Fermions and the Dirac Field[1]
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1 Non-relativistic fermions

Let's have another look at our non-relativistic field theory in which the particles interact with a potential $\mathcal{V}(\vec{x}-\vec{y})$. We will dress it up a little by giving the fields an index s that represents spin, but this is a rather trivial change. The spin index s can take values $-J, -J+1, \dots, J$ where J could be zero, 1/2, 1 *etc.* The action is

$$\begin{aligned} S[\psi^\dagger, \psi] &= \int dt\, d\vec{x}\, \Big\{ \psi_s^\dagger(t,\vec{x})[\tfrac{i}{2}\partial_t \psi_s(t,\vec{x})] - [\tfrac{i}{2}\partial_t \psi_s^\dagger(t,\vec{x})]\psi_s(t,\vec{x}) \\ &\qquad - \frac{1}{2m}\vec{\nabla}\psi_s^\dagger(t,\vec{x})\cdot\vec{\nabla}\psi_s(t,\vec{x}) \Big\} \\ &\quad - \int dt\, d\vec{x}\, d\vec{y}\; \tfrac{1}{2}\psi_s^\dagger(t,\vec{x})\psi_{s'}^\dagger(t,\vec{y})\,\psi_{s'}(t,\vec{y})\psi_s(t,\vec{x})\;\mathcal{V}(\vec{x}-\vec{y}). \end{aligned} \tag{1}$$

This action leads to the Schoödinger equation as the equation of motion,

$$i\frac{\partial}{\partial t}\,\psi_s(t,\vec{x}) = \left[-\frac{1}{2m}\nabla^2 + U(t,\vec{x})\right]\psi_s(t,\vec{x}), \tag{2}$$

where the potential energy at $\vec{x}$ depends on the fields at other points $\vec{y}$:

$$U(t,\vec{x}) = \int d\vec{y}\;\mathcal{V}(\vec{x}-\vec{y})\psi_{s'}^\dagger(t,\vec{y})\,\psi_{s'}(t,\vec{y})\;. \tag{3}$$

The hamiltonian derived from this action is

$$\begin{aligned} H &= \int d\vec{x}\;\frac{1}{2m}\vec{\nabla}\psi_s^\dagger(t,\vec{x})\cdot\vec{\nabla}\psi_s(t,\vec{x}) \\ &\quad + \int d\vec{x}\, d\vec{y}\;\tfrac{1}{2}\psi_s^\dagger(t,\vec{x})\psi_{s'}^\dagger(t,\vec{y})\,\psi_{s'}(t,\vec{y})\psi_s(t,\vec{x})\;\mathcal{V}(\vec{x}-\vec{y}). \end{aligned} \tag{4}$$

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(The order of operators in the last term is significant for the quantum theory.) There are two things to note. First, the terms containing $\partial_t \psi$ and $\partial_t \psi^\dagger$ drop out. Second, the canonical momenta π and $\pi^\dagger$ are just the old fields back again:

$$\pi = \frac{i}{2}\psi^\dagger \qquad \pi^\dagger = -\frac{i}{2}\psi \tag{5}$$

Thus we can't really use Poisson brackets to make commutators. This is typical for theories with an equation of motion that is first order in time derivatives instead of second order. However, we can just try the commutation relations

$$\begin{aligned} [\psi_s(t,\vec{x}), \psi_{s'}(t,\vec{y})] &= 0 \\ [\psi_s(t,\vec{x}), \psi^\dagger_{s'}(t,\vec{y})] &= \delta(\vec{x}-\vec{y})\,\delta_{ss'}. \end{aligned} \tag{6}$$

Then commuting the hamiltonian with the fields gives the right equations of motion.

For a free field (*i.e.* when the potential vanishes) we can solve for the time dependence:

$$\psi_s(t,\vec{x}) = (2\pi)^{-3}\int d\vec{k}\; e^{-iEt+i\vec{k}\cdot\vec{x}} b(s,\vec{k}), \tag{7}$$

Then the operators b have the commutation relations

$$[b(s,\vec{k}), b^\dagger(s',\vec{p})] = (2\pi)^3\,\delta(\vec{k}-\vec{p})\,\delta_{ss'}. \tag{8}$$

The evident interpretation is that $\psi_s(\vec{x},t)$ destroys a particle with spin s at position $\vec{x}$ (at time t), while $b(s,\vec{k})$ destroys a particle with momentum $\vec{k}$ and spin s.

