 (Irreducibility of strongly connecting kernels)**.** *Let X be a separable metric space and μ a strictly positive σ-finite Borel measure on X. Suppose that $G : X \times X \to \mathbb{R}_+$ is measurable, strongly connecting X, and the mapping $x \mapsto G(x, \cdot) \in L^1(\mu)$, is continuous, non-trivial, and bounded. Consider the $\mathbb{C}$-linear space $V = \mathcal{C}_\mathrm{b}(X, \mathbb{C})$ of complex continuous bounded functions on X as the complexification of the $\mathbb{R}$-linear Banach lattice space $V_\mathbb{R} = \mathcal{C}_\mathrm{b}(X, \mathbb{R})$ endowed with the pointwise partial order. Let $\hat{G} : V \to V$ be the integral operator corresponding to G. Then*

1. *$\hat{G}$ is a bounded, linear, positive operator; that is, $\hat{G}h \geq 0$ whenever $0 \leq h \in V_\mathbb{R}$ and $\hat{G} \neq 0$.*

2. $\hat{G}$ is σ-order continuous.
3. $\hat{G}$ is band irreducible.

For a proof see A.2.

Corollary 4.4 (Generalized Jentzsch-Perron theorem for integral operators)**.** *Let X, μ, G and V be as in Lemma 4.3. Consider the integral operator $\hat{G} \in \mathcal{L}(V)$ corresponding to G and let $\hat{G}^n$ be compact for some $n \in \mathbb{N}$. Then $r(\hat{G}) > 0$, and $r(\hat{G})$ is an eigenvalue of $\hat{G}$ of algebraic multiplicity one.*

Proof. Apply the generalized Jentzsch-Perron theorem for positive, σ-order continuous, band irreducible operators on Banach lattices [20, Theorem 6], which holds by Lemma 4.3. □

Lemma 4.5 (Spectrum of strongly connecting integral operators)**.** *Let X, μ and V be as in Lemma 4.3. Let $G : X \times X \to \mathbb{R}_+$ be measurable, satisfying the following.*
G1. G strongly connects X.
G2. $\|G(x,\cdot)\|_{L^1(\mu)} = G_o$ for some constant $G_o > 0$ and all $x \in X$.
G3. The mapping $x \mapsto G(x,\cdot) \in L^1(\mu)$ is continuous.
G4. The integral operator $\hat{G} : V \to V$ corresponding to G is power compact, that is, $\hat{G}^n$ is compact for some $n \in \mathbb{N}$.
Then $\hat{G} \in \mathcal{L}(V)$ satisfies
1. Both the spectral radius $r(\hat{G})$ and the operator norm $\|\hat{G}\|$ equal G_o.
2. G_o is an eigenvalue of $\hat{G}$ and the kernel of $(G_o - \hat{G})$ is one-dimensional, consisting of all constant functions.
3. $\sup\{\Re(\lambda) : \lambda \in \sigma(\hat{G}) \setminus \{G_o\}\} < G_o$.

Proof. Note that by assumptions (G2) and (G3), the integral operator $\hat{G} : V \to V$ is well-defined and continuous. In fact, it satisfies the assumptions of corollary 4.4, so that $r(\hat{G})$ is an eigenvalue of algebraic multiplicity one.

For any $h \in \mathcal{C}_\mathrm{b}(X,\mathbb{C})$ one can, using Hölder's inequality, estimate $\|\hat{G}h\|_\infty \leq G_o \|h\|_\infty$, so that $r(\hat{G}) \leq \|G\| \leq G_o$. On the other hand, the constant function $1 \in V$ is an eigenvector of $\hat{G}$ for the eigenvalue G_o, so that the first and second claims are verified.

Since $\hat{G}$ is power-compact, one has $\sigma(\hat{G}) = \sigma_p(\hat{G}) \cup \{0\}$ and the only point of adherence of $\sigma_p(\hat{G})$ is the origin. Thus $G_o > 0$ is isolated from $\sigma(\hat{G}) \setminus \{G_o\}$. Since $|\lambda| \leq G_o$ for any $\lambda \in \sigma(\hat{G})$, this implies the last claim. □

Example 4.6. Consider the m-dimensional torus $\mathbb{T}^m$ and the normalized Haar measure μ on it. Let $G : \mathbb{T}^m \times \mathbb{T}^m \to \mathbb{R}_+$ be of the form $G(x,y) = g(x-y)$ for some continuous $0 \neq g : \mathbb{T}^m \to \mathbb{R}_+$. Then G satisfies all assumptions of Lemma 4.5 (see Example 4.2(vi) on its strong connectivity). The point spectrum of the integral operator $\hat{G} : \mathcal{C}_\mathrm{b}(\mathbb{T}^m,\mathbb{C}) \to \mathcal{C}_\mathrm{b}(\mathbb{T}^m,\mathbb{C})$ is given by the set $\sigma_p(\hat{G}) = \{\tilde{g}(k) : k \in \mathbb{Z}^n\}$, with $\tilde{g}(k) = \int_{\mathbb{T}^m} g(x)e^{-i2\pi kx}\,dx$ being the k-th Fourier component of g and corresponding to the eigenfunction $e_k(x) = e^{i2\pi kx}$. Note that $\tilde{g}(0) = G_o$ and for $k \neq 0$ one has $|\tilde{g}(k)| < G_o$, since $g \geq 0$ and $g \neq 0$.

Example 4.7. Let $X = \{x_1, \ldots, x_n\}$ be a finite nonempty set, with the discrete metric and the counting measure μ. Then the space $V = \mathcal{C}_\mathrm{b}(X,\mathbb{C})$ can be naturally identified with $\mathbb{C}^n$. Let $G = (G_{ij})_{i,j=1}^n$ be a square matrix with non-negative entries, identified with the function $G : X \times X \to \mathbb{R}_+$ given by $G(x_i, x_j) = G_{ij}$. The latter satisfies assumptions (G3) and (G4) of Lemma 4.5, with the integral operator $\hat{G} : V \to V$ being simply the action of the matrix G on vectors in $\mathbb{C}^n$. Condition (G2) is equivalent to each row summing to G_o (in case $G_o = 1$, G is called a *stochastic matrix* [22]). Condition (G1) is equivalent to the irreducibility of the matrix G (see Example 4.2(ii)). Finally, statement (2) is nothing more than the Perron-Frobenius theorem for irreducible non-negative matrices [22, §8.4.4].

4.2. **Stability of synchrony in the field model.** The auxiliary statement 4.8 gives a technical spectral result for the linearized dynamics around the limit cycle of synchrony, and can be seen as an adaptation of [18, §2.5] to our system. The auxiliary statement 4.9 provides sufficient conditions for the stability of sub-manifolds in a special class of abstract dynamical systems, by looking at the contraction properties of their stroboscopic map [24]. In Theorem 4.10 we apply these two statements to the field model and give sufficient conditions for the existence and stability of synchrony in oscillator networks.

Auxiliary statement 4.8. *Let X be a separable metric space and μ a strictly positive finite Borel measure on X. Let $G : X \times X \to \mathbb{R}_+$ satisfy all assumptions of Lemma 4.5 and assume in addition that $\|G(\cdot, y)\|_{L^1(\mu)} = G_o$ for all $y \in X$. Let $\psi, I : S^1 \to \mathbb{R}$ be continuously differentiable and $\omega > 0$. Let $\phi : \mathbb{R} \to S^1$ be continuously differentiable, strictly increasing, T-periodic for some minimal $T > 0$ and satisfying the ODE $\dot{\phi}(t) = \omega + G_o\psi(\phi(t))I(\phi(t))$. Consider the time-dependent, bounded linear operator*

$$\hat{\mathcal{H}}(t) = G_o\psi'(\phi(t))I(\phi(t))\,\mathrm{Id} + \psi(\phi(t))I'(\phi(t))\hat{G}$$

on the Banach-space $V = \mathcal{C}_\mathrm{b}(X, \mathbb{C})$, with $\hat{G} \in \mathcal{L}(V)$ being the integral operator corresponding to G. Then the following hold:

1. *For every initial value $h(t_o) \in V$, the differential equation $\frac{dh}{dt}(t) = \mathcal{H}(t)h(t)$ admits a global, maximal solution $h \in \mathcal{C}^1(\mathbb{R}, V)$, given by*
$$h(t) = U(t, t_o)h(t_o) := \exp\left[A(t, t_o)G_o - B(t, t_o)\hat{G}\right] h(t_o), \tag{29}$$
where
$$\begin{aligned} A(t, t_o) &= \int_{t_o}^{t} \psi'(\phi(s))I(\phi(s))\, ds, \\ B(t, t_o) &= -\int_{t_o}^{t} \psi(\phi(s))I'(\phi(s))\, ds. \end{aligned} \tag{30}$$
2. *The operator $K = U(t_o + T, t_o) : V \to V$ is independent of t_o and given by $K = \exp[B(G_o - \hat{G})]$, where*
$$B := B(t_o + T, t_o) = \int_{S^1} \frac{\psi'(\varphi)I(\varphi)}{\omega + G_o\psi(\varphi)I(\varphi)}\, d\varphi.$$
3. *The linear sub-Banach space $V_2 = \left\{h \in V : \int h\, d\mu = 0\right\}$ is $\hat{G}$- and flow-invariant, that is, $\hat{G}(V_2) \subseteq V_2$ and $U(t, t_o)(V_2) \subseteq V_2$ for all $t, t_o \in \mathbb{R}$.*
4. *If $B < 0$ then $r(K|_{V_2}) < 1$.*

Proof.

1. Existence and uniqueness follow from the continuity of the mapping $t \mapsto \mathcal{H}(t) \in \mathcal{L}(V)$. Differentiation of (29) verifies the assertion.

2. Substituting $d\varphi = \frac{d\phi(s)}{ds} ds$ in (30) yields

$$\begin{aligned} B(t_o+T,t_o) =& -\int_{S^1} \frac{\psi(\varphi)I'(\varphi)}{\omega+G_o\psi(\varphi)I(\varphi)}\,d\varphi \\ &= -\frac{1}{G_o}\int_{S^1}\frac{d}{d\varphi}\ln\left[\omega+G_o\psi(\varphi)I(\varphi)\right]\,d\varphi \\ &\quad+\int_{S^1}\frac{\psi'(\varphi)I(\varphi)}{\omega+G_o\psi(\varphi)I(\varphi)}\,d\varphi \\ &= \int_{S^1}\frac{\psi'(\varphi)I(\varphi)}{\omega+G_o\psi(\varphi)I(\varphi)}\,d\varphi = A(t_o+T,t_o). \end{aligned} \tag{31}$$

Independence of $B(t_o+T,t_o)$ and $U(t_o+T,t_o)$ from t_o is apparent from (31).

3. Since the linear functional $h \mapsto \int h\,d\mu$ is continuous on V, V_2 is indeed a Banach space. The rest follows from the easily verifiable fact that $\int \hat{G}h\,d\mu = G_o \int h\,d\mu$. Thus

$$\int U(t,t_o)h\,d\mu = \exp\left[G_o\left(A(t,t_o)-B(t,t_o)\right)\right]\int h\,d\mu$$

for all $h \in V$.

