The Dirac Field[1]

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1 The Dirac field and Lorentz invariance

Much of the content of this section is in Peskin and Schroeder, Sections 3.1 and 3.2. You should study these sections.

Among the finite dimensional representations of the Lorentz group that we have found is the $(1/2,0)\oplus(0,1/2)$ representation. The objects on which this representation acts are called Dirac spinors. They have four components. The Lorentz transformation generators are 4×4 matrices as follows:

$$
\begin{aligned}
J_k &= \begin{pmatrix} \frac{1}{2}\sigma_k & 0 \\ 0 & \frac{1}{2}\sigma_k \end{pmatrix} \\
K_k &= \begin{pmatrix} -\frac{i}{2}\sigma_k & 0 \\ 0 & \frac{i}{2}\sigma_k \end{pmatrix} .
\end{aligned}
\tag{1}
$$

Here I follow the notation of Peskin and Schroeder, Eqs. (3.26) and (3.27). This puts the $(0,1/2)$ part on the top and the $(1/2,0)$ part on the bottom.

Exercise. Verify explicitly the Lorentz group algebra for these matrices.

There are some more matrices that are of interest with respect to Dirac spinors. These are the gamma matrices, γ^μ, with $\mu = 0,1,2,3$. We define

$$
\begin{aligned}
\gamma^0 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\gamma^k &= \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} .
\end{aligned}
\tag{2}
$$

The gamma matrices transform have the following commutation relations with the generators of the Lorentz group

$$
[\gamma^\mu, M_{\alpha\beta}] = (\mathcal{M}_{\alpha\beta})^\mu{}_\nu\,\gamma^\nu . \tag{3}
$$

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where, following our previous notation

$$M_{ij} = \epsilon_{ijk} J_k \qquad M_{i0} = K_i \tag{4}$$

and

$$(\mathcal{M}_{\alpha\beta})^\mu{}_\nu = i(\delta^\mu_\alpha\, g_{\beta\nu} - \delta^\mu_\beta\, g_{\alpha\nu}). \tag{5}$$

This relation implies

$$(1 + i\tfrac{1}{2}\theta^{\alpha\beta} M_{\alpha\beta})\gamma^\mu (1 - i\tfrac{1}{2}\theta^{\alpha\beta} M_{\alpha\beta}) = (\delta^\mu_\nu - i\tfrac{1}{2}\theta^{\alpha\beta}(\mathcal{M}_{\alpha\beta})^\mu{}_\nu)\gamma^\nu + \cdots. \tag{6}$$

In turn, this means that

$$U(\Lambda)^{-1}\gamma^\mu\, U(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu. \tag{7}$$

We interpret this as saying that the gamma matrices transform as a four-vector under Lorentz transformations Λ. For some insight as to why one might use these words, see the exercise below.

Exercise. Verify explicitly the commutation relation between the gamma matrices and the generators of the Lorentz group in the Dirac representation.

Exercise. Let $\mathcal{U}$ be a Dirac spinor and suppose that $\mathcal{U}$ obeys the equation

$$p_\mu \gamma^\mu\, \mathcal{U} = m\, \mathcal{U}. \tag{8}$$

Consider a Lorentz transformation Λ and let $\mathcal{U}' = U(\Lambda)\mathcal{U}$ and $p'^\mu = \Lambda^\mu{}_\nu p^\nu$ be the Lorentz transformed versions of $\mathcal{U}$ and p^μ. Show that then

$$p'_\mu \gamma^\mu\, \mathcal{U}' = m\, \mathcal{U}'. \tag{9}$$

The Lorentz group generators can be expressed in terms of the gamma matrices as

$$M_{\alpha\beta} = \frac{i}{4}\,[\gamma_\alpha, \gamma_\beta]. \tag{10}$$

Exercise. Verify that the Lorentz group generators can be expressed in terms of the gamma matrices as given above.