Now, what happens if we change the commutation relations to anticommutation relations? Using the notation $\{A,B\} = AB + BA$, we try

$$\begin{aligned} \{\psi_s(t,\vec{x}), \psi_{s'}(t,\vec{y})\} &= 0 \\ \{\psi_s(t,\vec{x}), \psi^\dagger_{s'}(t,\vec{y})\} &= \delta(\vec{x}-\vec{y})\,\delta_{ss'}. \end{aligned} \tag{9}$$

The corresponding anticommutation relations in momentum space are

$$\begin{aligned} \{b(s,\vec{k}), b(s',\vec{p})\} &= 0 \\ \{b(s,\vec{k}), b^\dagger(s',\vec{p})\} &= (2\pi)^3\,\delta(\vec{k}-\vec{p})\,\delta_{ss'}. \end{aligned} \tag{10}$$

To check on time dependence, we use the standard relation

$$i \frac{\partial}{\partial t} \psi_s(t, \vec{x}) = -[H, \psi_s(t, \vec{x})]. \tag{11}$$

To compute a commutator when we are given anticommutators, we need to use

$$[A, BC] = ABC + BAC - BAC - BCA = \{A, B\}C - B\{A, C\}. \tag{12}$$

and we need to be careful with the signs. Then we recover the equations of motion (2).

Exercise: Show that we get the right equation of motion for ψ.

Consider a two particle state,

$$|p_1, s_1; p_2, s_2\rangle = b^\dagger(p_2, s_2) b^\dagger(p_1, s_1)|0\rangle. \tag{13}$$

Since the b operators anticommute, we have

$$|p_1, s_1; p_2, s_2\rangle = -|p_2, s_2; p_1, s_1\rangle. \tag{14}$$

Similarly, multiparticle states

$$|p_1, s_1; p_2, s_2; \cdots; p_N, s_N\rangle \tag{15}$$

are completely antisymmetric under $\{p_i, s_i\} \leftrightarrow \{p_j, s_j\}$. That is, the particles are fermions.

Let's make an operator that creates a discrete state with wave function ϕ,

$$b_\phi^\dagger = (2\pi)^{-3} \int d\vec{k} \; \sum_s \phi(\vec{k}, s)\, b^\dagger(\vec{k}, s) \tag{16}$$

Then

$$b_\phi^\dagger b_\phi^\dagger = 0 \tag{17}$$

so we can't put two fermions into the same state.

The method for making perturbation theory for Green functions (or for the scattering matrix) is almost the same for fermions as for bosons. In the definition of a Green function, we need some signs in the definition of time ordered products. We set

$$\begin{aligned} T\psi(x)\psi^\dagger(y) &= \psi(x)\psi^\dagger(y) \qquad x^0 > y^0 \\ &= -\psi^\dagger(y)\psi(x) \qquad y^0 > x^0. \end{aligned} \tag{18}$$

The general rule is that the T product is the product of field operators with latest times to the left and a sign obtained by treating fermion fields as if they anticommuted with all other fermion fields. (If we had boson fields too, the sign would be comoputed by letting fermion fields anticommute with other fermion fields and commute with boson fields and letting the boson fields commute with each other.)

When we use Wick's theorem, we get produce factors of the propagator for our field. We have the usual non-relativistic propagator with a $\delta_{ss'}$ added,

$$\begin{aligned} D_F(x-y) &= \langle 0|T\psi_s(x)\psi^\dagger_{s'}(y)|0\rangle \\ &= (2\pi)^{-4}\int dE\, dk\ e^{-iE(x^0-y^0)+i\vec{k}\cdot(\vec{x}-\vec{y})}\frac{i\delta_{ss'}}{E-\vec{k}^2/(2m)+i\epsilon}. \end{aligned} \tag{19}$$

This gives Feynman rules

- Label the lines by their momenta and energy, using momentum and energy conservation at each vertex.

- For each loop, there will be one energy and momentum that is not constrained by energy and momentum conservation. Supply an integration
$$(2\pi)^{-4}\int dE\, d\vec{p}. \tag{20}$$

- To each line associate a propagator
$$\frac{i\delta_{ss'}}{E-\vec{k}^2/(2m)+i\epsilon} \tag{21}$$

- For each vertex representing the action of the potential, associate a factor
$$-i\tilde{\mathcal{V}}(\vec{q})\ \delta_{s_1 s_1'}\,\delta_{s_2 s_2'} \tag{22}$$
where $\tilde{\mathcal{V}}(\vec{q})$ is the Fourier transform of $\mathcal{V}(\vec{x})$, $\vec{q}$ is the momentum transferred by the potential, and the spins for the particle scattered at one end of the potential are $\{s_1, s_1'\}$ while the spins for the particle scattered at one end of the potential are $\{s_2, s_2'\}$.