4. Since $B(G_o - \hat{G}|_{V_2}) : V_2 \to V_2$ is a bounded operator, we know that

$$\begin{aligned} \sigma(K|_{V_2}) &= \exp\left[\sigma\left(B(G_o-\hat{G}|_{V_2})\right)\right] = \exp\left[B(G_o-\sigma(\hat{G}|_{V_2}))\right] \\ &= \exp\left[B(G_o-\{0\}\cup\sigma_p(\hat{G}|_{V_2}))\right]. \end{aligned}$$

In the last step we used the fact that $\hat{G}$ is power-compact. By Lemma 4.5 one has $r(\hat{G}) = G_o$ and the only eigenvectors of $\hat{G}$ in V corresponding to the eigenvalue G_o are the constant non-trivial functions, which are not in V_2. By Lemma 4.5(3) it follows that $r(K|_{V_2}) < 1$. □

Auxiliary statement 4.9. *Let V_1, V_2 be Banach spaces and $(\mathcal{H}_1,\mathcal{H}_2), (\mathcal{H}_{1,o},\mathcal{H}_{2,o}) : V_1\times V_2 \to V_1 \times V_2$ functions satisfying the following:*

1. *$\mathcal{H} := (\mathcal{H}_1,\mathcal{H}_2) : V_1\times V_2\to V_1\times V_2$ is Lipschitz continuous.*
2. *$\mathcal{H}_{2,o}(v_1,v_2) = \mathcal{H}_{2,o}(v_1)v_2$, with $\mathcal{H}_{2,o}(v_1) : V_2 \to V_2$ being a bounded, linear operator for each $v_1 \in V_1$.*
3. *The mapping $V_1 \to \mathcal{L}(V_2)$, $v_1 \mapsto \mathcal{H}_{2,o}(v_1)$, is Lipschitz continuous and bounded.*
4. *$[(\mathcal{H}_1,\mathcal{H}_2)(v_1,v_2) - (\mathcal{H}_{1,o},\mathcal{H}_{2,o})(v_1,v_2)] \in o(v_2)$ as $v_2 \to 0$, uniformly in $v_1 \in V_1$.*
5. *$\mathcal{H}_{1,o}(v_1,v_2) = \mathcal{H}_{1,o}(v_1)$ only depends on v_1.*

Let $U := ((U_1,U_2)(t,t_o))_{t\ge t_o}$ be the autonomous flow generated by $\mathcal{H} := (\mathcal{H}_1,\mathcal{H}_2)$ on $V_1\times V_2$. Suppose that there exists some period $T > 0$ such that the propagator $K : V_2 \to V_2$ from time t_o to $t_o + T$ induced on V_2 by the non-autonomous ODE $\dot{v}_2(t) = \mathcal{H}_{2,o}(U_1(t,t_o)(v_1,0))v_2(t)$ does not depend on the fixed $v_1 \in V_1$. Furthermore, suppose that there exists an $n_o \in \mathbb{N}$ such that $\|K^{n_o}\| < 1$.

Then the U-invariant sub-space $V_1 \times \{0\} \subseteq V_1 \times V_2$ is locally exponentially stable. That is, there exist constants $A, \beta, \delta > 0$ such that $\|U_2(t,t_o)(v_1,v_2)\| \le Ae^{-\beta(t-t_o)}\|v_2\|$ for all $t \ge t_o$, $v_1 \in V_1$ and $v_2 \in V_2$, provided that $\|v_2\| \le \delta$.

For a proof of this assertion see A.3.

We are now ready to state our main theorem regarding the stability of synchrony in the field model (3). We consider X to be a separable metric space and μ a finite, strictly positive Borel measure on X. We assume all oscillators to have the intrinsic frequency $\omega > 0$, response function $\psi \in \mathcal{C}^2(S^1, \mathbb{R})$ and pulse $I \in \mathcal{C}^2(S^1, \mathbb{R})$. Oscillators shall be connected through the sufficiently well-behaved coupling kernel $G : X \times X \to \mathbb{R}_+$. The field model then becomes the ODE

$$\frac{d\theta}{dt}(t,x) = \omega + \psi(\theta(t,x)) \int_X G(x,y) I(\theta(t,y))\, d\mu(y) \tag{32}$$

in the variable $\theta \in \mathcal{C}_\mathrm{b}(X, S^1)$. Without loss of generality, we regard (32) as an ODE on the Banach-space $\mathcal{C}_\mathrm{b}(X, \mathbb{R})$, by identifying ψ and I with their pullbacks on $\mathbb{R}$ (see also Remark 4.11 below). For $\theta \in \mathcal{C}_\mathrm{b}(X, \mathbb{R})$ we denote by $\overline{\theta} = \int \theta d\mu/\mu(X)$ the mean phase and by $\theta_\mathrm{v} = \theta - \overline{\theta}$ the variation about it.

Theorem 4.10 (Local exponential stability of synchrony)**.** *Consider the field model (32). Let $G : X \times X \to \mathbb{R}_+$ satisfy all assumptions of Lemma 4.5, and assume in addition that $\|G(\cdot, y)\|_{L^1(\mu)} = G_1$ $\forall y \in X$ for some constant $G_1 \in \mathbb{R}$. Also assume that $\omega + \psi(\varphi) G_o I(\varphi) > 0$ for all $\varphi \in S^1$. Let $\phi : \mathbb{R} \to S^1$ satisfy the autonomous ODE*

$$\frac{d\phi}{dt}(t) = \omega + \psi(\phi(t)) I(\phi(t)) G_o. \tag{33}$$

Note that ϕ is T-periodic for some minimal $T > 0$. Then $\theta(t,x) := \phi(t)$ is a (so-called synchronized) solution to (32). Assume furthermore that

$$B := \int_{S^1} \frac{\psi'(\varphi) I(\varphi)}{\omega + G_o \psi(\varphi) I(\varphi)}\, d\varphi < 0. \tag{34}$$

Then synchrony is locally exponentially stable. More precisely, there exist constants $A, \beta > 0$ and $0 < \delta < 1/2$ such that $\|\theta_\mathrm{v}(t,\cdot)\|_\infty \le A e^{-\beta t} \|\theta_\mathrm{v}(0,\cdot)\|_\infty$ for all $t \ge 0$, provided that $\|\theta_\mathrm{v}(0,\cdot)\|_\infty \le \delta$ and $\theta \in \mathcal{C}^1(\mathbb{R}, \mathcal{C}_\mathrm{b}(X,\mathbb{R}))$ solves (32).

Proof. We make use of the auxiliary statement 4.9. As a preliminary, let us note that by Tonelli and Fubini's theorems and the assumptions on G, one has

$$\begin{aligned}\mu(X) G_o &= \int_X \int_X G(x,y)\, d\mu(y)\, d\mu(x) \\ &= \int_X \int_X G(x,y)\, d\mu(x)\, d\mu(y) = \mu(X) G_1,\end{aligned}$$

so that $G_1 = G_o$. We also note the approximation

$$\psi(\theta(x)) = \psi(\overline{\theta}) + \psi'(\overline{\theta})(\theta(x) - \overline{\theta}) + o(\|\theta(\cdot) - \overline{\theta}\|_\infty)$$

for any $\theta \in \mathcal{C}_\mathrm{b}(X, \mathbb{R})$, with the error scaling down uniformly in $\overline{\theta}$. The uniformity of the error follows from the uniform continuity of ψ'. A similar approximation holds for $I(\theta(x))$. We may thus write (32) as

$$\begin{aligned}\frac{d\theta}{dt}(t,x) = \omega &+ \psi(\overline{\theta}(t)) I(\overline{\theta}(t)) G_o + \psi'(\overline{\theta}(t)) I(\overline{\theta}(t)) G_o \cdot (\theta(t,x) - \overline{\theta}(t)) \\ &+ \psi(\overline{\theta}(t)) I'(\overline{\theta}(t)) \int_X G(x,y) \cdot (\theta(t,y) - \overline{\theta}(t))\, d\mu(y) \\ &+ o(\|\theta(t,\cdot) - \overline{\theta}(t)\|_\infty),\end{aligned} \tag{35}$$

using the fact that $\|G(x,\cdot)\|_{L^1(\mu)} = G_o$ for all $x \in X$. Using (35), we find that

$$\begin{aligned}\frac{d\overline{\theta}}{dt}(t) &= \frac{1}{\mu(X)} \int_X \frac{d\theta}{dt}(t,x)\; d\mu(x) \\ &= \omega + \psi(\overline{\theta}(t)) I(\overline{\theta}(t)) G_o + o(\|\theta(t,\cdot) - \overline{\theta}(t)\|_\infty),\end{aligned} \tag{36}$$

where we have used the fact that $\|G(\cdot,y)\|_{L^1(\mu)} = G_o$ for all $y \in X$, and to exchange integration and differentiation, we used the fact that $\frac{d}{dt}\theta(t,\cdot)$ is a derivative in $\mathcal{C}_\mathrm{b}(X,\mathbb{R})$. Combining (35) with (36) leads to the coupled system of ODEs

$$\begin{aligned} \frac{d\overline{\theta}}{dt}(t) =& \omega + \psi(\overline{\theta}(t))I(\overline{\theta}(t))G_o + o(\|\theta_\mathrm{v}\|_\infty), \\ \frac{d\theta_\mathrm{v}}{dt}(t,\cdot) =& \left[\psi'(\overline{\theta}(t))I(\overline{\theta}(t))G_o + \psi(\overline{\theta}(t))I'(\overline{\theta}(t))\hat{G}\right]\theta_\mathrm{v}(t,\cdot) + o(\|\theta_\mathrm{v}\|_\infty), \end{aligned} \tag{37}$$

in the variables

$$\begin{aligned} \overline{\theta} &\in V_1 := \{f : X \to \mathbb{R} : \mathrm{const}\} \cong \mathbb{R}, \\ \theta_\mathrm{v} &\in V_2 := \left\{f \in \mathcal{C}_\mathrm{b}(X,\mathbb{R}) : \textstyle\int f \; d\mu = 0\right\}. \end{aligned}$$

Note that the error in (37) depends on both $\overline{\theta}$ and θ_v but scales down with $\|\theta_\mathrm{v}\|_\infty$ uniformly in $\overline{\theta}$. Observe the similarity of the first part of (37) to the ODE (33) for the common phase in case of synchrony. It reveals that, up to first order in the variation θ_v, the mean phase $\overline{\theta}$ advances as if the network was in total synchrony. Note that the mapping

$$V_1 \to \mathcal{L}(V_2), \qquad \overline{\theta} \mapsto \mathcal{H}_{2,o}(\overline{\theta}) := \psi'(\overline{\theta})I(\overline{\theta})G_o + \psi(\overline{\theta})I'(\overline{\theta})\hat{G}$$

is Lipschitz continuous and bounded. Any solution $\phi \in \mathcal{C}^1(\mathbb{R}, V_1)$ of the ODE

$$\frac{d\phi}{dt}(t) = \mathcal{H}_{1,o}(\phi(t)) := \omega + \psi(\phi(t))I(\phi(t))G_o$$

is T-periodic if projected to S^1. Let $K|_{V_2} : V_2 \to V_2$ be the propagator induced by the non-autonomous ODE $\dot{\theta}_\mathrm{v}(t,\cdot) = \mathcal{H}_{2,o}(\phi(t))\theta_\mathrm{v}(t,\cdot)$ from time t_o to time $t_o + T$, as described in the auxiliary statement 4.8. By 4.8(2), $K|_{V_2}$ is independent of the initial value $\phi(t_o)$ and initial time t_o. By 4.8(4), $K|_{V_2}$ has a spectral radius smaller than 1, so that $\|(K|_{V_2})^{n_o}\| < 1$ for some sufficiently large $n_o \in \mathbb{N}_o$. Identify $\mathcal{C}_\mathrm{b}(X,\mathbb{R}) = V_1 \oplus V_2$ by means of the decomposition $\theta = \overline{\theta} + \theta_\mathrm{v}$. Identify $(\mathcal{H}_1, \mathcal{H}_2) : V_1 \times V_2 \to V_1 \times V_2$ as the Lipschitz-continuous function appearing on the right hand side of (32), that is with $\mathcal{H}_i : V_1 \times V_2 \to V_i$ $(i \in \{1,2\})$ so that $\frac{d\overline{\theta}}{dt} = \mathcal{H}_1(\theta)$ and $\frac{d\theta_\mathrm{v}}{dt} = \mathcal{H}_2(\theta)$. Then the dynamics (37) satisfy the assumptions of the auxiliary statement 4.9. By the latter, the flow-invariant sub-space $V_1 \times \{0_{V_2}\}$ is indeed stable in the way postulated above. □

Remark 4.11. The stability result of Theorem 4.10 holds at first instance for solutions of the field equation in $\mathcal{C}_\mathrm{b}(X,\mathbb{R})$. But in fact it remains valid in an equivalent form for solutions in $\mathcal{C}_\mathrm{b}(X,S^1)$ if one replaces $\|\theta_\mathrm{v}(t,\cdot)\|_\infty$ by $\operatorname{diam}\theta(t,X)$: There exist constants $A, \beta > 0$ and $0 < \delta < 1/2$ such that $\operatorname{diam}\theta(t,X) \le Ae^{-\beta t}\operatorname{diam}\theta(0,X)$ for all $t \ge 0$, provided that $\operatorname{diam}\theta(0,X) \le \delta$ and $\theta \in \mathcal{C}^1(\mathbb{R}, \mathcal{C}_\mathrm{b}(X,S^1))$ solves the field equation (32). To see this, note that for any $\theta_o \in \mathcal{C}(X,S^1)$ with $\operatorname{diam}\theta_o(X) < 1/2$, there exists a lift $\tilde{\theta}_o \in \mathcal{C}_\mathrm{b}(X,\mathbb{R})$ with respect to the covering map $\Pi_\mathrm{c} : \mathbb{R} \to S^1$ such that $\|(\tilde{\theta}_o)_\mathrm{v}\|_\infty \le \operatorname{diam}\theta_o(X)$. If $\theta \in \mathcal{C}^1(\mathbb{R}, \mathcal{C}_\mathrm{b}(X,S^1))$ has initial value $\theta(0,\cdot) = \theta_o$ then there exists a unique lift $\tilde{\theta} \in \mathcal{C}(\mathbb{R} \times X, \mathbb{R})$ of θ with $\tilde{\theta}(0,\cdot) = \tilde{\theta}_o$. This follows from the homotopy lifting property of covering maps [31, §6.2.3]. Since the covering map Π_c is a local diffeomorphism and a local isometry, one has $\partial_t\tilde{\theta}(t,x) = \partial_t\theta(t,x)$ and $\tilde{\theta} \in \mathcal{C}^1(\mathbb{R}, \mathcal{C}_\mathrm{b}(X,\mathbb{R}))$ satisfies the field equation. Finally, the exponential decay of $\|\tilde{\theta}_\mathrm{v}(t,\cdot)\|_\infty$ as $t \to \infty$ implies the exponential decay of $\operatorname{diam}\theta(t,X)$.