The gamma matrices obey the algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\, g^{\mu\nu}, \tag{11}$$

where $\{a, b\} = ab + ba$ is the anticommutator. If we make any linear transformation on the Dirac spinors, $\mathcal{U} \to \mathcal{U}' = A\mathcal{U}$ where A is a 4×4 matrix whose inverse exists, then we get new gamma matrices, $\gamma^\mu \to \gamma'^\mu = A\gamma^\mu A^{-1}$. That is, $\gamma'^\mu \mathcal{U}' = A\gamma^\mu \mathcal{U}$. In fact, one often chooses a different basis for the Dirac spinors, thus doing exactly this, and getting different gamma matrices. Evidently the new gamma matrices will obey the same algebra. Also the new Lorentz group generators will be related to the new gamma matrices by the same relation as above. We will get the same commutation relations for $[\gamma^\mu, M_{\alpha\beta}]$ and $[M_{\alpha\beta}, M_{\gamma\delta}]$. You will not be surprized to learn that one can derive these commutation relations from $\{\gamma^\mu, \gamma^\nu\} = 2\, g^{\mu\nu}$. Thus all you ever really need to know is $\{\gamma^\mu, \gamma^\nu\} = 2\, g^{\mu\nu}$.

Exercise. Verify

$$[\gamma^\mu, M_{\alpha\beta}] = (\mathcal{M}_{\alpha\beta})^\mu{}_\nu\, \gamma^\nu \tag{12}$$

using $\{\gamma^\mu, \gamma^\nu\} = 2\, g^{\mu\nu}$.

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2 The Dirac equation

Dirac proposed that, to describe electrons, one should use a field $\Psi(x)$ that transorms under the Lorentz group as described above. Furthermore, he proposed that in the absence of any interactions, the field should obey the covariant equation

$$(i\partial_\mu \gamma^\mu - m)\Psi(x) = 0. \tag{13}$$

3 Free particle solutions of the Dirac equation

This material is discussed in Peskin and Schroeder, so we can be brief. We seek the free particle solutions of the Dirac equation in momentum space:

$$(p_\mu \gamma^\mu - m)\,\mathcal{U}(p, s) = 0. \tag{14}$$

This equation implies $p_\mu p^\mu = m^2$:

$$0 = (p_\nu \gamma^\nu + m)(p_\mu \gamma^\mu - m)\mathcal{U}(p,s) = (p_\mu p^\mu - m^2)\mathcal{U}(p,s). \tag{15}$$

In this section, we take $p^0 > 0$. There are also solutions with $p^0 < 0$, which we discuss later.

We begin with the solutions for $p = p_0 = (m, \vec{0})$. Then the equation is

$$\begin{pmatrix} -m & m \\ m & -m \end{pmatrix} \mathcal{U}(p_0, s) = 0. \tag{16}$$

This says that the top two components of $\mathcal{U}$ equal the bottom two components, so the solution is

$$\mathcal{U}(p_0, s) = \sqrt{m} \begin{pmatrix} \xi(s) \\ \xi(s) \end{pmatrix}, \tag{17}$$

where

$$\xi(+\tfrac{1}{2}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \xi(-\tfrac{1}{2}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{18}$$

These are solutions because the top components and the bottom components are both the same, namely $\xi(s)$. The normalization factor $\sqrt{m}$ will prove convenient later. The particular solutions are chosen so that

$$J_3\,\mathcal{U}(p_0, s) = s\,\mathcal{U}(p_0, s), \tag{19}$$

where, we recall

$$J_3 = \begin{pmatrix} \frac{1}{2}\sigma_3 & 0 \\ 0 & \frac{1}{2}\sigma_3 \end{pmatrix}. \tag{20}$$

Now we can use Wigner construction to generate solutions $\mathcal{U}(p,s)$ from the solutions $\mathcal{U}(p_0, s)$. We choose a standard boost $\Lambda_0(p)$ that transforms p_0 into p:

$$\Lambda_0(p) = \exp(-i\omega\,\hat{p}\cdot\vec{\mathcal{K}}), \tag{21}$$

where $\hat{p} = \vec{p}/|\vec{p}|$ and

$$\cosh\omega = E/m, \qquad \sinh\omega = |\vec{p}|/m. \tag{22}$$

In order to generate spinors $\mathcal{U}(p,s)$ that correspond to the states

$$|p,s\rangle = \exp(-i\omega\,\hat{p}\cdot\vec{K})|p_0, s\rangle, \tag{23}$$

we take

$$\mathcal{U}(p,s) = \exp(-i\omega\,\hat{p}\cdot\vec{K})\mathcal{U}(p_0,s). \tag{24}$$

In the upper equation here, K_i is a quantum operator, while in the second equation is is the 4×4 matrix given in Eq. (1). I hope that using the same symbol for these two objects will not cause confusion.