- There is a minus sign between graphs that are identical except for exchange of two fermion lines.

The last point is the only place where we see the difference between fermions and bosons.

2 Dirac fermions

Now we turn to the theory of relativistic spin 1/2 particles. We use a Dirac field $\psi_\alpha(x)$. Here the index α takes the values 1,2,3,4 and is usually suppressed in the notation. Recall that a Dirac field transforms under the Lorentz group according to the $(1/2,0)\oplus(0,1/2)$ representation. We need the gamma matrices, which satisfy

$$\{\gamma^\mu,\gamma^\nu\} = 2\, g^{\mu\nu}. \tag{23}$$

Also, we will choose gamma matrices with

$$(\gamma^0)^\dagger = \gamma^0, \qquad\qquad (\gamma^j)^\dagger = -\gamma^j \;\; \text{for } j = 1,2,3. \tag{24}$$

so that

$$\gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu. \tag{25}$$

We use the notation

$$A_\mu\gamma^\mu = \not{A}. \tag{26}$$

The field ψ is complex. Instead of the adjoint, it is useful to use

$$\bar\psi = \psi^\dagger\gamma^0. \tag{27}$$

The equations given above are just the barest summary of the information you need on Dirac spinors and gamma matrices. You should consult the notes from last quarter for more.

For a free Dirac field, we use the action

$$S[\psi,\bar\psi] = \int d^4x\,\left\{\bar\psi(x)\gamma^\mu\tfrac{i}{2}(\partial_\mu\psi(x)) - \tfrac{i}{2}(\partial_\mu\bar\psi(x))\gamma^\mu\psi(x) - m\,\bar\psi(x)\psi(x)\right\}. \tag{28}$$

Then the equations of motion are

$$0 = \frac{\delta S[\psi,\bar\psi]}{\delta\bar\psi(x)} = (i\not{\partial} - m)\psi(x) \tag{29}$$

and

$$0 = \frac{\delta S[\psi,\bar\psi]}{\delta\psi(x)} = -i(\partial_\mu\bar\psi(x))\gamma^\mu - \bar\psi(x)\,m \tag{30}$$

That is, we get the free Dirac equation and its "bar" version, which is a little messy to write unless we invent some special notation to indicate a derivative operator that operates to the left.

The hamiltonian is

$$H = \int d\vec{x} \left\{ -\bar{\psi}(x)\gamma^j \tfrac{i}{2}(\partial_j \psi(x)) + \tfrac{i}{2}(\partial_j \bar{\psi}(x))\gamma^j \psi(x) + m\,\bar{\psi}(x)\psi(x) \right\}. \quad (31)$$

where the implied sum is over $j = 1, 2, 3$. Note that the $\partial/(\partial t)$ terms cancel. We need some anticommutation relations. Let's try

$$\begin{aligned} \{\psi(\vec{x},t), \psi(\vec{y},t)\} &= 0 \\ \{\psi(\vec{x},t), \bar{\psi}(\vec{y},t)\} &= \gamma^0\,\delta(\vec{x}-\vec{y}). \end{aligned} \quad (32)$$

In fact, this works.

Exercise: Show that the anticommutation relations (32) and the hamiltonian (31) give us the Dirac equation as the equation of motion for $\psi(x)$.

What if we have more than one fermion field? For instance, we might have several fermion fields ψ_I and several boson fields ϕ_J. Then we would let each ψ_I anticommute with all the other ψ_I and $\psi_J^\dagger$ fields except for having a delta function for $\{\psi_I, \psi_J^\dagger\}$ with $I = J$. The ψ_I and $\psi_J^\dagger$ fields would commute with all the ϕ_J fields and with the conjugate π_J fields. Finally, the ϕ_J and π_J fields commute with each other except for having a delta function for $\{\phi_I, \pi_J\}$ with $I = J$. Similar considerations apply to the definition of time ordered products of fields. The definition is that the field operators with later time arguments go on the left, with a sign determined by moving fields past each other as if all fermion fields anticommuted with each other and commuted with boson fields, while all boson fields commuted with each other. For example, if $x_1^0 > x_2^0 > x_3^0$ then

$$T\bar{\psi}(x_3)\psi(x_2)\psi(x_1) = -\psi(x_1)\psi(x_2)\bar{\psi}(x_3). \quad (33)$$