Remark 4.12. Key to the proof of the theorem is the exponential stability of the iterated dynamical system $(V_2, ((K|_{V_2})^n)_{n\in\mathbb{N}_0})$, given by the fact that the spectrum $\sigma(K|_{V_2}) = \exp[B \cdot (G_o - \{0\} \cup \sigma_p(\hat{G}|_{V_2}))]$ lies in the open unit disc in the complex plane (see the auxiliary statement 4.8). The subspace $V_2 \subseteq V$ corresponds to the stable subspace of the linear propagator $K : V \to V$, mapping phase fields $\theta(t_o)$ at time t_o to phase fields $\theta(t_o + T)$ at time $t_o + T$. It is a complement to the one-dimensional neutral subspace $V_1 = \{f \in V : f \equiv \text{const}\}$ of constant functions. The latter corresponds to uniform phase shifts of the oscillator population, under which the limit cycle of synchrony is invariant.

Example 4.6 gives a hint on how the *strength* of stability might depend on the detailed shape of G. For that example, where $X = \mathbb{T}^m$ and $G(x, y) = g(x - y)$ for some $g \in \mathcal{C}(\mathbb{T}^m, \mathbb{R}_+)$, we know in connection with the proof of 4.8(4) that the point spectrum of $\hat{G}|_{V_2} : V_2 \to V_2$ is the Fourier-spectrum of g, the value G_o excluded. The stronger any non-trivial modes are represented in the latter (more precisely, the greater the real part of their eigenvalues), the closer the eigenvalues of $K|_{V_2}$ will be to the boundary of the unit disc. On the other hand, if g is constant on $\mathbb{T}^m$ then all eigenvalues of $\hat{G}|_{V_2}$ are zero and the spectral radius of $K|_{V_2}$ takes the value e^{BG_o} (B being strictly negative).

Remark 4.13. It is worth comparing Theorem 4.10 to the equivalent asymptotic stability result by [18, §2.5] for the special case of finitely many oscillators, that is, for X finite with μ as the counting measure. Their condition (ii) corresponds to our condition $\|G(x, \cdot)\|_{L^1(\mu)} = \|G(\cdot, y)\|_{L^1(\mu)} = G_o > 0 \ \forall x, y \in X$ with G strongly connecting X (also see Example 4.2(ii)), their condition (iii) to our condition $\omega + \psi(\varphi)G_o I(\varphi) > 0 \ \forall \varphi \in S^1$, their condition (iv) to our condition $G(x, y) \geq 0$ $\forall x, y \in X$ and their condition (v) to our condition $B < 0$. Apart from the greater generality achieved in Theorem 4.10, the latter allows us to study the local stability of synchrony in the fluid model. This will be the subject of section 4.3.

4.3. **Stability of synchrony in the fluid model.** This section starts with the auxiliary statement 4.14 on the stability of certain orbits in abstract dynamical systems. We use this rather technical statement to prove Theorem 4.16, by showing how our fluid model for coupled oscillator networks fits into the context of 4.14 and, under sufficient conditions, satisfies its assumptions. This will lead to a local stability statement for synchrony in the fluid model.

Auxiliary statement 4.14. *Let Γ_o be a nonempty set and Γ a collection of mappings $\gamma : [0, \infty) \to \Gamma_o$, henceforth referred to as orbits. Let $\{D_\alpha\}_{\alpha\in A}$ be a family of functions $D_\alpha : \Gamma_o \to \mathbb{R}_+$ such that for any orbit $\gamma \in \Gamma$, the mappings $t \mapsto D_\alpha(\gamma(t))$ are differentiable and the family $\{D_\alpha(\gamma(\cdot))\}_{\alpha\in A}$ is equicontinuous. Let $D_{\text{loc}} = \sup_{\alpha\in A} D_\alpha$ be defined pointwise. Let V_1, V_2 be Banach spaces, $V = V_1 \times V_2$, and $\mathcal{H}_o : V \to V$ a Lipschitz-continuous function. Let $\Theta_1 : \Gamma_o \to V_1$, $\Theta_2 : \Gamma_o \to V_2$, and $\mathcal{E} : \Gamma_o \to V$ be such that for any orbit $\gamma \in \Gamma$, the mappings $t \mapsto \Theta_1(\gamma(t))$ and $t \mapsto \Theta_2(\gamma(t))$ are differentiable, the mapping $t \mapsto \mathcal{E}(\gamma(t))$ is continuous, and all three of them satisfy*

$$\frac{d}{dt}(\Theta_1(\gamma(t)), \Theta_2(\gamma(t))) = \mathcal{H}_o[\Theta_1(\gamma(t)), \Theta_2(\gamma(t))] + \mathcal{E}(\gamma(t)).$$

Assume that $\mathcal{E}(\gamma_o) \in O(D_{\text{loc}}(\gamma_o))$ as $D_{\text{loc}}(\gamma_o) \to 0$, $\gamma_o \in \Gamma_o$. Define $D_{\text{gl}} = 2\|\Theta_2\| + 2D_{\text{loc}}$ and assume the following stability conditions:

C1. There exists an $\varepsilon > 0$ such that whenever $D_{\rm gl}(\gamma(t)) \le \varepsilon$, one has $\frac{d}{dt} D_\alpha(\gamma(t)) \le 0$ for all $\alpha \in A$ and any orbit $\gamma \in \Gamma$.

C2. There exists a period $T > 0$ and a constant $r < 1$ such that the following statement holds: Whenever $\gamma \in \Gamma$ and $t_o \ge 0$ are such that $D_{\rm gl}(\gamma(t)) \le \varepsilon$ for all $t \in [t_o, t_o + T]$, one has $D_\alpha(\gamma(t_o + T)) \le r D_\alpha(\gamma(t_o))$ for all $\alpha \in A$.

C3. The dynamical system induced by $\mathcal{H}_o$ on $V = V_1 \times V_2$ is locally exponentially stable in the second coordinate. That is to say, there exist constants $A_o, \beta_o, \delta_o > 0$ such that whenever $(\Theta_{o1}(t), \Theta_{o2}(t))$ satisfies the ODE $\frac{d}{dt}(\Theta_{o1}(t), \Theta_{o2}(t)) = \mathcal{H}_o [\Theta_{o1}(t), \Theta_{o2}(t)]$ in V and the initial condition $\|\Theta_{o2}(0)\| \le \delta_o$, one has $\|\Theta_{o2}(t)\| \le A_o e^{-\beta_o t} \|\Theta_{o2}(0)\|$ for all $t \ge 0$.

Then there exist constants $A, \beta, \delta > 0$ such that for any orbit $\gamma \in \Gamma$ satisfying $D_{\rm gl}(\gamma(0)) \le \delta$, one has $D_{\rm gl}(\gamma(t)) \le A e^{-\beta t} D_{\rm gl}(\gamma(0))$ for all $t \ge 0$ and every $D_\alpha(\gamma(t))$ ($\alpha \in A$) is decreasing in t.

For a proof see A.4.

Remark 4.15. In the proof of the theorem below we will be making use of the following useful fact (sometimes known as *law of mass conservation*): Suppose $\rho : [t_o, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and continuously differentiable in the 2nd argument, with $\partial_t \rho$ existing and continuous on $[t_o, \infty) \times \mathbb{R}$. Let $\mathrm{v} : [t_o, \infty) \times \mathbb{R} \to \mathbb{R}$ be continuous in the 1st argument and continuously differentiable in the 2nd argument. Suppose ρ satisfies the continuity equation

$$\partial_t \rho(t, \vartheta) = -\partial_\vartheta \left[\rho(t, \vartheta) \cdot \mathrm{v}(t, \vartheta)\right].$$

Let $t_1 \ge t_o$ and let $\theta^p : [t_o, t_1] \to \mathbb{R}$ be a continuously differentiable solution to the ODE $\dot\theta^p(t) = \mathrm{v}(t, \theta^p(t))$. Then

$$\int_{t_o}^{t_1} (\rho \mathrm{v})(t, \theta^p(t_o))\, dt = \int_{\theta^p(t_o)}^{\theta^p(t_1)} \rho(t_1, \vartheta)\, d\vartheta. \tag{38}$$

To see this, note that

$$\begin{aligned}
&\int_{\theta^p(t_o)}^{\theta^p(t_1)} \rho(t_1, \vartheta)\, d\vartheta \\
&= \int_{t_o}^{t_1} \frac{d}{dt} \int_{\theta^p(t_o)}^{\theta^p(t)} \rho(t, \vartheta)\, d\vartheta\, dt \\
&= \int_{t_o}^{t_1} \left[\dot\theta^p(t) \rho(t, \theta^p(t)) + \int_{\theta^p(t_o)}^{\theta^p(t)} (\partial_t \rho)(t, \vartheta)\, d\vartheta\right] dt \\
&= \int_{t_o}^{t_1} \left[\dot\theta^p(t) \rho(t, \theta^p(t)) - \int_{\theta^p(t_o)}^{\theta^p(t)} \partial_\vartheta \left[\rho(t, \vartheta) \cdot \mathrm{v}(t, \vartheta)\right] d\vartheta\right] dt \\
&= \int_{t_o}^{t_1} \left[\dot\theta^p(t) \rho(t, \theta^p(t)) - \rho(t, \theta^p(t))\ \mathrm{v}(t, \theta^p(t)) + \rho(t, \theta^p(t_o)) \mathrm{v}(t, \theta^p(t_o))\right] dt \\
&= \int_{t_o}^{t_1} \rho(t, \theta^p(t_o)) \mathrm{v}(t, \theta^p(t_o))\, dt.
\end{aligned}$$

An interpretation of (38) is that the time-integrated probability flux through $\theta^p(t_o)$ from time t_o to t_1 equals the *mass* between $\theta^p(t_o)$ and $\theta^p(t_1)$ at time t_1.