Using Eq. (1), we have

$$-i\omega\,\hat{p}\cdot\vec{K} = \frac{\omega}{2}\begin{pmatrix} -\hat{p}\cdot\vec{\sigma} & 0 \\ 0 & \hat{p}\cdot\vec{\sigma} \end{pmatrix}. \tag{25}$$

Then

$$\exp(-i\omega\,\hat{p}\cdot\vec{K}) = \cosh(\omega/2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sinh(\omega/2)\begin{pmatrix} -\hat{p}\cdot\vec{\sigma} & 0 \\ 0 & \hat{p}\cdot\vec{\sigma} \end{pmatrix}. \tag{26}$$

Using Eq. (22) we have

$$\cosh(\omega/2) = \frac{\sqrt{E+m}}{\sqrt{2m}}, \qquad \sinh(\omega/2) = \frac{\sqrt{E-m}}{\sqrt{2m}}, \tag{27}$$

or

$$\cosh(\omega/2) = \frac{E+m}{\sqrt{2m(E+m)}}, \qquad \sinh(\omega/2) = \frac{|\vec{p}\,|}{\sqrt{2m(E+m)}}. \tag{28}$$

Thus

$$\exp(-i\omega\,\hat{p}\cdot\vec{K}) = \frac{1}{\sqrt{2m(E+m)}}\begin{pmatrix} E+m-\vec{p}\cdot\vec{\sigma} & 0 \\ 0 & E+m+\vec{p}\cdot\vec{\sigma} \end{pmatrix}. \tag{29}$$

Thus our solutions are

$$\mathcal{U}(p,s) = \frac{1}{\sqrt{2(E+m)}}\begin{pmatrix} [E+m-\vec{p}\cdot\vec{\sigma}]\,\xi(s) \\ [E+m+\vec{p}\cdot\vec{\sigma}]\,\xi(s) \end{pmatrix}. \tag{30}$$

Exercise. Prove that

$$\left(\frac{[E+m\pm\vec{p}\cdot\vec{\sigma}]}{\sqrt{2(E+m)}}\right)^2 = E\pm\vec{p}\cdot\vec{\sigma}. \tag{31}$$

Thus one might write a more compact looking form of the result for $\mathcal{U}$ as

$$\mathcal{U}(p,s) = \begin{pmatrix} \sqrt{E - \vec{p}\cdot\vec{\sigma}}\,\xi(s) \\ \sqrt{E + \vec{p}\cdot\vec{\sigma}}\,\xi(s) \end{pmatrix}, \tag{32}$$

as claimed in Peskin and Schroeder.

Exercise. Prove directly that Eq. (30) is a solution of the Dirac equation for any choice of the two component spinor ξ.

There is another set of solutions that is sometimes useful. Since, as you saw in the exercise, Eq. (30) is a solution of the Dirac equation for any choice of the two component spinor ξ, we could replace $\xi(s)$ by $w(\hat{p},s)$ where

$$\tfrac{1}{2}\,\hat{p}\cdot\vec{\sigma}\,w(\hat{p},s) = s\,w(\hat{p},s). \tag{33}$$

then

$$\mathcal{U}(p,s) = \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} [E + m - 2s\,|\vec{p}|]\,w(\hat{p},s) \\ [E + m + 2s\,|\vec{p}|]\,w(\hat{p},s) \end{pmatrix}. \tag{34}$$

The nice feature of these solutions is that

$$\hat{p}\cdot\vec{J}\,\mathcal{U}(p,s) = s\,\mathcal{U}(p,s). \tag{35}$$

Thus these spinors correspond to states that are eigenstates the helicity operator:

$$\hat{p}\cdot\vec{J}\,|p,s\rangle = s\,|p,s\rangle. \tag{36}$$

Note that in order to get these states, we need a different "standard boost" from the one that we have been using.

The helicity eigenstates are especially useful in the case $m = 0$. Then Eq. (34) becomes

$$\mathcal{U}(p,s) = \sqrt{\frac{E}{2}} \begin{pmatrix} [1 - 2s]\,w(\hat{p},s) \\ [1 + 2s]\,w(\hat{p},s) \end{pmatrix}. \tag{37}$$

That is

$$\begin{aligned} \mathcal{U}(p,\tfrac{1}{2}) &= \sqrt{2E} \begin{pmatrix} 0 \\ w(\hat{p},s) \end{pmatrix}, \\ \mathcal{U}(p,-\tfrac{1}{2}) &= \sqrt{2E} \begin{pmatrix} w(\hat{p},s) \\ 0 \end{pmatrix}. \end{aligned} \tag{38}$$

Thus the upper two components are used for helicity $-1/2$, “left handed,” while the lower two components are used for helicity $+1/2$, “right handed.” Neutrinos are left handed. (I suppose the neutrino mass is zero here, contrary to what seem to be the facts.)

