Renormalization of Quantum Electrodynamics[1]

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1 The action

Most of the calculations that one does in quantum field theory beyond the leading order lead to ultraviolet divergences. In these notes, we see how to deal with that.

The first step is to look at the action. (We consider quantum electrodynamics since it is perhaps the most important relativistic quantum field theory, but the ideas are quite general.) The action is

$$S = \int d^d x \left[\bar\psi_0(x)\{ i\partial\!\!\!/ - Q e_0 A\!\!\!/(x) - m_0 \}\psi_0(x) - \tfrac{1}{4} F_0^{\mu\nu}(x) F_{0,\mu\nu}(x) \right]. \qquad (1)$$

Here I have called the fields ψ_0 and A_0^μ, with $F_0^{\mu\nu}(x) \equiv \partial^\mu A_0^\nu - \partial^\nu A_0^\mu$. These are the "bare fields." Also, the coupling is e_0, the "bare coupling" and the mass is m_0, the "bare mass." This is like a corperate reorganization, where every job gets a new title.

I have also anticipated that we may want to operate in

$$d = 4 - 2\epsilon \qquad (2)$$

dimensions of space-time. Thus we have $\int d^d x$.

All of this business with dimensional regulation is not so important for the moment. What is important is that we recogize that the bare quantities may not be upon which to base our perturbation theory. Instead, we can define renormalized quantities by

$$\begin{aligned} \psi_0 &= \tilde\mu^{-\epsilon}\sqrt{Z_\psi}\,\psi \\ A_0^\mu &= \tilde\mu^{-\epsilon}\sqrt{Z_A}\,A^\mu \\ m_0 &= Z_m\, m \\ g_0 &= \tilde\mu^{\epsilon} Z_g\, g. \end{aligned} \qquad (3)$$

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In these relations, I have introduced a quantity $\tilde{\mu}$ with the dimensions of mass (or momentum or inverse distance). The idea is this: S is dimensionless, while $d^d x$ has dimensions $m^{-d} = m^{-4} \times m^{2\epsilon}$. Thus the bare fields and the bare coupling have dimensions that depend on ϵ. On the other hand, I want the ψ, A^μ, m and g to have fixed dimensions, independent of ϵ and I want the various factors Z to be dimensionless. Thus I put the proper dimensionful factor equal to a power of $\tilde{\mu}$ in front of each Z.

Then

$$\begin{aligned} S &= \tilde{\mu}^{-2\epsilon} \int d^d x \Big[\bar{\psi}(x) \{ Z_\psi i\not{\partial} - Z_\psi \sqrt{Z_A} Z_e Q e \not{A}(x) - Z_\psi Z_m m \} \psi(x) \\ &\qquad - \tfrac{1}{4} Z_A F^{\mu\nu}(x) F_{\mu\nu}(x) \Big]. \end{aligned} \tag{4}$$

This doesn't seem very helpful, but we can make it even worse:

$$\begin{aligned} S &= \tilde{\mu}^{-2\epsilon} \int d^d x \Big[\bar{\psi}(x) \{ i\not{\partial} - m \} \psi(x) - \tfrac{1}{4} F^{\mu\nu}(x) F_{\mu\nu}(x) \\ &\qquad - \bar{\psi}(x) Q e \not{A}(x) \psi(x) \\ &\qquad + (Z_\psi - 1)\ \bar{\psi}(x) i\not{\partial} \psi(x) \\ &\qquad - (Z_\psi \sqrt{Z_A} Z_e - 1)\ \bar{\psi}(x) Q e \not{A}(x) \psi(x) \\ &\qquad - (Z_\psi Z_m - 1)\ \bar{\psi}(x) m \psi(x) \\ &\qquad - (Z_A - 1) \tfrac{1}{4} F^{\mu\nu}(x) F_{\mu\nu}(x) \Big]. \end{aligned} \tag{5}$$

Although this looks like a mess, it does something good. We can treat the first two terms as the unperturbed action that provides the basis for perturbation theory. All the other terms are treated as perturbations. This includes the terms proportional to powers of various Zs minus 1.

These terms are treated as perturbations and generate "counterterms" in the Feynman diagram expansion of the theory. Let's write the rules in momentum space. I should mention that when we go from position space to momentum space in the *renormalized* theory we will defne the Fourier transform by

$$\tilde{\mu}^{-2\epsilon} \int d^{4-2\epsilon} x \ e^{ik\cdot x} \tag{6}$$

so that the Fourier transformed renormalized fields have fixed dimensions. Then each integral over a loop momentum will come out to have the form

$$\tilde{\mu}^{2\epsilon} \int \frac{d^{4-2\epsilon} k}{(2\pi)^{4-2\epsilon}} \tag{7}$$

which is just right to keep the dimensionality fixed if we add loops.

Here are the Feynman rules. We have the usual interaction

$$-ieQ\gamma^\mu \tag{8}$$

but now we have counterterms

- Electron field strength renormalization counterterm

$$i(Z_\psi - 1)\not{k} \tag{9}$$

- Electron mass counterterm

$$-i(Z_\psi Z_m - 1)m \tag{10}$$

- Photon field strength renormalization counterterm

$$i(Z_A - 1)(-g^{\mu\nu}q^2 + q^\mu q^\nu) \tag{11}$$

- Vertex counterterm

$$-i(Z_\psi\sqrt{Z_A}Z_e - 1)\,eQ\gamma^\mu \tag{12}$$

2 Integrals

When we compute Feynman diagrams in d dimensions, we will need the integral

$$I = \int \frac{d^d k}{(2\pi)^d}\,\frac{1}{[k^2 - \Lambda^2 + i\epsilon]^A}. \tag{13}$$

Let's see how to do this.