We are now ready to state our main theorem on the stability of synchrony in the fluid model. We assume X to be a separable metric space and μ a finite Borel measure on X. We consider a measurable coupling kernel $G : X \times X \to \mathbb{R}_+$, such that the mapping $x \mapsto G(x,\cdot) \in L^1(\mu)$ is uniformly continuous and bounded. All oscillators shall have the intrinsic frequency $\omega > 0$, the response function $\psi \in \mathcal{C}^2(S^1,\mathbb{R})$ and the pulse $I \in \mathcal{C}^2(S^1,\mathbb{R}_+)$. Let Ω_{o,S^1} and Ω_{S^1} be the function spaces defined in (8) and (9), respectively. We consider the continuity equation

$$\partial_t \rho(t,x,\vartheta) = -\partial_\vartheta \Big[\rho(t,x,\vartheta)\,[\omega + \psi(\vartheta) S_{S^1}(x,\rho(t,\cdot,\cdot))]\Big], \tag{39}$$

with the stimulus

$$S_{S^1}(x,\rho_o) = \int_X G(x,y) \int_{S^1} \rho_o(y,\varphi) I(\varphi)\, d\varphi\, d\mu(y) \tag{40}$$

defined for $x \in X$ and $\rho_o \in \Omega_{o,S^1}$. Also consider the collection of orbits

$$\Omega_{\mathrm{CE},S^1} := \{\rho \in \Omega_{S^1} : \partial_\vartheta \rho(0,\cdot,\cdot) \in \mathcal{C}_{\mathrm{u,b}}(X \times S^1,\mathbb{R}) \ \wedge\ \rho \text{ solves } (39)\}. \tag{41}$$

We shall temporarily identify I and ψ with their pullbacks on $\mathbb{R}$. The theorem and its proof emphasize the connection to the corresponding field model, i.e. the dynamical system on the Banach space $V = \mathcal{C}_{\mathrm{b}}(X,\mathbb{R})$ induced by the ODE

$$\frac{d}{dt}\theta(t,\cdot) = \mathcal{H}_o(\theta(t,\cdot)). \tag{42}$$

Here, $\mathcal{H}_o : V \to V$ is the Lipschitz-continuous function defined by

$$\mathcal{H}_o(\theta(t,\cdot))(x) := \omega + \psi(\theta(t,x)) \int_X G(x,y) I(\theta(t,y))\, d\mu(y). \tag{43}$$

We shall denote $\overline{\theta}(t) = \int \theta(t,x)\ d\mu(x)/\mu(X)$ and $\theta_{\mathrm{v}} = \theta - \overline{\theta}$.

Theorem 4.16 (Stability of synchrony against perturbations with small support)**.** *Consider the fluid model* (39) *and assume the following:*

C1. The derivative ψ' is strictly negative on some circular arc containing $\operatorname{supp} I$.

C2. Synchrony is locally exponentially stable in the corresponding field model (42)*, that is, there exist constants $A_o, \beta_o > 0$ and $0 < \delta_o < 1/2$ such that*

$$\|\theta_{\mathrm{v}}(t,\cdot)\|_\infty \le A_o e^{-\beta_o t} \|\theta_{\mathrm{v}}(0,\cdot)\|_\infty$$

for all $t \ge 0$ whenever $\|\theta_{\mathrm{v}}(0,\cdot)\|_\infty \le \delta_o$ (see Theorem 4.10).

Then synchrony is locally exponentially stable in the space Ω_{CE,S^1}, defined in (41)*. That is to say, there exist constants $A, \beta > 0$ and $0 < \delta < 1/2$, such that for any orbit $\rho \in \Omega_{\mathrm{CE},S^1}$ with $\operatorname{diam}_{\mathrm{ph}} \rho(0,\cdot,\cdot) \le \delta$, one has $\operatorname{diam}_{\mathrm{ph}} \rho(t,\cdot,\cdot) \le Ae^{-\beta t} \operatorname{diam}_{\mathrm{ph}} \rho(0,\cdot,\cdot)$ for $t \ge 0$.*

Proof of the theorem. The proof consists of preparations necessary for an application of the auxiliary statement 4.14. We start with a short remark on I and G. The stability assumption on the field model, I can not be trivial and in particular $\|I\|_{L^1(S^1)} > 0$. By condition (C1) also $I\psi$ is non-trivial. Furthermore, by the stability condition on the field model, there exists a solution of the type $\theta(t,x) = \phi(t)$ (synchrony) for some appropriate continuously differentiable ϕ. This implies that $\|G(x,\cdot)\|_{L^1(\mu)}$ is independent of $x \in X$ and in fact equal to some constant $G_o > 0$.

Consider the continuity equation

$$\partial_t \rho(t,x,\vartheta) = -\partial_\vartheta \Big[\rho(t,x,\vartheta)\,[\omega + \psi(\vartheta) S_{\mathbb{R}}(x,\rho(t,\cdot,\cdot))]\Big] \tag{44}$$

for densities $\rho : \mathbb{R} \times X \times \mathbb{R} \to \mathbb{R}_+$ instead of $\mathbb{R} \times X \times S^1 \to \mathbb{R}_+$, with $S_\mathbb{R}$ defined as in (40) with the integration domain S^1 replaced by $\mathbb{R}$. Recall that by Theorem 3.9 every orbit in Ω_{CE,S^1} is a wrapping of an orbit of class

$$\Omega_{\mathrm{CE},\mathbb{R}} := \{\rho \in \Omega_\mathbb{R} : \partial_\vartheta \rho(0,\cdot,\cdot) \in \mathcal{C}_{\mathrm{u,b}}(X \times \mathbb{R}, \mathbb{R}) \\ \wedge\ \operatorname{diam}_{\mathrm{ph}} \rho(0,\cdot,\cdot) < \infty \ \wedge\ \rho \text{ solves } (44)\},$$

with $\Omega_\mathbb{R}$ defined in (9). For technical convenience, we shall prove the theorem for orbits in $\Omega_{\mathrm{CE},\mathbb{R}}$ instead. Also recall that, by Theorem 3.9, every orbit $\rho \in \Omega_{\mathrm{CE},\mathbb{R}}$ is in fact in $\mathcal{C}^1([0,\infty), \mathcal{C}_{\mathrm{u,b}}(X \times \mathbb{R}, \mathbb{R}))$ with $\partial_\vartheta \rho(t,\cdot,\cdot)$ being continuously differentiable and bounded for all $t \geq 0$. Furthermore $\operatorname{supp}_{\mathrm{ph}} \rho(t,\cdot,\cdot) \subseteq \left\{\vartheta \in \mathbb{R} : d\left(\vartheta,\ \operatorname{supp}_{\mathrm{ph}} \rho(0,\cdot,\cdot)\right) \leq Ct\right\}$ for all $t \geq 0$ and some constant $C > 0$.

For any fixed orbit $\rho \in \Omega_{\mathrm{CE},\mathbb{R}}$ and $x \in X$ consider the non-autonomous equation of motion

$$\frac{d\theta_x^p(t)}{dt} = \mathrm{v}(t, x, \theta_x^p(t)) := \omega + \psi(\theta_x^p(t)) S_\mathbb{R}(x, \rho(t,\cdot,\cdot)) \tag{45}$$

in the real variable θ_x^p, whose solutions we shall refer to as *point orbits*. We begin with some regularity statements:

R.1. Every $\rho_o \in \Omega_{o,\mathbb{R}}$ is also in $\mathcal{C}_\mathrm{b}(X, L^1(\mathbb{R},\mathbb{R}))$. To see this, note that $\rho_o \in \mathcal{C}_\mathrm{b}(X, \mathcal{C}_\mathrm{b}(\mathbb{R},\mathbb{R}))$. Furthermore, by [12] every sequence $(f_n)_n \in \mathcal{C}_\mathrm{b}(\mathbb{R},\mathbb{R})$ with $\|f_n\|_{L^1(\mathbb{R})} = 1\ \forall n$, converging uniformly to some $f \in \mathcal{C}_\mathrm{b}(\mathbb{R},\mathbb{R})$ with $\|f\|_{L^1(\mathbb{R})} = 1$, also converges to f in $L^1(\mathbb{R},\mathbb{R})$.

R.2. By the auxiliary statement 3.8(1), for any $\rho_o \in \Omega_{o,\mathbb{R}}$ the mapping $x \mapsto S_\mathbb{R}(x, \rho_o)$ is well-defined and continuous. Furthermore, for any orbit $\rho \in \Omega_\mathbb{R}$ and fixed $x \in X$, the mapping $t \mapsto S_\mathbb{R}(x, \rho(t,\cdot,\cdot))$ is continuous. Finally, $S_\mathbb{R}$ is bounded on $X \times \Omega_{o,\mathbb{R}}$.

R.3. The velocity field $\mathrm{v} : [0,\infty) \times X \times \mathbb{R} \to \mathbb{R}$ defined in (45) is bounded by $\mathrm{v}_{\max} := \omega + \|\psi\|_\infty G_o \|I\|_\infty$, continuous in (t,ϑ) and Lipschitz-continuous in ϑ, while the Lipschitz constant can be chosen to be independent of $t \in [0,\infty)$ or $x \in X$, and in fact of the orbit ρ itself. A similar statement holds for $\partial_\vartheta \mathrm{v}$. Thus the initial value problem for the equation of motion (45) has, for any intermediate value at $t_1 \geq 0$, one maximal corresponding point orbit $[0,\infty) \to \mathbb{R}$.

For any orbit $\rho \in \Omega_{\mathrm{CE},\mathbb{R}}$ and any corresponding point orbit θ_x^p, one finds as in Remark 3.1 the representation

$$\rho(t, x, \theta_x^p(t)) = \exp\left[-\int_0^t \psi'(\theta_x^p(\tau)) S_\mathbb{R}(x, \rho(\tau,\cdot,\cdot))\, d\tau\right] \rho(0, x, \theta_x^p(0)).$$

This shows that $\rho(t, x, \theta_x^p(t))$ is either zero for all $t \geq 0$ or non-zero for all $t \geq 0$, depending on the initial position $\theta_x^p(0)$. We thus conclude that any two point orbits $\theta_x^f(\cdot), \theta_x^b(\cdot)$ initially *enclosing* $\rho(0,x,\cdot)$, that is $\operatorname{supp} \rho(0,x,\cdot) \subseteq [\theta_x^b(0), \theta_x^f(0)]$, do so for all $t \geq 0$. For any $\rho_o \in \Omega_{o,\mathbb{R}}$ with $\operatorname{diam}_{\mathrm{ph}} \rho_o < \infty$ and $x \in X$, define

$$\Theta(\rho_o)(x) = \int_\mathbb{R} \rho_o(x,\vartheta)\vartheta\, d\vartheta.$$

Then the mapping $x \mapsto \Theta(\rho_o)(x)$ is well-defined and by (R.1) continuous and bounded. Furthermore, for any orbit $\rho \in \Omega_{\mathrm{CE},\mathbb{R}}$, the mapping $[0,\infty) \to \mathcal{C}_\mathrm{b}(X,\mathbb{R})$, $t \mapsto \Theta(\rho(t,\cdot,\cdot))$, is continuous. This follows from the definition of $\Omega_{\mathrm{CE},\mathbb{R}}$ and the fact that the boundaries of $\operatorname{supp}_{\mathrm{ph}} \rho(t,\cdot,\cdot)$ grow at most linearly with time. In fact, the mapping is for similar reasons differentiable with derivative

$$\frac{d}{dt}\Theta(\rho(t,\cdot,\cdot))(x) = \int_{\mathbb{R}} (d_t\rho)(t,x,\vartheta)\,\vartheta\,d\vartheta, \tag{46}$$

with $d_t\rho$ being the time-derivative of ρ in $\mathcal{C}^1([0,\infty),\mathcal{C}_{\mathrm{u,b}}(X\times\mathbb{R},\mathbb{R}))$. By continuity of the map $t \mapsto (d_t\rho)(t) \in \mathcal{C}_{\mathrm{u,b}}(X\times\mathbb{R},\mathbb{R})$, the map $t \mapsto \frac{d}{dt}\Theta(\rho(t)) \in \mathcal{C}_{\mathrm{b}}(X,\mathbb{R})$ is continuous. We conclude that $\Theta(\rho) \in \mathcal{C}^1([0,\infty),\mathcal{C}_{\mathrm{b}}(X,\mathbb{R}))$ for any orbit $\rho \in \Omega_{\mathrm{CE},\mathbb{R}}$. Using (44) and (46) one finds that $\Theta(\rho)$ satisfies the ODE

$$\begin{aligned}\frac{d}{dt}\Theta(\rho(t,\cdot,\cdot))(x) &= -\int_{\mathbb{R}} \vartheta\cdot\partial_\vartheta(\rho\mathrm{v})(t,x,\vartheta)\,d\vartheta \overset{(2)}{=} \int_{\mathbb{R}} (\rho\mathrm{v})(t,x,\vartheta)\,d\vartheta \\ &= \omega + S_{\mathbb{R}}(x,\rho(t,\cdot,\cdot))\int_{\mathbb{R}} \rho(t,x,\vartheta)\psi(\vartheta)\,d\vartheta,\end{aligned} \tag{47}$$