4 Negative energy solutions of the Dirac equation

We have found solutions $\mathcal{U}(p,s)$ of the free Dirac equation

$$(k_\mu \gamma^\mu - m)\mathcal{U}(k,s) = 0. \tag{39}$$

These are the solutions for $k^2 = m^2$, $k^0 > 0$. What if we want $k^2 = m^2$, $k^0 < 0$? Well, we will want that. It's easy. First, let $p^\mu = -k^\mu$. Then $p^2 = m^2$, $p^0 > 0$ and we want to solve

$$(p_\mu \gamma^\mu + m)\mathcal{V}(p,s) = 0. \tag{40}$$

Note the change in sign of m. The notation is standard: $\mathcal{V}(p,s)$ refers to an antiparticle with momentum p and spin s. We can't see the relation with antiparticles yet – for now it's just notation.

We solve the equation using a symmetry. There is a matrix C with the property

$$C^{-1}\gamma^\mu C = -(\gamma^\mu)^*. \tag{41}$$

With our choce of gamma matrices, this matrix is

$$C = C^{-1} = -i\gamma^2. \tag{42}$$

Another property of C is that it is unitary:

$$C^\dagger = C^{-1}. \tag{43}$$

Let us define

$$\mathcal{V}(p,s) = C\,\mathcal{U}(p,s)^*. \tag{44}$$

Then

$$\begin{aligned} p_\mu\gamma^\mu\mathcal{V}(p,s) &= CC^{-1}p_\mu\gamma^\mu C\,\mathcal{U}(p,s)^* \\ &= C[-p_\mu\gamma^\mu\mathcal{U}(p,s)]^* \\ &= C[-m\mathcal{U}(p,s)]^* \\ &= -m\mathcal{V}(p,s). \end{aligned} \tag{45}$$

Thus $\mathcal{V}(p,s)$ solves the equation. We will find it useful later on to construct the antiparticle solutions this way.

Exercise. With $C = C^{-1} = -i\gamma^2$, show that $CC = 1$ and $C\gamma^\mu C = -(\gamma^\mu)^*$.

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5 Covariant inner product

Suppose that we have two Dirac spinors $\mathcal{A}$ and $\mathcal{B}$. We would like to have an inner product something like $\mathcal{A}^\dagger\mathcal{B}$. However, this is not Lorentz invariant:

$$\mathcal{A}^\dagger\mathcal{B} \to \mathcal{A}^\dagger \exp(+\tfrac{i}{2}\theta^{\alpha\beta}M^\dagger_{\alpha\beta}) \exp(-\tfrac{i}{2}\theta^{\alpha\beta}M_{\alpha\beta})\mathcal{B}. \tag{46}$$

This would be $\mathcal{A}^\dagger\mathcal{B}$ if the Lorentz group generator matrices $M_{\alpha\beta}$ were hermitian, but they are not: $M^\dagger_{\alpha\beta} \neq M_{\alpha\beta}$. Recall that

$$M_{\alpha\beta} = \tfrac{i}{4}[\gamma_\alpha, \gamma_\beta]. \tag{47}$$

Thus $M_{\alpha\beta}$ would be hermitian if the γ^μ matrices were hermitian, but they are not. In fact

$$(\gamma^0)^\dagger = \gamma^0 \;, \qquad (\gamma^j)^\dagger = -\gamma^j \quad j = 1,2,3. \tag{48}$$

[Note: one usually takes this equation to be an extra requirement for the gamma matrices besides the anticommutation relations.] From this relation together with the anticommutation relations, we get

$$\gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu. \tag{49}$$

Thus

$$\gamma^0(M_{\alpha\beta})^\dagger\gamma^0 = M_{\alpha\beta}. \tag{50}$$

Thus we find that $\mathcal{A}^\dagger\gamma^0\mathcal{B}$ is Lorentz invariant:

$$\begin{aligned}
\mathcal{A}^\dagger\gamma^0\mathcal{B} &\to \mathcal{A}^\dagger \exp(+\tfrac{i}{2}\theta^{\alpha\beta}M^\dagger_{\alpha\beta})\gamma^0 \exp(-\tfrac{i}{2}\theta^{\alpha\beta}M_{\alpha\beta})\mathcal{B} \\
&= \mathcal{A}^\dagger\gamma^0\gamma^0 \exp(+\tfrac{i}{2}\theta^{\alpha\beta}M^\dagger_{\alpha\beta})\gamma^0 \exp(-\tfrac{i}{2}\theta^{\alpha\beta}M_{\alpha\beta})\mathcal{B} \\
&= \mathcal{A}^\dagger\gamma^0 \exp(+\tfrac{i}{2}\theta^{\alpha\beta}M_{\alpha\beta}) \exp(-\tfrac{i}{2}\theta^{\alpha\beta}M_{\alpha\beta})\mathcal{B} \\
&= \mathcal{A}^\dagger\gamma^0\mathcal{B}.
\end{aligned} \tag{51}$$