We can solve the free Dirac equation for the field to get

$$\Psi(x) = \frac{1}{(2\pi)^3}\int \frac{d\vec{k}}{2k^0}\sum_s \left\{ e^{-ik_\mu x^\mu}\mathcal{U}(k,s)\,b(k,s) + e^{+ik_\mu x^\mu}\mathcal{V}(k,s)\,d^\dagger(k,s) \right\}, \quad (34)$$

where $\mathcal{U}(k,+\frac{1}{2})$ and $\mathcal{U}(k,-\frac{1}{2})$ are the two solutions of the momentum space Dirac equation,

$$(p_\mu \gamma^\mu - m)\,\mathcal{U}(p,s) = 0. \quad (35)$$

The spinors $\mathcal{V}(p,s)$ are solutions of

$$(p_\mu \gamma^\mu + m)\mathcal{V}(p,s) = 0. \tag{36}$$

We have chosen the normalization

$$\bar{\mathcal{U}}(p,s)\gamma^\mu \mathcal{U}(p,s) = 2p^\mu, \qquad \bar{\mathcal{V}}(p,s)\gamma^\mu \mathcal{V}(p,s) = 2p^\mu. \tag{37}$$

The anticommutation relations for the field $\psi(x)$ translates into

$$\begin{aligned}
\{b(k,s), b(p,s')\} &= 0 \\
\{d(k,s), d(p,s')\} &= 0 \\
\{b(k,s), d(p,s')\} &= 0 \\
\{b^\dagger(k,s), b^\dagger(p,s')\} &= 0 \\
\{d^\dagger(k,s), d^\dagger(p,s')\} &= 0 \\
\{b^\dagger(k,s), d^\dagger(p,s')\} &= 0 \\
\{b(k,s), d^\dagger(p,s')\} &= 0 \\
\{d(k,s), b^\dagger(p,s')\} &= 0 \\
\{b(k,s), b^\dagger(p,s')\} &= (2\pi)^3\, 2\omega(\vec{p})\, \delta(\vec{k}-\vec{p})\, \delta_{ss'} \\
\{d(k,s), d^\dagger(p,s')\} &= (2\pi)^3\, 2\omega(\vec{p})\, \delta(\vec{k}-\vec{p})\, \delta_{ss'}.
\end{aligned} \tag{38}$$

Commuting the hamiltonian with the b and d operators, we find

$$\begin{aligned}
[H, b(k,s)] &= -\omega(\vec{k})\, b(k,s) \\
[H, d(k,s)] &= -\omega(\vec{k})\, d(k,s)
\end{aligned} \tag{39}$$

Thus we interpret b and d as annihilation operators. We call the particle that b annihilates the particle and the particle that d annihilates the antiparticle. The operators $b^\dagger$ and $d^\dagger$ are then particle and antiparticle creation operators, respectively.

We postulate the existence of a vacuum state $|0\rangle$ with zero energy and momentum and with

$$\begin{aligned}
b(k,s)|0\rangle &= 0 \\
d(k,s)|0\rangle &= 0.
\end{aligned} \tag{40}$$

Then we can make states with particles by using the creation operators,

$$\begin{aligned}
b^\dagger(k,s)|0\rangle &= |e^-, k, s\rangle \\
d^\dagger(k,s)|0\rangle &= |e^+, k, s\rangle.
\end{aligned} \tag{41}$$

Here I have supposed that $psi(x)$ is the electron field and have inserted appropriate names for the particles in the labels for the states.

We can make a two electron state with two electron creation operators,

$$b^\dagger(k_1,s_1)b^\dagger(k_2,s_2)|0\rangle = |e^-,k_1,s_1;e^-,k_2,s_2\rangle. \tag{42}$$

Similarly

$$d^\dagger(k_1,s_1)d^\dagger(k_2,s_2)|0\rangle = |e^+,k_1,s_1;e^+,k_2,s_2\rangle. \tag{43}$$

Because of the anticommutation relations, we have

$$|e^-,k_1,s_1;e^-,k_2,s_2\rangle = -|e^-,k_2,s_2;e^-,k_1,s_1\rangle. \tag{44}$$

That is, electrons are fermions. (So are positrons.) By the way, we also get

$$|e^-,k_1,s_1;e^+,k_2,s_2\rangle = -|e^+,k_2,s_2;e^-,k_1,s_1\rangle. \tag{45}$$

if we define the first of these states as $b^\dagger(k_1,s_1)d^\dagger(k_2,s_2)|0\rangle$ and the second as $d^\dagger(k_2,s_2)b^\dagger(k_1,s_1)|0\rangle$. However, since the particles are not identical, we can just agree to a convention to, say, use the first of these states for an e^+e^- pair. That is, the sign in Eq. (44) is important, the sign in Eq. (45) is not.