where partial integration is used in step (2). Take any $\rho_o \in \Omega_{o,\mathbb{R}}$ with $\operatorname{diam}\operatorname{supp}\rho_o(y,\cdot) < \infty$ for some $y \in X$. Then since I' is uniformly continuous, one can approximate

$$\int_{\mathbb{R}} \rho_o(y,\vartheta)I(\vartheta)\,d\vartheta = I(\Theta(\rho_o)(y)) + o([\operatorname{diam}\operatorname{supp}\rho_o(y,\cdot)]), \tag{48}$$

with the error term on the right hand side depending on $\rho_o(y,\cdot)$ but scaling down as $\operatorname{diam}\operatorname{supp}\rho_o(y,\cdot) \to 0$. A similar estimate holds for $\int_{\mathbb{R}} \rho_o(y,\vartheta)\psi(\vartheta)\,d\vartheta$. Using (48) and the fact that $\sup_x \|G(x,\cdot)\|_\infty < \infty$, for any $\rho_o \in \Omega_{o,\mathbb{R}}$ and $x \in X$ one finds

$$S_{\mathbb{R}}(x,\rho_o) = \int_X G(x,y)I(\Theta(\rho_o)(y))\,d\mu(y) + o\Big(\sup_{y\in Y}\operatorname{diam}\operatorname{supp}\rho_o(y,\cdot)\Big), \tag{49}$$

with the error term on the right hand side depending on ρ_o and x but scaling down as $\sup_y \operatorname{diam}\operatorname{supp}\rho_o(y,\cdot) \to 0$ uniformly in x. Since $S_{\mathbb{R}}$ is bounded on $X\times\Omega_{o,\mathbb{R}}$, using (47) and (49) we conclude that for any orbit $\rho \in \Omega_{\mathrm{CE},\mathbb{R}}$, $\Theta(\rho)$ satisfies the ODE

$$\frac{d}{dt}\Theta(\rho(t,\cdot,\cdot)) = \mathcal{H}_o(\Theta(\rho(t,\cdot,\cdot))) + \mathcal{E}_o(\rho(t,\cdot,\cdot)), \tag{50}$$

with $\mathcal{H}_o : \mathcal{C}_{\mathrm{b}}(X,\mathbb{R}) \to \mathcal{C}_{\mathrm{b}}(X,\mathbb{R})$ given by (43) and the term $\mathcal{E}_o(\rho(t,\cdot,\cdot))$ being of order $o(\sup_y \operatorname{diam}\operatorname{supp}\rho(t,y,\cdot))$. We point out that comparing (50) to (28) reveals the following relationship: The mean phase field $\Theta(\rho(t,\cdot,\cdot))$ evolves similarly to the phase field in the field model, up to leading order in the phase distribution's diameter $\operatorname{diam}_{\mathrm{ph}}\rho(t,\cdot,\cdot)$. We now proceed with an important statement about the speed of decay of density diameters.

Claim. *There exist constants $\varepsilon > 0$, $0 < r < 1$, and a period $T > 0$ such that for any orbit $\rho \in \Omega_{\mathrm{CE},\mathbb{R}}$ and any corresponding family $\{\theta_x^b(t), \theta_x^f(t)\}_{x\in X}$ of point orbits enclosing $\rho(t,\cdot,\cdot)$, that is $\operatorname{supp}\rho(t,x,\cdot) \subseteq [\theta_x^b(t), \theta_x^f(t)]$ for all $t \geq 0$ and $x \in X$, one has:*

(i) If

$$2\left\|\Theta(\rho(t,\cdot,\cdot)) - \overline{\Theta}(\rho(t,\cdot,\cdot))\right\|_\infty + 2\sup_{x\in X}\left|\theta_x^f(t) - \theta_x^b(t)\right| \leq \varepsilon \tag{51}$$

holds for some t, then $\frac{d}{dt}\big(\theta_x^f(t) - \theta_x^b(t)\big) \leq 0$ for all $x \in X$.

(ii) If (51) holds for all $t \in [t_o, t_o+T]$ and some $t_o \geq 0$, then

$$\big(\theta_x^f(t_o+T) - \theta_x^b(t_o+T)\big) \leq r\big(\theta_x^f(t_o) - \theta_x^b(t_o)\big)$$

for all $x \in X$.

Roughly speaking, the density diameter decreases with time provided it is small already, and does so at an exponential rate.

Proof of statement. By condition (C1) there exist $0 < \varepsilon_I < \varepsilon_\psi < 1/2$ and $\vartheta_o \in S^1$ such that $\operatorname{supp} I \subseteq B_{\varepsilon_I}(\vartheta_o)$ and ψ' is strictly negative on $B_{\varepsilon_\psi}(\vartheta_o)$. Without loss of generality we may assume $\vartheta_o = 0$. Furthermore, there exists a constant $\alpha > 0$ such that $\psi(\vartheta_1) - \psi(\vartheta_2) \leq -\alpha(\vartheta_1 - \vartheta_2)$ for any $-\varepsilon_\psi \leq \vartheta_2 \leq \vartheta_1 \leq +\varepsilon_\psi$. Choose $\varepsilon = (\varepsilon_\psi - \varepsilon_I)/2 > 0$ and denote $\theta^b(t) = \inf_{x\in X} \theta^b_x(t)$. Then whenever condition (51) is met one has $\theta^b(t) \leq \theta^b_x(t) \leq \theta^f_x(t) \leq \theta^b(t) + \varepsilon$ and $\operatorname{supp} \rho(t, x, \cdot) \subseteq [\theta^b(t), \theta^b(t) + \varepsilon]$ for all $x \in X$. The former stems from the fact that $\Theta(\rho(t, \cdot, \cdot))(x)$ is always between $\theta^b_x(t)$ and $\theta^f_x(t)$. Denote $L_x(t) = \theta^f_x(t) - \theta^b_x(t)$ and suppose that (51) is satisfied at some time $t \geq t_o$. Then

$$\begin{aligned} \frac{d}{dt} L_x(t) &= S_{\mathbb{R}}(x, \rho(t, \cdot, \cdot)) \left[\psi(\theta^f_x(t)) - \psi(\theta^b_x(t))\right] \\ &\leq -\alpha L_x(t) S_{\mathbb{R}}(x, \rho(t, \cdot, \cdot)) \end{aligned} \tag{52}$$

for all $x \in X$. The equality in (52) follows from (45). To see the inequality, consider the following complementary cases:

- Either $B_{\varepsilon_I}(n) \cap \left[\theta^b(t), \theta^b(t) + \varepsilon\right] = \emptyset$ for all $n \in \mathbb{Z}$, in which case one has $S_{\mathbb{R}}(x, \rho(t, \cdot, \cdot)) = 0$ for all $x \in X$,
- or $B_{\varepsilon_I}(n) \cap \left[\theta^b(t), \theta^b(t) + \varepsilon\right] \neq \emptyset$ for some $n \in \mathbb{Z}$, in which case $\left[\theta^b_x(t), \theta^f_x(t)\right] \subseteq n + [-\varepsilon_\psi, +\varepsilon_\psi]$ and therefore $\psi(\theta^f_x(t)) - \psi(\theta^b_x(t)) \leq -\alpha L_x(t)$ for all $x \in X$.

In both cases the inequality in (52) and thus the first part of the claim are verified. We also see that whenever (51) is satisfied for all $t \in [t_o, t_o + T]$, the estimate

$$L_x(t_o + T) \leq \exp\left[-\alpha \int_{t_o}^{t_o+T} S_{\mathbb{R}}(x, \rho(t, \cdot, \cdot))\, dt\right] L_x(t_o) \tag{53}$$

holds. Now choose $T = 4/\omega$. Then one can estimate

$$\int_{t_o}^{t_o+T} S_{\mathbb{R}}(x, \rho(t, \cdot, \cdot))\, dt \geq \min\left\{\frac{1}{\|\psi\|_\infty},\ \frac{G_o}{\mathrm{v}_{\max}} \|I\|_{L^1(S^1)}\right\} > 0. \tag{54}$$

To see this, consider the two complementary cases:

- During $[t_o, t_o + T]$, every point orbit $\theta^b_x(t)$ $(x \in X)$ has advanced by at least 2. By 1-periodicity in ϑ of the corresponding velocity field $\mathrm{v} : (t, x, \vartheta) \mapsto \mathrm{v}(t, x, \vartheta)$ defined in (45), this implies that any point orbit $\theta^p_x(t)$ (with arbitrary initial value) has during $[t_o, t_o + T]$ advanced by at least 1. Consequently, by Remark 4.15 one can estimate

$$\int_{t_o}^{t_o+T} \rho(t, y, \vartheta) \mathrm{v}(t, y, \vartheta)\, dt \geq \int_{\vartheta}^{\vartheta+1} \rho(t_o + T, y, \varphi)\, d\varphi$$

for all $\vartheta \in \mathbb{R}$. This implies, for any $y \in X$,

$$\begin{aligned}
&\int_{\mathbb{R}} I(\vartheta) \int_{t_o}^{t_o+T} \rho(t, y, \vartheta)\, dt\, d\vartheta \\
&\geq \frac{1}{\mathrm{v}_{\max}} \int_{\mathbb{R}} I(\vartheta) \int_{t_o}^{t_o+T} \rho(t, y, \vartheta) \mathrm{v}(t, y, \vartheta)\, dt\, d\vartheta \\
&\geq \frac{1}{\mathrm{v}_{\max}} \int_{\mathbb{R}} I(\vartheta) \int_{\vartheta}^{\vartheta+1} \rho(t_o + T, y, \varphi)\, d\varphi\, d\vartheta \\
&= \frac{1}{\mathrm{v}_{\max}} \int_{\mathbb{R}} \rho(t_o + T, y, \varphi) \int_{\varphi-1}^{\varphi} I(\vartheta)\, d\vartheta\, d\varphi \\
&= \frac{\|I\|_{L^1(S^1)}}{\mathrm{v}_{\max}} \int_{\mathbb{R}} \rho(t_o + T, y, \varphi)\, d\varphi = \frac{\|I\|_{L^1}}{\mathrm{v}_{\max}}
\end{aligned}$$

This allows us to estimate

$$\begin{aligned}
&\int_{t_o}^{t_o+T} S_{\mathbb{R}}(x, \rho(t, \cdot, \cdot))\, dt \\
&= \int_X G(x, y) \int_{\mathbb{R}} I(\vartheta) \int_{t_o}^{t_o+T} \rho(t, y, \vartheta)\, dt\, d\vartheta\, d\mu(y) \\
&\geq \|G(x, \cdot)\|_{L^1(S^1)} \frac{\|I\|_{L^1}}{\mathrm{v}_{\max}} = \frac{G_o}{\mathrm{v}_{\max}} \|I\|_{L^1(S^1)},
\end{aligned}$$

which verifies (54).

- There exists an $x \in X$ so that the point orbit $\theta_x^b(t)$ has advanced by less than 2 during $[t_o, t_o + T]$. Since $|\theta_y^b(t) - \theta_x^b(t)| \leq \varepsilon < 1/2$ for all $y \in X$ and $t \in [t_o, t_o + T]$, this implies that all point orbits $\theta_y^b(t)$ have advanced by less than 3 during $[t_o, t_o + T]$. In particular

$$\begin{aligned}
3 &\geq \int_{t_o}^{t_o+T} \left[\omega - \|\psi\|_\infty S_{\mathbb{R}}(y, \rho(t, \cdot, \cdot))\right] dt \\
&= 4 - \|\psi\|_\infty \int_{t_o}^{t_o+T} S_{\mathbb{R}}(y, \rho(t, \cdot, \cdot))\, dt,
\end{aligned}$$

so that $\int_{t_o}^{t_o+T} S_{\mathbb{R}}(y, \rho(t, \cdot, \cdot))\, dt \geq 1/\|\psi\|_\infty$ for all $y \in X$. This also verifies (54).