There is a nice notation for this:

$$\bar{\mathcal{A}} = \mathcal{A}^\dagger \gamma^0. \tag{52}$$

Thus the Lorentz invariant inner product is

$$\bar{\mathcal{A}}\mathcal{B}. \tag{53}$$

Note that $\bar{\mathcal{A}}\gamma^\mu\mathcal{B}$ transforms as a four-vector:

$$\begin{aligned} \bar{\mathcal{A}}\gamma^\mu\mathcal{B} &\to \bar{\mathcal{A}}\exp(+\tfrac{i}{2}\,\theta^{\alpha\beta}M_{\alpha\beta})\gamma^\mu\exp(-\tfrac{i}{2}\,\theta^{\alpha\beta}M_{\alpha\beta})\mathcal{B} \\ &= \left[\exp(+\tfrac{i}{2}\,\theta^{\alpha\beta}\mathcal{M}_{\alpha\beta})\right]^\mu{}_\nu\;\bar{\mathcal{A}}\gamma^\nu\mathcal{B}. \end{aligned} \tag{54}$$

The same kind of analysis shows that $\bar{\mathcal{A}}\gamma^\mu\gamma^\nu\mathcal{B}$, $\bar{\mathcal{A}}\gamma^\mu\gamma^\nu\gamma^\rho\mathcal{B}$, *etc.* transform as tensors.

Here is another nice notation that we will use from now on:

$$a_\mu\gamma^\mu = \not{a}. \tag{55}$$

6 Normalization for solutions of the Dirac equation

From the preceeding section, we see that $\bar{\mathcal{U}}(p,s)\mathcal{U}(p,s)$ is given by

$$\bar{\mathcal{U}}(p,s')\mathcal{U}(p,s) = \bar{\mathcal{U}}(p_0,s')U(\Lambda_0)^{-1}U(\Lambda_0)\mathcal{U}(p_0,s) = \bar{\mathcal{U}}(p_0,s')\mathcal{U}(p_0,s), \tag{56}$$

where Λ_0 is the special Lorentz thansformation that transforms p_0 to p. With the spinors $\mathcal{U}(p_0,s)$ that we have chosen, we have $\bar{\mathcal{U}}(p_0,s')\mathcal{U}(p_0,s) = 2m\delta_{ss'}$. Thus our spinors have the normalization

$$\bar{\mathcal{U}}(p,s')\mathcal{U}(p,s) = 2m\,\delta_{ss'}. \tag{57}$$

for the antiparticle spinors we have

$$\begin{aligned} \bar{\mathcal{V}}(p,s')\mathcal{V}(p,s) &= \mathcal{U}(p,s')^{*\dagger}C^\dagger\gamma^0 C\mathcal{U}(p,s)^* \\ &= \mathcal{U}(p,s')^{*\dagger}C^{-1}\gamma^0 C\mathcal{U}(p,s)^* \\ &= -\mathcal{U}(p,s')^{*\dagger}(\gamma^0)^*\mathcal{U}(p,s)^* \\ &= -[\mathcal{U}(p,s')^\dagger\gamma^0\mathcal{U}(p,s)]^* \\ &= -[\bar{\mathcal{U}}(p,s')\mathcal{U}(p,s)]^* \\ &= -[2m\,\delta_{ss'}]^* \\ &= -2m\,\delta_{ss'}. \end{aligned} \tag{58}$$