3 Scattering with Dirac particles

Let's try an interacting theory that involves a Dirac field. The easiest thing to try is the Dirac field coupled to a massless scalar field. (This would be quantum electrodynamics if we made replaced the massless scalar field by a massless vector field and chose a suitable interaction term.) We try

$$\begin{aligned} S[\psi,\bar\psi,\phi] &= \int d^4x \Big\{ \bar\psi(x)\gamma^\mu \tfrac{i}{2}(\partial_\mu\psi(x)) - \tfrac{i}{2}(\partial_\mu\bar\psi(x))\gamma^\mu\psi(x) - m\,\bar\psi(x)\psi(x) \\ &\quad + \tfrac12 \partial_\mu\phi(x)\partial^\mu\phi(x) - g\phi(x)\,\bar\psi(x)\psi(x)\Big\}. \end{aligned} \tag{46}$$

We can construct Green functions for this theory using our by now familiar techniques. The main new feature is the free Dirac propagator

$$S_F(x-y) = \langle 0|T\psi(x)\bar\psi(y)|0\rangle. \tag{47}$$

Note the hidden Dirac indices:

$$S_F(x-y)_{\alpha\beta} = \langle 0|T\psi_\alpha(x)\bar\psi_\beta(y)|0\rangle. \tag{48}$$

Also, recall the meaning of the T-product for fermion fields. The propagator is $\langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle$ for $x^0 > y^0$ and $-\langle 0|\bar{\psi}_\beta(y)\psi_\alpha(x)|0\rangle$ for $y^0 > x^0$.

To evaluate this, insert the Fourier expansion for a free field in the case, say, $x^0 > y^0$. We use the relation

$$\sum_s \mathcal{U}(k,s)\bar{\mathcal{U}}(k,s) = \not{k} + m. \tag{49}$$

We get

$$\begin{aligned}
S_F(x-y)_{\alpha\beta} &= \langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle \\
&= (2\pi)^{-6}\int \frac{d\vec{k}}{2\omega(\vec{k})}\,\frac{d\vec{p}}{2\omega(\vec{p})}\sum_{ss'} e^{-ik\cdot x+ip\cdot y} \\
&\quad \times\langle 0|\mathcal{U}_\alpha(k,s)b(k,s)\,d^\dagger(p,s')\bar{\mathcal{U}}_\beta(p,s)|0\rangle \\
&= (2\pi)^{-3}\int \frac{d\vec{k}}{2\omega(\vec{k})}\,e^{-ik\cdot(x-y)}\sum_s \mathcal{U}_\alpha(k,s)\bar{\mathcal{U}}_\beta(k,s) \\
&= (2\pi)^{-3}\int \frac{d\vec{k}}{2\omega(\vec{k})}\,e^{-ik\cdot(x-y)}(\not{k}+m)_{\alpha\beta} \\
&= (2\pi)^{-4}\int d^4\vec{k}\,e^{-ik\cdot(x-y)}\frac{i\,(\not{k}+m)_{\alpha\beta}}{k^2-m^2+i\epsilon}
\end{aligned} \tag{50}$$

Let's also try this for the other ordering of the times, using

$$\sum_s \mathcal{V}(k,s)\bar{\mathcal{V}}(k,s) = \not{k} - m. \tag{51}$$

We get

$$\begin{aligned}
S_F(x-y)_{\alpha\beta} &= -\langle 0|\bar{\psi}_\beta(y)\psi_\alpha(x)|0\rangle \\
&= -(2\pi)^{-6}\int \frac{d\vec{k}}{2\omega(\vec{k})}\,\frac{d\vec{p}}{2\omega(\vec{p})}\sum_{ss'} e^{+ik\cdot x-ip\cdot y} \\
&\quad \times\langle 0|d(p,s')\bar{\mathcal{V}}_\beta(p,s)\,\mathcal{V}_\alpha(k,s)d^\dagger(k,s)|0\rangle \\
&= -(2\pi)^{-3}\int \frac{d\vec{k}}{2\omega(\vec{k})}\,e^{+ik\cdot(x-y)}\sum_s \mathcal{V}_\alpha(k,s)\bar{\mathcal{V}}_\beta(k,s) \\
&= -(2\pi)^{-3}\int \frac{d\vec{k}}{2\omega(\vec{k})}\,e^{+ik\cdot(x-y)}(\not{k}-m)_{\alpha\beta} \\
&= -(2\pi)^{-4}\int d^4\vec{k}\,e^{+ik\cdot(x-y)}\frac{i\,(\not{k}-m)_{\alpha\beta}}{k^2-m^2+i\epsilon}
\end{aligned}$$

$$= \quad (2\pi)^{-4} \int d^4\vec{k}\, e^{-ik\cdot(x-y)} \frac{i\,(\not{k}+m)_{\alpha\beta}}{k^2-m^2+i\epsilon}. \tag{52}$$

We thus get the Feynman rules for Green functions in momentum space.