Choosing $r = \exp\left[-\alpha \cdot \min\left\{1/\|\psi\|_\infty, G_o \|I\|_{L^1(S^1)}/\mathrm{v}_{\max}\right\}\right]$ proves by (53) the second part of the claim. □

We finish the proof by putting the above into the context of the auxiliary statement 4.14. Define the set

$$\begin{aligned}
\Gamma_o = \{&(\rho_o, (\theta_x^b)_{x \in X}, (\theta_x^f)_{x \in X}) \in \Omega_{o,\mathbb{R}} \times \mathbb{R}^X \times \mathbb{R}^X \\
&: \mathrm{diam}_{\mathrm{ph}}\, \rho_o < \infty \ \wedge \ \mathrm{supp}\, \rho_o(x, \cdot) \subseteq [\theta_x^b, \theta_x^f]\ \forall x \in X\}
\end{aligned}$$

and Γ as the collection of orbits $\gamma = (\rho(t, \cdot, \cdot), (\theta_x^b(t))_x, (\theta_x^f(t))_x)_{t \geq 0} \subseteq \Gamma_o$ satisfying the following:

- ρ is of class $\Omega_{\mathrm{CE},\mathbb{R}}$.
- θ_x^b and θ_x^f are point orbits corresponding to and enclosing ρ.

For each $x \in X$ define the function

$$D_x : \Gamma_o \to \mathbb{R}, \quad D_x\big(\rho_o, (\theta_y^b)_{y\in X}, (\theta_y^f)_{y\in X}\big) := \theta_x^f - \theta_x^b.$$

Then for any orbit $\gamma \in \Gamma$, the mappings $t \mapsto D_x(\gamma(t))$ are differentiable with a derivative bounded by $2\mathrm{v}_{\max}$, so that the family $\{D_x(\gamma(\cdot))\}_{x\in X}$ is equicontinuous. Define the functions

$$\begin{aligned}
&\Theta_1 : \Gamma_o \to V_1 = \mathbb{R}, \quad \Theta_1\big(\rho_o, (\theta_x^b)_x, (\theta_x^f)_x\big) = \overline{\Theta}(\rho_o) = \mu(\Theta(\rho_o))/\mu(X),\\
&\Theta_2 : \Gamma_o \to V_2 = \left\{ f \in \mathcal{C}_\mathrm{b}(X, \mathbb{R}) : \textstyle\int f \, d\mu = 0 \right\},\\
&\Theta_2\big(\rho_o, (\theta_x^b)_x, (\theta_x^f)_x\big) = \Theta(\rho_o) - \Theta_1(\rho_o),\\
&\mathcal{E} : \Gamma_o \to V_1 \times V_2 \cong \mathcal{C}_\mathrm{b}(X, \mathbb{R}), \quad \mathcal{E}\big(\rho_o, (\theta_x^b)_x, (\theta_x^f)_x\big) := \mathcal{E}_o(\rho_o).
\end{aligned}$$

Note that we identify $\mathcal{C}_\mathrm{b}(X, \mathbb{R}) \cong V_1 \times V_2$ by means of the decomposition $\theta = \overline{\theta} + \theta_\mathrm{v}$ for $\theta \in \mathcal{C}_\mathrm{b}(X, \mathbb{R})$. As seen above, for any orbit $\gamma \in \Gamma$ the mapping $t \mapsto (\Theta_1(\gamma(t)), \Theta_2(\gamma(t)))$ is continuously differentiable and satisfies by (50) the ODE

$$\frac{d}{dt}\left(\Theta_1(\gamma(t)), \Theta_2(\gamma(t))\right) = \mathcal{H}_o\left(\Theta_1(\gamma(t)), \Theta_2(\gamma(t))\right) + \mathcal{E}(\gamma(t)),$$

with $\mathcal{H}_o : V \to V$ being the Lipschitz continuous function defined in (43). This ensures in particular the continuity of $t \mapsto \mathcal{E}(\gamma(t))$. Furthermore, we have seen that $\mathcal{E}(\gamma_o)$ is of order $o(\sup_{x\in X} d_x(\gamma_o))$ uniformly in $\gamma_o \in \Gamma_o$. Conditions (C1) and (C2) of 4.14 are satisfied by the auxiliary statement above. Condition 4.14(C3) is satisfied by the stability assumption (C2) of this theorem on the field model. Therefore, the auxiliary statement 4.14 can be readily applied. Observe that $\operatorname{diam}_\mathrm{ph} \rho_o \le 2\,\|\Theta_2(\gamma_o)\| + 2\sup_{x\in X} D_x(\gamma_o)$ for any $\gamma_o = \big(\rho_o, (\theta_x^b)_x, (\theta_x^f)_x\big) \in \Gamma_o$ and that for any $\rho_o \in \Omega_{o,\mathbb{R}}$ with $\operatorname{diam}_\mathrm{ph} \rho_o < \infty$ there exists a $\gamma_o = \big(\rho_o, (\theta_x^b)_x, (\theta_x^f)_x\big) \in \Gamma_o$ such that $2\,\|\Theta_2(\gamma_o)\| + 2\sup_{x\in X} D_x(\gamma_o) \le 4 \operatorname{diam}_\mathrm{ph} \rho_o$. This translates the results of 4.14 to the claim of this theorem. $\square$

By Theorem 3.9, any initial state $\rho_o \in \Omega_{o,S^1}$ with uniformly continuous and bounded ρ_o and $\partial_\vartheta \rho_o$ corresponds to a global, maximal solution of the continuity equation (39) within Ω_{CE,S^1}. The theorem ensures the synchronization of the network, provided initial states are smooth enough and have an adequately small bandwidth ($\operatorname{diam}_\mathrm{ph} \rho_o < \delta$). Condition (C1) is similar to findings in finite networks of weakly coupled spiking oscillators [18, §2.4], which identify a negative response derivative ψ' at the spiking point as a sufficient condition for the stability of synchrony. Our results are an extension to the fluid model, i.e. the thermodynamic limit, for arbitrarily strong coupling but for identical oscillators. Also compare (C1) to the condition in the stability Theorem 4.10 for the field model, demanding that the scalar product (34) defining the constant B be strictly negative. If condition (C1) of Theorem 4.16 is satisfied, B will indeed be strictly negative. Figure 3 gives an example of a response function ψ and pulse I satisfying the two conditions.

The condition of a bounded initial density bandwidth can be translated to a condition on the upper bound in the inter-network variation of external perturbations, which keep the network within the basin of attraction of synchrony. Let the oscillator phase density satisfy the continuity equation (39) and suppose that at time 0 each oscillator is subject to a random, spike-like external stimulus, independent of other oscillators. Specifically, let the network be in the state $\rho(0^-, \cdot, \cdot)$ at time 0^-. At each point $x \in X$ let $\hat{F}(x)$ be a real random variable representing an external random stimulus acting on oscillators at x. We model the perturbative effects of

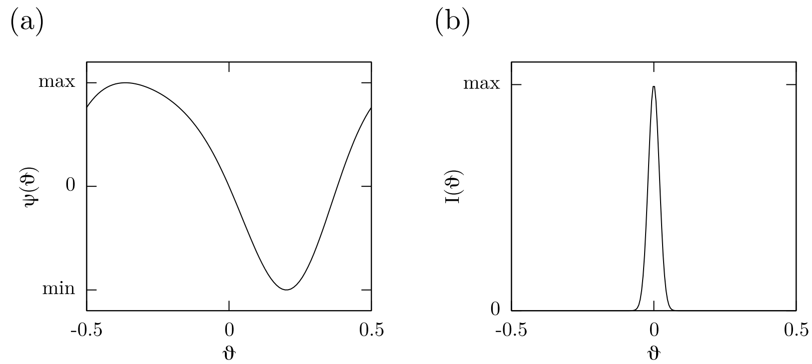

FIGURE 3. On Theorem 4.16. (a) Example response function $\psi : S^1 \to \mathbb{R}$ and (b), example pulse $I : S^1 \to \mathbb{R}_+$ satisfying condition (C1) of Theorem 4.16. The localization of the pulse within the region $\{\psi' < 0\}$ results in the local stability of network synchrony.

the stimulus $\hat{F}$ on the network, by setting $\rho(0^+, x, \cdot)$ to be the probability density on S^1 of the random variable $\hat{\theta} + \psi(\hat{\theta})\hat{F}(x)$, with $\hat{\theta}$ being some random variable on S^1 distributed with probability density $\rho(0^-, x, \cdot)$ and independent of $\hat{F}(x)$. If the network at time 0^- was synchronized, that is $\rho(0^-, x, \vartheta) = \delta(\vartheta - \phi_o)$ for some common phase $\phi_o \in S^1$ and all $x \in X$, the new density $\rho(0^+, x, \cdot)$ will correspond to the distribution of the random variable $\phi_o + \psi(\phi_o)\hat{F}(x)$. If furthermore there exist an $\varepsilon_F > 0$ and $F_o \in \mathbb{R}$ such that for every $x \in X$ one has $|\hat{F}(x) - F_o| \le \varepsilon_F$ almost surely, then all oscillator phases will be within an interval of width $2\,|\psi(\phi_o)|\,\varepsilon_F$. Let $f(x, \cdot)$ be the (sufficiently smooth) probability density for $\hat{F}(x)$. If $\delta > 0$ is as postulated by this theorem, then the condition $\varepsilon_F \le \delta/(2\,\|\psi\|_\infty)$ ensures the asymptotic recovery of synchrony after any such external perturbation.

5. Conclusion.

In this article we have studied two generalizations of the Winfree model for networks of phase oscillators distributed on a σ-finite Borel measure space over a separable metric space X. The oscillator coupling strength is described by a coupling kernel G. The first model is the differential equation (28) for the phase field $\theta(t, x)$. For a special subclass of this model, we have studied the local stability of synchrony against distortions within a certain smoothness class. The three key conditions for stability, as described in Theorem 4.10, concern the symmetry properties of the coupling kernel G, the strong connectivity of G, and the relation of the response derivative ψ' to the pulse I.

These results generalize previous findings on finite oscillator networks (see Remark 4.13). Furthermore, they demonstrate that network synchronization, even in the continuum limit, is a question of network connectivity and oscillator interaction dynamics, separated from the underlying space geometry (e.g. ring or infinite chain). The strong network connectivity, as defined in 4.1, may appear to be a quite abstract condition. But examples such as 4.2(iv) and (v) show that in many practical cases, strong connectivity reduces to the existence of short-range connections between oscillators, such as a non-vanishing local synapse density in nervous tissue.

The second model considered is the fluid model, defined as the continuity equation (4) for the oscillator phase density. In Theorems 3.6, 3.7 and 3.9 we have proven the existence and uniqueness of solutions within a certain function class and for certain initial values. In Theorem 4.16 we have shown the local stability of synchrony for networks of identical oscillators on finite measure spaces. The conditions are that synchrony be locally exponentially stable in the corresponding field model and ψ' be strictly negative on a circular arc enclosing $\operatorname{supp} I$. To the best of our knowledge, no similar results have been obtained for the fluid model, not even for all-to-all coupling (i.e. $G(x,y) \equiv \text{const}\ \forall x,y \in X$). This might be due to the fact that the synchronized solution is, strictly speaking, a rotating Dirac-distribution, therefore difficult to treat using classical perturbation methods.

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Appendix A. Supplemental proofs.

A.1: Proof of the auxiliary statement 3.8

Let $G_{\max} = \sup_{x\in X} \|G(x,\cdot)\|_{L^1(\mu)}$. Since for any $\rho \in \Omega_o$ and $x \in X$

$$\int_X |G(x,y)| \int_K |\rho(y,\varphi)I(y,\varphi)|\, d\kappa(\varphi)\, d\mu(y) \le G_{\max} \|I\|_\infty ,$$

the stimulus $S : X \times \Omega_o \to \mathbb{K}$ is well-defined and bounded.