This normalization may seem a little strange: the right hand side is zero if m vanishes, and for $\mathcal{V}$ we have a negative normalization. However, there is another way to write the same thing. Consider $\bar{\mathcal{U}}(p,s')\gamma^\mu\mathcal{U}(p,s)$. For a particle at rest we find by explicit calculation $\bar{\mathcal{U}}(p,s')\gamma^\mu\mathcal{U}(p,s) = 2p_0^\mu\delta ss'$ where $p_0^0 = m$ and $p_0^j = 0$. Then for an arbitrary p^μ,

$$\begin{aligned}
\bar{\mathcal{U}}(p,s')\gamma^\mu\mathcal{U}(p,s) &= \bar{\mathcal{U}}(p_0,s')U(\Lambda_0)^{-1}\gamma^\mu U(\Lambda_0)\mathcal{U}(p_0,s) \\
&= \Lambda^\mu{}_\nu\bar{\mathcal{U}}(p_0,s')\gamma^\nu\mathcal{U}(p_0,s) \\
&= \Lambda^\mu{}_\nu 2p_0^\nu\delta_{ss'} \\
&= 2p^\mu\delta_{ss'}.
\end{aligned} \tag{59}$$

We see that the $m\to 0$ limit of our spinors is sensible. For $\bar{\mathcal{V}}(p,s')\gamma^\mu\mathcal{V}(p,s)$ we have

$$\begin{aligned}
\bar{\mathcal{V}}(p,s')\gamma^\mu\mathcal{V}(p,s) &= \mathcal{U}(p,s')^{*\dagger}C^\dagger\gamma^0 C\mathcal{U}(p,s)^* \\
&= \mathcal{U}(p,s')^{*\dagger}C^{-1}\gamma^0 CC^{-1}\gamma^\mu C\mathcal{U}(p,s)^* \\
&= +\mathcal{U}(p,s')^{*\dagger}(\gamma^0)^*(\gamma^\mu)^*\mathcal{U}(p,s)^* \\
&= [\mathcal{U}(p,s')^\dagger\gamma^0\gamma^\mu\mathcal{U}(p,s)]^* \\
&= [\bar{\mathcal{U}}(p,s')\gamma^\mu\mathcal{U}(p,s)]^* \\
&= [2p^\mu\,\delta_{ss'}]^* \\
&= 2p^\mu\,\delta_{ss'}.
\end{aligned} \tag{60}$$

Here there is no minus sign.

7 Spinor sums

In practical calculations, one often needs the quantity

$$\sum_s \mathcal{U}(p,s)\bar{\mathcal{U}}(p,s). \tag{61}$$

This is easy to calculate since we know about Lorentz invariance. First, let's see what it is for $p = p_0 = (m,0,0,0)$. Explicit calculation gives

$$\sum_s \mathcal{U}(p_0,s)\bar{\mathcal{U}}(p_0,s) = m\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = m\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = \not{p}_0 + m. \tag{62}$$

Now we apply Lorentz invariance:

$$\begin{aligned}
\sum_s \mathcal{U}(p,s)\bar{\mathcal{U}}(p,s) &= \sum_s U(\Lambda_0(p))\mathcal{U}(p_0,s)\bar{\mathcal{U}}(p_0,s)U(\Lambda_0(p))^{-1} \\
&= U(\Lambda_0(p))\sum_s \mathcal{U}(p_0,s)\bar{\mathcal{U}}(p_0,s)U(\Lambda_0(p))^{-1} \\
&= U(\Lambda_0(p))(\not{p}_0 + m)U(\Lambda_0(p))^{-1} \\
&= (p_0)_\mu \; U(\Lambda_0(p))\gamma^\mu U(\Lambda_0(p))^{-1} + m \\
&= (p_0)_\mu \; [\Lambda_0(p)^{-1}]^\mu{}_\nu \gamma^\nu + m \\
&= p_\nu \gamma^\nu + m \\
&= \not{p} + m. \qquad (63)
\end{aligned}$$

For the spin sum with antiparticle solutions, we can use our construction recipe. Using the properties of C we find that

$$\sum_s \mathcal{V}(p,s)\bar{\mathcal{V}}(p,s) = \not{p} - m. \qquad (64)$$

Notice that whenever a calculation turns out to need only one of these spinor sums it does *not* need the spinors themselves.

Exercise. Use the properties of the matrix C to show that $\sum_s \mathcal{V}(p,s)\bar{\mathcal{V}}(p,s)$ $= \not{p} - m$.

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8 Another gamma matrix and parity transformations

We define

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3. \qquad (65)$$

If we write this as

$$\gamma_5 = -\frac{i}{4!}\,\epsilon_{\alpha\beta\gamma\delta}\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta. \qquad (66)$$

and recall that $\epsilon_{\alpha\beta\gamma\delta}$ is a tensor, we see that γ_5 is a scalar in the sense that

$$U(\Lambda)^{-1}\gamma_5 U(\Lambda) = \gamma_5 \qquad (67)$$

for a proper Lorentz transformation Λ. The commutation relation of γ_5 with the γ^μ is

$$\{\gamma_5, \gamma^\mu\} = 0. \tag{68}$$

For our representation of the gamma matrices,

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{69}$$

Thus it is the unit matrix on the $(0, \frac{1}{2})$ part of Dirac spinors, while it is minus a unit matrix on the $(\frac{1}{2}, 0)$ part.