- For each ψ in the Green function, there is an electron propagator with its arrow pointing toward the point that represents ψ.
- For each $\bar{\psi}$ in the Green function, there is an electron propagator with its arrow pointing away from the point that represents $\bar{\psi}$.
- For each ϕ in the Green function, there is a boson propagator ending at the point that represents ϕ.
- Label the lines by their momenta and energy, using momentum and energy conservation at each vertex.
- For each loop, there will be one energy and momentum that is not constrained by energy and momentum conservation. Supply an integration

$$(2\pi)^{-4} \int d^4p. \tag{53}$$

- To each electron line associate a propagator

$$\frac{i(\not{k}+m)}{k^2-m^2+i\epsilon} \tag{54}$$

  where k^μ is the momentum in the direction of the electron-number arrow.

- To each boson line associate a propagator

$$\frac{i}{k^2+i\epsilon} \tag{55}$$

- For each vertex representing a $\phi\bar{\psi}\psi$ interaction associate a factor

$$-ig1 \tag{56}$$

  where 1 here represents a unit matrix in the spinor space associated with the electron lines.

- There is a minus sign between graphs that are identical except for exchange of two fermion lines.

- There is a minus sign for each fermion loop.

This last rule that associates a minus sign for each fermion loop comes from counting the minus signs in Wick's theorem with fermions.

How do we calculate a scattering matrix element in this theory? We need to adapt the LSZ formula to the present circumstance. Recall that we defined

$$\langle\Omega|\phi(0)|k\rangle = \sqrt{Z} \tag{57}$$

in the case of a boson field. For a fermion field, it's a little more complicated. Consider first a free field. To compute

$$\langle 0|\psi_\alpha(0)|e^-, k, s\rangle \tag{58}$$

in free field theory, we insert the momentum space expansion of $\psi_\alpha(0)$. This gives

$$\langle 0|\psi_\alpha(0)|e^-, k, s\rangle = \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{2\omega(\vec{k})}\sum_{s'}\mathcal{U}_\alpha(p,s')\langle 0|b(p,s')|e^-, k, s\rangle \tag{59}$$

Evaluating the matrix element gives

$$\langle 0|\psi(0)|e^-, k, s\rangle = \mathcal{U}_\alpha(k,s). \tag{60}$$

In the interacting theory we must get the same thing except for a multiplicative constant because of Lorentz invariance. That is

$$\langle\Omega|\psi_\alpha(0)|e^-, k, s\rangle = \sqrt{Z_\psi}\,\mathcal{U}_\alpha(k,s). \tag{61}$$

Similarly,

$$\begin{aligned}
\langle e^-, k, s|\bar{\psi}_\alpha(0)|\Omega\rangle &= \sqrt{Z_\psi}\,\bar{\mathcal{U}}_\alpha(k,s) \\
\langle e^+, k, s|\psi_\alpha(0)|\Omega\rangle &= \sqrt{Z_\psi}\,\mathcal{V}_\alpha(k,s) \\
\langle\Omega|\bar{\psi}_\alpha(0)|e^+, k, s\rangle &= \sqrt{Z_\psi}\,\bar{\mathcal{V}}_\alpha(k,s).
\end{aligned} \tag{62}$$

Let's use these formulas in our derivation for the LSZ formula. We start with a Green function for an electron and a positron coming into the scattering and an electron, a positron, and a scalar particle coming out. The Green function is

$$
\begin{aligned}
(2\pi)^4\,\delta^{(4)}\Big(\sum k_j - \sum q_j\Big)\ \tilde{G}(q_1,q_2,q_3;k_1,k_2) \\
&= \int dy_1\,dy_2\,dy_3\,dx_1\,dx_2\ \exp(-i\sum k_j\cdot x_j + i\sum q_j\cdot y_j) \\
&\times\langle\Omega|T\psi(y_1)\bar{\psi}(y_2)\phi(y_3)\bar{\psi}(x_1)\psi(x_2)|\Omega\rangle
\end{aligned} \tag{63}
$$