1. For any $\rho \in \Omega_o$ and $x_1, x_2 \in X$ one can by Hölder estimate

$$\begin{aligned}|S(x_1,\rho) - S(x_2,\rho)| \le& \|G(x_1,\cdot) - G(x_2,\cdot)\|_{L^1(\mu)} \|I\|_\infty \\ \le& \omega_G(d(x_1,x_2)) \|I\|_\infty ,\end{aligned}$$

with ω_G being the modulus of continuity of the mapping $x \mapsto G(x,\cdot) \in L^1(\mu)$. Therefore $S(\cdot,\rho) \in \mathcal{C}_{\mathrm{u,b}}(X,\mathbb{K})$. Now fix $x \in X$ and let $(\rho_n)_n \subseteq \Omega_o$ be a sequence converging to $\rho \in \Omega_o$. By [12] we have $\|\rho_n(y,\cdot) - \rho(y,\cdot)\|_{L^1(\kappa)} \overset{n\to\infty}{\longrightarrow} 0$ for every $y \in X$. This implies that

$$\int_K \rho_n(y,\varphi)I(y,\varphi)\, d\kappa(\varphi) \overset{n\to\infty}{\longrightarrow} \int_K \rho(y,\varphi)I(y,\varphi)\, d\kappa(\varphi)$$

for every $y \in X$. Since $\left|\int_K \rho_n(y,\varphi)I(y,\varphi)\, d\kappa(\varphi)\right| \le \|I\|_\infty$ and $G(x,\cdot) \in L^1(\mu)$, by Lebesgue's dominated convergence theorem one has $S(x,\rho_n) \overset{n\to\infty}{\longrightarrow} S(x,\rho)$.

2. For any $x_1, x_2 \in X$ and $\rho_1, \rho_2 \in \Omega_o$ one can estimate

$$\begin{aligned}&|S(x_1,\rho_1) - S(x_2,\rho_2)| \\ \le& \int_X |G(x_1,y)| \int_K |\rho_1(y,\varphi) - \rho_2(y,\varphi)|\, |I(y,\varphi)|\, d\kappa(\varphi)\, d\mu(y) \\ &+ \int_X |G(x_1,y) - G(x_2,y)| \int_K |\rho_2(y,\varphi)I(y,\varphi)|\, d\kappa(\varphi)\, d\mu(y) \\ \le& G_{\max} \|I\|_\infty \kappa(K) \|\rho_1 - \rho_2\|_\infty + \omega_G(d(x_1,x_2)) \|I\|_\infty ,\end{aligned}$$

which shows that $S \in \mathcal{C}_{\mathrm{u,b}}(X \times \Omega_o, \mathbb{K})$. It also shows that S is Lipschitz continuous in Ω_o with Lipschitz constant $L_S = G_{\max} \|I\|_\infty \kappa(K)$. $\square$

A.2: Proof of Lemma 4.3
The continuity of the operator $\hat{G} : V \to V$ follows from the fact that $\sup_{x\in X} \|G(x,\cdot)\|_{L^1(\mu)} < \infty$. Non-negativity is clear since G is real and non-negative. Non-triviality of $\hat{G}$ follows from the non-triviality of the mapping $x \mapsto G(x,\cdot) \in L^1(\mu)$. Its σ-order continuity follows from Lebesgue's dominated convergence theorem. It remains to show its band irreducibility.

Let $\{0\} \subsetneq \mathcal{B}_{\mathbb{C}} \subseteq V$ be a $\hat{G}$-invariant band in $V = V_{\mathbb{R}} + iV_{\mathbb{R}}$. We show that $\mathcal{B}_{\mathbb{C}} = V$. For that it suffices to consider its real part $\mathcal{B} \supsetneq \{0\}$, a $\hat{G}$-invariant band in $V_{\mathbb{R}}$, and show that $\mathcal{B} = V_{\mathbb{R}}$. We begin by showing the existence of a function $0 \le f \in \mathcal{B}$, strictly positive on a dense subset of X.

> *Proof.* By assumption, there exists some $0 \neq h \in \mathcal{B}$. Since $\mathcal{B}$ is a band, we may without loss assume $h \ge 0$. Then by continuity, h is strictly positive on some non-empty open subset of X. Since μ is strictly positive, this means $\|h\|_{L^1(\mu)} > 0$. Define the family of functions $0 \le f_n := \hat{G}^n h \in \mathcal{B}$ for $n \in \mathbb{N}$. By strong connectivity of G, the union $\bigcup_{n\in\mathbb{N}_0} f_n^{-1}((0,\infty])$ has full measure, and by strict positivity of the measure, it is dense in X. We rescale $\tilde{f}_n := \frac{1}{(n+1)^2} f_n/(1+\|f_n\|_\infty)$ and consider the pointwise sum $f := \sum_{k=0}^{\infty} \tilde{f}_k$. Then f is the uniform limit of the sequence of partial sums $\sum_{k=0}^{n} \tilde{f}_k$, so that $f \in \mathcal{C}_{\mathrm{b}}(X,\mathbb{R})$. It is also the supremum of the family $\{\tilde{f}_k\}_{k\in\mathbb{N}_0}$ in $V_{\mathbb{R}}$, so that by band property (B2), $f \in \mathcal{B}$. Furthermore, $f \ge 0$ is strictly positive on a dense subset of X.

We finish the proof by showing that every $h \in V_{\mathbb{R}}$ is in $\mathcal{B}$. Since $\mathcal{B}$ is a sub-lattice vector space, it suffices to show that the positive part of h is within $\mathcal{B}$, so that we may assume $h \ge 0$. For each $x \in X$ define $h_x = \frac{h(x)}{f(x)} f$ if $f(x) > 0$ and $h_x = 0$ if $f(x) = 0$. Then $0 \le h_x \in \mathcal{B}$. Furthermore $h_x(x) = h(x)$ whenever $f(x) > 0$. For each $x \in X$ define $\tilde{h}_x = \min(h_x, h)$. Then $\tilde{h}_x \in \mathcal{B}$ by band property (B1), since it is dominated by h_x. By construction, $\tilde{h}_x \le h$ for all $x \in X$ and $\tilde{h}_x(x) = h(x)$ whenever $f(x) > 0$. Consequently, h is an upper bound for all $\{\tilde{h}_y\}_{y\in X}$ and $h(x)$ is the supremum of the values $\{\tilde{h}_y(x)\}_{y\in X}$ whenever $f(x) > 0$. Since h is continuous and f is strictly positive on a dense subset, h is the smallest upper bound for $\{\tilde{h}_y\}_{y\in X}$ in $V_{\mathbb{R}}$, which by band property (B2) implies $h \in Bc$. $\square$

A.3: Proof of the auxiliary statement 4.9
We start with two remarks:

1. $(\mathcal{H}_{1,o}, \mathcal{H}_{2,o})$ generates a flow $(U_{1,o}, U_{2,o})$ on $V_1 \times V_2$. To see this, note that $\mathcal{H}_{1,o}(v_1) = \mathcal{H}_1(v_1, 0)$ for all $v_1 \in V_1$ by assumptions (4) and (5). By assumption (1), $\mathcal{H}_{1,o}$ is Lipschitz-continuous, so the flow $U_{1,o}(t,t_o) : V_1 \to V_1$ is well-defined. Since by assumption (3) $\mathcal{H}_{2,o} : V_1 \times V_2 \to V_2$ is continuous, for each fixed $v_1 \in V_1$ and $t_o \in \mathbb{R}$ the mapping $(t, v_2) \mapsto \mathcal{H}_{2,o}(U_{1,o}(t,t_o)(v_1))v_2$ is continuous. Furthermore, by assumption (3) the mapping $(t, v_2) \mapsto \mathcal{H}_{2,o}(U_{1,o}(t,t_o)(v_1))v_2$ is Lipschitz continuous in the second argument, with a time-independent Lipschitz constant. Therefore, for each $v_1 \in V_1$ and $t_o \le t \in \mathbb{R}$, the propagator $U_{2,o}(t,t_o)(v_1,\cdot) : V_2 \to V_2$ is well-defined and a bounded linear operator on V_2.

2. Since $\mathcal{H}_2(v_1, 0) = 0$ for all $v_1 \in V_1$, the subspace $V_1 \times \{0\}$ is indeed U-invariant, that is $(U_1, U_2)(t, t_o)(v_1, 0) \in V_1 \times \{0\}$ for any $v_1 \in V_1$ and $t_o \leq t \in \mathbb{R}$. Thus $U_1(t, t_o)(v_1, 0) = U_{1,o}(t, t_o)(v_1)$. Consequently, Kv_2 is simply $U_{2,o}(t_o + T, t_o)(v_1, v_2)$ for any arbitrary $v_1 \in V_1$.

Since the flow is autonomous we may fix the initial time t_o without loss of generality. We claim that

$$[(U_1, U_2)(t, t_o)(v_1, v_2) - (U_{1,o}, U_{2,o})(t, t_o)(v_1, v_2)] \in o(v_2)$$

as $v_2 \to 0$, uniformly in $v_1 \in V_1$ and $t \in [t_o, t_o + T]$.

Proof of claim. Define $\eta_1 = \mathcal{H}_1 - \mathcal{H}_{1,o}$ and $\eta_2 = \mathcal{H}_2 - \mathcal{H}_{2,o}$. By Lipschitz continuity of $(\mathcal{H}_1, \mathcal{H}_2)$ there exists a constant $C > 0$, depending only on T and the Lipschitz-constant of $(\mathcal{H}_1, \mathcal{H}_2)$, such that

$$\begin{aligned} \|(U_1, U_2)(t, t_o)(v_1, v_2) - (U_1, U_2)(t, t_o)(v_1, 0)\| \leq & C \|(v_1, v_2) - (v_1, 0)\| \\ = & C \|v_2\| . \end{aligned}$$

for all $(v_1, v_2) \in V_1 \times V_2$ and $t \in [t_o, t_o + T]$. By flow invariance of $V_1 \times \{0\}$ in particular,

$$\|U_2(t, t_o)(v_1, v_2)\| \leq C \|v_2\| . \tag{55}$$

We can therefore estimate

$$\|\eta_i((U_1, U_2)(t, t_o)(v_1, v_2))\| \in o(v_2) \tag{56}$$

for $i \in \{1, 2\}$, uniformly in $v_1 \in V_1$ and $t \in [t_o, t_o + T]$. Let $\Omega \subseteq V_2$ be an open neighbourhood of the origin such that $(\mathcal{H}_{1,o}, \mathcal{H}_{2,o})$ is Lipschitz continuous on $V_1 \times \Omega$ with Lipschitz constant L_o. Such a neighbourhood exists by assumptions (2) and (3). Any solution $(v_1(t), v_2(t))$ to the ODE $\frac{d}{dt}(v_1(t), v_2(t)) = (\mathcal{H}_1, \mathcal{H}_2)(v_1(t), v_2(t))$ in $V_1 \times V_2$ satisfies

$$\frac{d}{dt}(v_1(t), v_2(t)) = (\mathcal{H}_{1,o}, \mathcal{H}_{2,o})(v_1(t), v_2(t)) + (\eta_1, \eta_2)(v_1(t), v_2(t)).$$

By Grönwall's inequality, this implies

$$\begin{aligned} &\|(U_1, U_2)(t, t_o)(v_1, v_2) - (U_{1,o}, U_{2,o})(t, t_o)(v_1, v_2)\| \\ &\leq T e^{T L_o} \sup_{s \in [t_o, t_o + T]} \|(\eta_1, \eta_2)((U_1, U_2)(s, t_o)(v_1, v_2))\| \end{aligned} \tag{57}$$

for $t \in [t_o, t_o + T]$ and $(v_1, v_2) \in V_1 \times V_2$, as long as both $(U_1, U_2)(t, t_o)(v_1, v_2)$ and $(U_{1,o}, U_{2,o})(t, t_o)(v_1, v_2)$ are within $V_1 \times \Omega$ for all $t \in [t_o, t_o + T]$. By (55) we can guarantee that $(U_1, U_2)(t, t_o)(v_1, v_2) \in V_1 \times \Omega$ for all $t \in [t_o, t_o + T]$ and all $v_1 \in V_1$ by choosing v_2 arbitrarily small. On the other hand, one can estimate $\|U_{2,o}(t, t_o)(v_1, v_2)\| \leq e^{TM} \|v_2\|$ for all $t \in [t_o, t_o + T]$, with $M := \sup_{u \in V_1} \|\mathcal{H}_{2,o}(u)\| < \infty$. We can thus also guarantee $(U_{1,o}, U_{2,o})(t, t_o)(v_1, v_2) \in V_1 \times \Omega$ for all $t \in [t_o, t_o + T]$ and all $v_1 \in V_1$ by choosing v_2 adequately small. Therefore the estimate (57) holds for all $v_1 \in V_1$ and $t \in [t_o, t_o + T]$ provided that v_2 is chosen sufficiently small. By (56) the supremum on the right hand side of (57) is of order $o(v_2)$, uniformly in $v_1 \in V_1$. □