What if Λ is a parity transformation, Λ_P? Call the correspoding matrix $U(\Lambda_P) = \mathcal{P}$. We need

$$\begin{aligned} \mathcal{P} M_{0i} \mathcal{P} &= -M_{0i} \\ \mathcal{P} M_{ij} \mathcal{P} &= +M_{ij} \end{aligned} \tag{70}$$

in order to have a representation of the Lorentz group including parity. Since $M_{\alpha\beta} = \frac{i}{4}[\gamma_\alpha, \gamma_\beta]$ we see that

$$\mathcal{P} = \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{71}$$

is what we need. This is what we anticipated in our general construction: $\mathcal{P}$ needs to intechange the $(0, \frac{1}{2})$ and the $(\frac{1}{2}, 0)$ representations of the Lorentz group.

Now we see that

$$\mathcal{P} \gamma_5 \mathcal{P} = -\gamma_5. \tag{72}$$

That is, γ_5 is a pseudoscalar (in the sense of having this transformation law).

9 Decomposition of Dirac matrices

Let Γ be a 4×4 matrix, which we think of as something we sould apply to Dirac spinors. Then Γ can be decomposed in the form

$$\Gamma = a + b_\mu \gamma^\mu + c_{\mu\nu} \sigma^{\mu\nu} + d_\mu \gamma_5 \gamma^\mu + e \gamma_5 \tag{73}$$

where the coefficients $a, \ldots, e$ are all numbers rather than matrices and where

$$\sigma^{\mu\nu} = \tfrac{i}{2}[\gamma^\mu, \gamma^\nu]. \tag{74}$$

The proof is that all of the matrices are linearly independent (one could just check explicitly) and that there are 16 of them. This representation decomposes the space of 4×4 matrices acting on Dirac spinors into irreducible representations of the Lorentz group.

10 Solutions in coordinate space

The free Dirac equation in coordinate space is

$$(i\partial\!\!\!/ - m)\Psi(x) = 0. \tag{75}$$

Using our previously found spinor solutions in momentum space, we can write the solutions as

$$\Psi(x) = \frac{1}{(2\pi)^3} \int \frac{d\vec{k}}{2k^0} \sum_s \left\{ e^{-ik_\mu x^\mu} \mathcal{U}(k,s)\, a(k,s) + e^{+ik_\mu x^\mu} \mathcal{V}(k,s)\, b^*(k,s) \right\}. \tag{76}$$

Here $\Psi(x)$ will solve the Dirac equation for any choice of the coefficients $a(k,s)$ and $b^*(k,s)$. For now, these coefficients are numbers that tell how the solultion $\Psi(x)$ is made of solutions for particular momenta and spins. Later, $\Psi(x)$ becomes an operator – the quantum field – and $a(k,s)$ and $b^*(k,s)$ become operators: $a(k,s)$ destroys an electron of momentum $\vec{k}$ and spin s while $b^*(k,s)$ creates a positron of momentum $\vec{k}$ and spin s

11 Interaction with the electromagnetic field

For an electron interacting with an electromagnetic field, Dirac proposed the following:

$$(i\partial\!\!\!/ - Qe A\!\!\!/(x) - m)\Psi(x) = 0. \tag{77}$$

Here e is the proton charge ($e = +|e|$), $Q = -1$ for an electron, and $A^\mu(x)$ is the electromagnetic potential, with

$$\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) = F_{\mu\nu}(x). \tag{78}$$

You could be worried about using the potential in the equation instead of the field strength tensor $F_{\mu\nu}(x)$. However, if you were worried because

$F_{\mu\nu}(x)$ is gauge invariant while A_μ is not, you should note that the Dirac equation is gauge invariant. Consider a gauge transformation