Following our previous derivation, the part of the Green function with poles is given by

$$
\begin{aligned}
(2\pi)^4\,\delta^{(4)}\Big(\sum k_j - \sum q_j\Big)\ \tilde{G}(q_1,q_2,q_3;k_1,k_2) \\
\sim\ & \int dy_1\,dy_2\,dy_3\,\prod_j \theta(y_j^0 > T)\exp(i\sum q_j\cdot y_j) \\
&\times\prod_{i=1}^{3}(2\pi)^{-3}\int\frac{d\vec{p}_i}{2\omega(\vec{p}_i)}\sum_{s_i'} \\
&\times\langle\Omega|T\{\psi(y_1)\bar{\psi}(y_2)\phi(y_3)\}|e^-,p_1,s_1';e^+,p_2,s_2';p_3\rangle_{\rm out} \\
&\times\prod_{j=1}^{2}(2\pi)^{-3}\int\frac{d\vec{l}_j}{2\omega(\vec{l}_j)}\sum_{s_j} \\
&\times\int dx_1\,dx_2\,\prod_j\theta(x_j^0 < -T)\exp(-i\sum k_j\cdot x_j) \\
&\times\ {}_{\rm in}\langle e^-,l_1,s_1;e^+,l_2,s_2|T\{\bar{\psi}(x_1)\psi(x_2)\}|\Omega\rangle \\
&\times\ {}_{\rm out}\langle e^-,p_1,s_1';e^+,p_2,s_2';p_3|e^-,l_1,s_1;e^+,l_2,s_2\rangle_{\rm in}
\end{aligned} \tag{64}
$$

Here I have omitted states that don't contribute from the sums over intermediate in and out states. Since we want just the contributions that give poles, we can replace the matrix element

$$
\langle\Omega|T\{\psi(y_1)\bar{\psi}(y_2)\phi(y_3)\}|e^-,p_1,s_1';e^+,p_2,s_2';p_3\rangle_{\rm out} \tag{65}
$$

by

$$
\langle\Omega|\psi(y_1)|e^-,p_1,s_1'\rangle_{\rm out}\langle\Omega|\bar{\psi}(y_2)|e^+,p_2,s_2'\rangle_{\rm out}\langle\Omega|\phi(y_3)|p_3\rangle_{\rm out}. \tag{66}
$$

In turn, this is

$$
\exp(-i\sum_j p_j\cdot y_j)\langle\Omega|\psi(0)|e^-,p_1,s_1'\rangle_{\rm out}\langle\Omega|\bar{\psi}(0)|e^+,p_2,s_2'\rangle_{\rm out}\langle\Omega|\phi(0)|p_3\rangle_{\rm out}. \tag{67}
$$

Finally, using our $\sqrt{Z}$ formulas, this becomes

$$\exp(-i\sum_j p_j \cdot y_j)\sqrt{Z_\psi}\mathcal{U}(p_1,s_1')\sqrt{Z_\psi}\bar{\mathcal{V}}(p_2,s_2')\sqrt{Z_\phi} \tag{68}$$

Similarly, we can replace the matrix element

$$_{\rm in}\langle e^-,l_1,s_1;e^+,l_2,s_2|T\{\bar\psi(x_1)\psi(x_2)\}|\Omega\rangle \tag{69}$$

by

$$_{\rm in}\langle e^-,l_1,s_1|\bar\psi(x_1)|\Omega\rangle\ _{\rm in}\langle e^+,l_2,s_2|\psi(x_2)|\Omega\rangle \tag{70}$$

In turn this is

$$\exp(i\sum_i l_i \cdot x_j)_{\rm in}\langle e^-,l_1,s_1|\bar\psi(0)|\Omega\rangle\ _{\rm in}\langle e^+,l_2,s_2|\psi(0)|\Omega\rangle \tag{71}$$

Finally, this is

$$\exp(i\sum_i l_i \cdot x_j)\sqrt{Z_\psi}\bar{\mathcal{U}}(l_1,s_1)\sqrt{Z_\psi}\mathcal{V}(l_2,s_2) \tag{72}$$

Inserting these results and performing the itegrals gives

$$\begin{aligned}
&(2\pi)^4\,\delta^{(4)}\Big(\sum k_j - \sum q_j\Big)\ \tilde G(q_1,q_2,q_3;k_1,k_2)\\
&\qquad\sim \sum_{s_1',s_2',s_1,s_2}\frac{i\sqrt{Z_\psi}\mathcal{U}(q_1,s_1')}{q_1^2-M^2+i\epsilon}\ \frac{i\sqrt{Z_\psi}\bar{\mathcal{V}}(q_2,s_2')}{q_2^2-M^2+i\epsilon}\ \frac{i\sqrt{Z_\phi}}{q_3^2-M_\phi^2+i\epsilon}\\
&\qquad\quad\times\ _{\rm out}\langle e^-,q_1,s_1';e^+,q_2,s_2';q_3|e^-,k_1,s_1;e^+,k_2,s_2\rangle_{\rm in}\\
&\qquad\quad\times\frac{i\sqrt{Z_\psi}\bar{\mathcal{U}}(k_1,s_1)}{k_1^2-M^2+i\epsilon}\ \frac{i\sqrt{Z_\psi}\mathcal{V}(k_2,s_2)}{k_2^2-M^2+i\epsilon}
\end{aligned} \tag{73}$$

Here we should recall that the Green function carries Dirac indices, which belong to the Dirac spinors on the right hand side of this equation. I have written M_ϕ for the physical mass of the boson, which can be non-zero even though the mass that we put into the lagrangian was zero.