Define $f = U_2(t_o + T, t_o) - U_{2,o}(t_o + T, t_o)$. Then $U_2(t_o + T, t_o)(v_1, v_2) = Kv_2 + f(v_1, v_2)$ with $K \in \mathcal{L}(V_2)$ uniformly exponentially stable [16, Chapter II]. Since $f(v_1, v_2) \in o(v_2)$ uniformly in v_1, it follows easily that uniform exponential stability in V_2 is still valid for $U_2(t_o+T,t_o)$: There exist constants $\tilde{A},\beta,\tilde{\delta}>0$ such that whenever $\|v_2\|\leq\tilde{\delta}$, one has

$$\|U_2(t_o+nT,t_o)(v_1,v_2)\| \leq \tilde{A}e^{-\beta nT}\|v_2\| \tag{58}$$

for any $v_1\in V_1$ and all $n\in\mathbb{N}_0$. Choose $\delta=\tilde{\delta}/(C+1)$ and $A=C\tilde{A}e^{\beta T}$. Then for any $n\in\mathbb{N}_0$, $\tau\in[0,T]$, $v_1\in V_1$ and $v_2\in V_2$ with $\|v_2\|\leq\delta$ one has

$$\begin{aligned}
&\|U_2(t_o+nT+\tau,t_o)(v_1,v_2)\|\\
&\overset{(1)}{=}\|U_2(t_o+nT,t_o)\circ(U_1(t_o+\tau,t_o),U_2(t_o+\tau,t_o))(v_1,v_2)\|\\
&\overset{(2)}{\leq}\tilde{A}e^{-\beta nT}\|U_2(t_o+\tau,t_o)(v_1,v_2)\|\\
&\overset{(3)}{\leq}C\tilde{A}e^{-\beta nT}\|v_2\|\leq Ae^{-\beta(nT+\tau)}\|v_2\|.
\end{aligned}$$

In step (1) we used the fact that the flow U is autonomous. In step (2) we used (55) and (58). In step (3) we used again (55). This completes the proof. □

A.4: Proof of the auxiliary statement 4.14

As a preliminary, we begin with the following remark: *Let $[t_o,t_1]\subseteq\mathbb{R}$ be a compact interval. Let $\{f_\alpha\}_{\alpha\in A}\subseteq\mathcal{C}([t_o,t_1],\mathbb{R})$ be a family of equicontinuous functions differentiable on (t_o,t_1). Let $f:=\sup_{\alpha\in A}f_\alpha$ be defined pointwise (not necessarily finite). Suppose there exists some $M\in\mathbb{R}$ such that whenever $f(t)<M$ for some $t\in[t_o,t_1]$ one has $\frac{d}{dt}f_\alpha(t)\leq 0$ for all $\alpha\in A$. Then if $f(t_o)<M$, one has $f(t)<M$ for all $t\in[t_o,t_1]$.* This follows from the fact that the supremum of equicontinuous functions is continuous. We shall make use of it below.

Without loss of generality we assume $A_o\geq 1$, $r\leq\frac{1}{4}$, and $A_oe^{-\beta_oT}\leq\frac{1}{4}$ (otherwise increase T), and $A_o\delta_o\leq\varepsilon/8$ (otherwise decrease δ_o). We may also assume that there exists a constant $C_o>0$ such that $\|\mathcal{E}(\gamma_o)\|\leq C_oD_{\mathrm{loc}}(\gamma_o)$ whenever $D_{\mathrm{loc}}(\gamma_o)\leq\delta_o$ (otherwise decrease δ_o). In the following, we shall denote by $\Theta_o(t)=(\Theta_{o1}(t),\Theta_{o2}(t))$ the solutions of the ODE $\frac{d}{dt}\Theta_o(t)=\mathcal{H}_o(\Theta_o(t))$ in V. Let us fix some orbit $\gamma\in\Gamma$ and write $f(t)$ instead of $f(\gamma(t))$ for any function f defined on Γ_o, as for example $\Theta(t):=(\Theta_1(t),\Theta_2(t)):=(\Theta_1(\gamma(t)),\Theta_2(\gamma(t)))$.

Since $\mathcal{H}_o$ is Lipschitz continuous there exists a constant $C\geq 1$, depending only on the Lipschitz constant of $\mathcal{H}_o$, the period T, and C_o, such that for all $0\leq t_o\leq t\leq t_o+T$,

$$\|\Theta(t)-\Theta_o(t)\|\leq C\sup_{s\in[t_o,t]}D_{\mathrm{loc}}(s), \tag{59}$$

provided $\Theta(t_o)=\Theta_o(t_o)$ and $\sup_{s\in[t_o,t]}D_{\mathrm{loc}}(s)\leq\delta_o$. Now choose some $0<\delta_1<\delta_o/(4C)$.

Claim 01. *Suppose $\Theta(t_o)=\Theta_o(t_o)$, $D_{\mathrm{loc}}(t_o)<\delta_1$ and $\|\Theta_2(t_o)\|\leq\delta_o$ at some time $t_o\geq 0$. Then $\|\Theta(t)-\Theta_o(t)\|\leq\delta_o/4$ for all $t\in[t_o,t_o+T]$.*

Proof of claim. Assume the contrary. Then by continuity of $\Theta(t)$ and $\Theta_o(t)$ in t, there exists $t_1\in[t_o,t_o+T]$ such that $\|\Theta(t)-\Theta_o(t)\|<\delta_o/4$ for all $t\in[t_o,t_1)$ and $\|\Theta(t_1)-\Theta_o(t_1)\|=\delta_o/4$. Since $\|\Theta_{o2}(t_o)\|=\|\Theta_2(t_o)\|\leq\delta_o$, by condition (C3) one has $\|\Theta_{o2}(t)\|\leq\varepsilon/8$ for all $t\in[t_o,t_1]$. This implies

$$\|\Theta_2(t)\|\leq\|\Theta_{o2}(t)\|+\|\Theta_2(t)-\Theta_{o2}(t)\|\leq\frac{\varepsilon}{8}+\frac{\delta_o}{4}\leq\frac{\varepsilon}{4} \tag{60}$$

for all $t \in [t_o, t_1]$. Thus whenever $D_{\rm loc}(t) \leq \delta_1 \leq \varepsilon/8$ with $t \in [t_o, t_1]$, one has by (60) $D_{\rm gl}(t) \leq \varepsilon$, and therefore by condition (C1), $\frac{d}{dt} D_\alpha(t) \leq 0 \ \forall \alpha \in A$. Consequently, the initial condition $D_{\rm loc}(t_o) < \delta_1$ implies that $D_{\rm loc}(t) < \delta_1$ for all $t \in [t_o, t_1]$, by the preliminary remark above. By (59) this implies $\|\Theta(t) - \Theta_o(t)\| \leq C\delta_1 < \delta_o/4$ for all $t \in [t_o, t_1]$, which is a contradiction. $\square$

Claim 02. *Suppose $D_{\rm loc}(t_o) < \delta_1$ and $\|\Theta_2(t_o)\| \leq \delta_o$ at some time $t_o \geq 0$. Then $D_{\rm gl}(t) \leq \varepsilon$ for all $t \in [t_o, t_o + T]$.*

Proof of claim. Set $\Theta_o(t_o) = \Theta(t_o)$. By condition (C3) one then has $\|\Theta_{o2}(t)\| \leq \varepsilon/8$ for all $t \in [t_o, t_o + T]$. Since by claim 01 also $\|\Theta(t) - \Theta_o(t)\| \leq \varepsilon/8$ for all $t \in [t_o, t_o + T]$, this implies $\|\Theta_2(t)\| \leq \varepsilon/4$ for all $t \in [t_o, t_o + T]$. Thus whenever $D_{\rm loc}(t) \leq \delta_1 \leq \varepsilon/8$ with $t \in [t_o, t_o + T]$, one has $D_{\rm gl}(t) \leq \varepsilon$ and therefore by condition (C1) $\frac{d}{dt} D_\alpha(t) \leq 0 \ \forall \alpha \in A$. The initial condition $D_{\rm loc}(t_o) < \delta_1$ implies by the preliminary remark above that $D_{\rm loc}(t) < \delta_1$ for all $t \in [t_o, t_o + T]$. Thus $D_{\rm gl}(t) = 2\|\Theta_2(t)\| + 2D_{\rm loc(t)} \leq \varepsilon$ for all $t \in [t_o, t_o + T]$. $\square$

In the following, suppose $D_{\rm loc}(t_o) < \delta_1$ and $\|\Theta_2(t_o)\| \leq \delta_o$ at some time $t_o \geq 0$. Then by claim 02 and condition (C1), $D_{\rm loc}(t)$ is non-increasing on $[t_o, t_o + T]$, so $D_{\rm loc}(t_o + T) < \delta_1$. By condition (C3) $\|\Theta_{o2}(t_o + T)\| \leq A_o e^{-\beta_o T} \|\Theta_{o2}(t_o)\| \leq \delta_o/4$ and by claim 01 $\|\Theta_2(t_o + T) - \Theta_{o2}(t_o + T)\| \leq \delta_o/4$ if we set $\Theta_o(t_o) = \Theta(t_o)$. Therefore $\|\Theta_2(t_o + T)\| \leq \delta_o/2 \leq \delta_o$. By induction we conclude that $D_{\rm loc}(t_o + nT) < \delta_1$ as well as $\|\Theta_2(t_o + nT)\| \leq \delta_o$ for all $n \in \mathbb{N}_0$ and by claim 02 $D_{\rm gl}(t) \leq \varepsilon$ for all $t \geq t_o$. By condition (C1) this implies that all $D_\alpha(t)$ $(\alpha \in A)$ are non-increasing with t and by condition (C2) that $D_\alpha(t_o + (n+1)T) \leq D_\alpha(t_o + nT)/4$ for all $n \in \mathbb{N}_0$ and $\alpha \in A$.

Through similar reasoning as above and by using estimate (59), we find that

$$\begin{aligned} \|\Theta_2(t_o + (n+1)T)\| \leq & A_o e^{-\beta_o T} \|\Theta_2(t_o + nT)\| + C D_{\rm loc}(t_o + nT) \\ \leq & \frac{1}{4} \|\Theta_2(t_o + nT)\| + C D_{\rm loc}(t_o + nT). \end{aligned} \tag{61}$$

Denoting $a_n = \|\Theta_2(t_o + nT)\|$ and $b_n = 4CD_{\rm loc}(t_o + nT)$, inequality (61) reads

$$a_{n+1} \leq \frac{a_n}{4} + \frac{b_n}{4}.$$

Note that also $b_{n+1} \leq \frac{a_n}{4} + \frac{b_n}{4}$, so that $(a_{n+1} + b_{n+1}) \leq (a_n + b_n)/2$ and therefore $(a_n + b_n) \leq (a_o + b_o)/2^n$. Consequently,

$$\begin{aligned} \frac{1}{2} D_{\rm gl}(t_o + nT) \leq & \|\Theta_2(t_o + nT)\| + 4CD_{\rm loc}(t_o + nT) \\ \leq & \frac{1}{2^n} \big(\|\Theta_2(t_o)\| + 4CD_{\rm loc}(t_o) \big) \leq \frac{2C}{2^n} D_{\rm gl}(t_o), \end{aligned} \tag{62}$$

where we used the fact that $C \geq 1$. Similarly to (61), for any $n \in \mathbb{N}_0$ and $\tau \in [0, T]$ we may estimate

$$\begin{aligned} D_{\rm gl}(t_o + nT + \tau) = & 2D_{\rm loc}(t_o + nT + \tau) + 2\|\Theta_2(t_o + nT + \tau)\| \\ \leq & 2D_{\rm loc}(t_o + nT) + 2A_o e^{-\beta_o \tau} \|\Theta_2(t_o + nT)\| \\ & + 2CD_{\rm loc}(t_o + nT) \\ \leq & A_1 D_{\rm gl}(t_o + nT), \end{aligned} \tag{63}$$

with $A_1 > 0$ being some constant only depending on C and A_o. By combining (62) with (63) we find

$$D_{\rm gl}(t) \leq Ae^{-\beta(t-t_o)} D_{\rm gl}(t_o)$$

for all $t \geq t_o$, with $\beta = \frac{1}{T} \ln 2$ and $A > 0$ being some constant only depending on C and A_1. Choosing $\delta = \delta_1$ completes the proof. $\square$

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