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x). \tag{79}$$

Let $\Psi(x)$ transform according to

$$\Psi(x) \to \Psi'(x) = e^{-iQe\Lambda(x)}\,\Psi(x). \tag{80}$$

Then we see by simple substitution that if Ψ obeys the original equation then Ψ' obeys

$$(i\not\partial - Qe\not{A}'(x) - m)\Psi'(x) = 0. \tag{81}$$

An important special case is when $A^\mu(x)$ is a Coloumb potential produced by a charge Ze: $\vec{A}(x) = 0$ and

$$A^0(x) \equiv \phi(x) = \frac{Ze}{4\pi}\frac{1}{|\vec{x}|}. \tag{82}$$

Then the Dirac equation reads

$$(i\partial_0\gamma^0 + i\vec{\nabla}\cdot\vec{\gamma} - Qe\Phi(x)\gamma^0 - m)\Psi(x) = 0. \tag{83}$$

or

$$i\partial_0\Psi(x) = (Qe\Phi(x) - i\vec{\nabla}\cdot\gamma^0\vec{\gamma} + m\gamma^0)\Psi(x). \tag{84}$$

If we suppose that Ψ has an $\exp(-i\omega t)$ time dependence, then we have an eigenvalue equation to determine ω:

$$\omega\Psi(x) = (Qe\Phi(x) - i\vec{\nabla}\cdot\gamma^0\vec{\gamma} + m\gamma^0)\Psi(x). \tag{85}$$

12 Nonrelativistic limit

Somehow, the Dirac equation should reduce to the Schrödinger equation in the limit of small velocities. To see that this is what happens, we need to realize two things. First, the energy ω is mostly the rest mass m. Thus we should write $\omega = m + \delta\omega$ and try to find an aproximate eignevalue equation that gives $\delta\omega$. Second, some parts of Ψ are bigger than others in this limit.

Let us define the big part of Ψ to be

$$\Psi_B = \tfrac{1}{2}(1 + \gamma^0)\Psi. \tag{86}$$

The small part is

$$\Psi_S = \tfrac{1}{2}(1-\gamma^0)\Psi. \tag{87}$$

Thus $\Psi = \Psi_B + \Psi_S$. Note that the matrices $\frac{1}{2}(1\pm\gamma^0)$ are orthogonal projection operators:

$$\begin{aligned} \tfrac{1}{2}(1\pm\gamma^0)\tfrac{1}{2}(1\pm\gamma^0) &= \tfrac{1}{2}(1\pm\gamma^0) \\ \tfrac{1}{2}(1\pm\gamma^0)\tfrac{1}{2}(1\mp\gamma^0) &= 0. \end{aligned} \tag{88}$$

We multiply the Dirac equation by $\frac{1}{2}(1+\gamma^0)$ and use the γ matrix algebra to get an equation for Ψ_B:

$$\omega\Psi_B(x) = (Qe\Phi(x)+m)\Psi_B(x) - i\vec{\nabla}\cdot\vec{\gamma}\Psi_S(x) \tag{89}$$

or

$$\delta\omega\,\Psi_B(x) = Qe\Phi(x)\Psi_B(x) - i\vec{\nabla}\cdot\vec{\gamma}\Psi_S(x). \tag{90}$$

We multiply the Dirac equation by $\frac{1}{2}(1-\gamma^0)$ and use the γ matrix algebra to get an equation for Ψ_S

$$\omega\Psi_S(x) = (Qe\Phi(x)-m)\Psi_S(x) + i\vec{\nabla}\cdot\vec{\gamma}\,\Psi_B(x) \tag{91}$$

or

$$(2m+\delta\omega)\Psi_S(x) = Qe\Phi(x)\Psi_S(x) + i\vec{\nabla}\cdot\vec{\gamma}\,\Psi_B(x). \tag{92}$$

So far, this is exact. But in the Ψ_S equation we can neglect $\delta\omega$ and $Qe\Phi(x)$ compared to $2m$:

$$\Psi_S(x) \approx \frac{i}{2m}\vec{\nabla}\cdot\vec{\gamma}\,\Psi_B(x). \tag{93}$$

If we insert this into the equation for Ψ_B we get

$$\delta\omega\,\Psi_B(x) = Qe\Phi(x)\Psi_B(x) - i\vec{\nabla}\cdot\vec{\gamma}\frac{i}{2m}\vec{\nabla}\cdot\vec{\gamma}\,\Psi_B(x). \tag{94}$$

or

$$\delta\omega\,\Psi_B(x) = \left[-\frac{1}{2m}\vec{\nabla}^2 + Qe\Phi(x)\right]\Psi_B(x). \tag{95}$$

This is the nonrelativistic Schrödinger equation. Evidently, we could systematically improve the approximation so as to obtain an expansion in powers of $\delta\omega/m$, but we will not pursue this topic here.