This result is not yet in the form we need. Let's write $\tilde G$ in terms of the amputated Green function Γ. With an analysis similar to that used for the scalar field, we find that the full two point function at the one particle pole has the form

$$\tilde G(q) \sim \frac{iZ_\psi(\not q + M)}{q^2-M^2+i\epsilon} \tag{74}$$

Using this form, we factor a full propagator from each external leg of our Green function, leaving the amputated Green function,

$$\begin{aligned}\tilde{G}(q_1, q_2, q_3; k_1, k_2) &= \frac{iZ_\psi(\not{q}_1 + M)}{q_1^2 - M^2 + i\epsilon} \; \frac{iZ_\psi(-\not{q}_2 + M)}{q_2^2 - M^2 + i\epsilon} \; \frac{iZ_\phi}{q_3^2 - M_\phi^2 + i\epsilon} \\ &\quad \times \Gamma(q_1, q_2, q_3; k_1, k_2) \\ &\quad \times \frac{iZ_\psi(\not{k}_1 + M)}{k_1^2 - M^2 + i\epsilon} \; \frac{iZ_\psi(-\not{k}_2 + M)}{k_2^2 - M^2 + i\epsilon} \end{aligned} \tag{75}$$

Using what we know about the solutions $\mathcal{U}$ and $\mathcal{V}$ of the Dirac equation, this is

$$\begin{aligned}\tilde{G}(q_1, q_2, q_3; k_1, k_2) &= \sum_{s_1', s_2', s_1, s_2} \frac{iZ_\psi \mathcal{U}(q_1, s_1')\bar{\mathcal{U}}(q_1, s_1')}{q_1^2 - M^2 + i\epsilon} \; \frac{-iZ_\psi \mathcal{V}(q_2, s_2')\bar{\mathcal{V}}(q_2, s_2')}{q_2^2 - M^2 + i\epsilon} \\ &\quad \times \frac{iZ_\phi}{q_3^2 - M^2 + i\epsilon} \Gamma(q_1, q_2, q_3; k_1, k_2) \\ &\quad \times \frac{iZ_\psi \mathcal{U}(k_1, s_1)\bar{\mathcal{U}}(k_1, s_1)}{k_1^2 - M^2 + i\epsilon} \; \frac{-iZ_\psi \mathcal{V}(k_2, s_2)\bar{\mathcal{V}}(k_2, s_2)}{k_2^2 - M^2 + i\epsilon} \end{aligned} \tag{76}$$

Now we just match coefficients to get the result we want:

$$\begin{aligned} & {}_{\rm out}\langle e^-, q_1, s_1'; e^+, q_2, s_2'; q_3 | e^-, k_1, s_1; e^+, k_2, s_2 \rangle_{\rm in} \\ &\qquad = (2\pi)^4 \, \delta^{(4)}\Big(\sum k_j - \sum q_j\Big) \\ &\qquad\quad \times \sqrt{Z_\psi}\bar{\mathcal{U}}(q_1, s_1') \; \sqrt{Z_\psi}\mathcal{V}(q_2, s_2') \; \sqrt{Z_\phi} \\ &\qquad\quad \times \Gamma(q_1, q_2, q_3; k_1, k_2) \\ &\qquad\quad \times \sqrt{Z_\psi}\mathcal{U}(k_1, s_1) \; \sqrt{Z_\psi}\bar{\mathcal{V}}(k_2, s_2) \end{aligned} \tag{77}$$

That is, there is a momentum conserving delta function and there is a $\sqrt{Z}$ for each external line. There is the amputated Green function. There is also a Dirac spinor for each external line

- $\bar{\mathcal{U}}(q_1, s_1')$ for the outgoing electron.
- $\mathcal{U}(k_1, s_1)$ for the incoming electron.
- $\bar{\mathcal{V}}(k_1, s_1)$ for the incoming positron.
- $\mathcal{V}(q_2, s_2')$ for the outgoing positron.

The Dirac indices on the spinors are not explicitly indicated in this formula. They must match the corresponding indices on Γ and then the repeated Dirac indices are summed.