First, we rotate the k^0 integration to the imaginary axis, so $k^0 = i\kappa$. Then

$$I = \int \frac{d\kappa\, d^{d-1}\vec{k}}{(2\pi)^d}\,\frac{i}{[-\kappa^2 - \vec{k}^2 - \Lambda^2]^A} = \int \frac{d\kappa\, d^{d-1}\vec{k}}{(2\pi)^d}\,\frac{i(-1)^A}{[\kappa^2 + \vec{k}^2 + \Lambda^2]^A}. \tag{14}$$

We write this as an integration over a vector space with a euclidian metric ("E"):

$$I = i(-1)^A \int_E \frac{d^d k}{(2\pi)^d}\,\frac{1}{[k^2 + \Lambda^2]^A}. \tag{15}$$

Now use the integral representation of the gamma function to write

$$
\begin{aligned}
I &= \frac{i(-1)^A}{\Gamma(A)} \int_E \frac{d^d k}{(2\pi)^d} \int_0^\infty \frac{d\lambda}{\lambda}\ \lambda^A \exp\{-\lambda[k^2+\Lambda^2]\} \\
&= \frac{i(-1)^A}{\Gamma(A)} \int_0^\infty \frac{d\lambda}{\lambda}\ \lambda^A e^{-\lambda\Lambda^2} \int_E \frac{d^d k}{(2\pi)^d}\ \exp\{-\lambda k^2\}
\end{aligned}
\tag{16}
$$

Here is where we come to the meaning of d dimensions. If d is an integer, we have

$$
k^2 = \sum_{j=1}^{d} k_j^2 \tag{17}
$$

so

$$
\int_E \frac{d^d k}{(2\pi)^d}\ \exp\{-\lambda k^2\} = \left(\int_{-\infty}^{\infty} \frac{dk}{2\pi}\ \exp\{-\lambda k^2\}\right)^d \tag{18}
$$

We make this the definition if d is not an integer. Of course, we know the integral:

$$
\int_{-\infty}^{\infty} dx\ \exp\{-x^2\} = \sqrt{\pi}, \tag{19}
$$

so

$$
\int_{-\infty}^{\infty} \frac{dk}{2\pi}\ \exp\{-\lambda k^2\} = \frac{1}{\sqrt{4\pi\lambda}}. \tag{20}
$$

Thus

$$
\begin{aligned}
I &= \frac{i(-1)^A}{\Gamma(A)} \int_0^\infty \frac{d\lambda}{\lambda}\ \lambda^A e^{-\lambda\Lambda^2}\ [4\pi\lambda]^{-d/2} \\
&= \frac{i(-1)^A}{\Gamma(A)}\ [4\pi]^{-d/2} \int_0^\infty \frac{d\lambda}{\lambda}\ \lambda^{A-d/2} e^{-\lambda\Lambda^2} \\
&= \frac{i(-1)^A}{\Gamma(A)}\ [4\pi]^{-d/2}\, \Lambda^{d-2A}\ \Gamma(A-d/2)
\end{aligned}
\tag{21}
$$

That is

$$
\int \frac{d^d k}{(2\pi)^d}\ \frac{1}{[k^2-\Lambda^2+i\epsilon]^A} = \frac{i(-1)^A}{\Gamma(A)}\ [4\pi]^{-d/2}\ \frac{\Gamma(A-d/2)}{[\Lambda^2]^{A-d/2}}. \tag{22}
$$

Note that the power of Λ^2 follows by power counting. Recall that $\Gamma(z)$ has a pole at $z = 0$ (and also at positive integer values of z). Thus our integral has a pole at just the point where the power of Λ^2 is zero. Near $z = 0$ one has

$$\Gamma(z) = \frac{1}{z} - \gamma_E + \mathcal{O}(\epsilon). \tag{23}$$

where γ_E is the Euler constant, $0.577\cdots$.

Sometimes, one needs the integral

$$\int \frac{d^d k}{(2\pi)^d} \; \frac{k^\mu}{[k^2 - \Lambda^2 + i\epsilon]^A}. \tag{24}$$

This integral is zero because it is odd under $k^\mu \to -k^\mu$. Also, one can encounter

$$\int \frac{d^d k}{(2\pi)^d} \; \frac{k^\mu k^\nu}{[k^2 - \Lambda^2 + i\epsilon]^A}. \tag{25}$$

This is easy to evaluate by replacing

$$k^\mu k^\nu \to \frac{1}{d} \; g^{\mu\nu} \; k^2 \tag{26}$$

and writing k^2 as $(k^2 - \Lambda^2) + \Lambda^2$.

3 The electron propagator

The 1PI graph has a name: $-i\Sigma$. We have

$$-i\Sigma(p) = -e^2 \tilde{\mu}^{2\epsilon} \int \frac{d^{4-2\epsilon} q}{(2\pi)^{4-2\epsilon}} \; \frac{\gamma^\mu(\not{p} - \not{q} + m)\gamma_\mu}{[(p-q)^2 - m^2 + i\epsilon][q^2 + i\epsilon]} \tag{27}$$

Using a Feynman parameter α this becomes

$$-i\Sigma(p) = -e^2 \int_0^1 d\alpha \; \tilde{\mu}^{2\epsilon} \int \frac{d^{4-2\epsilon} q}{(2\pi)^{4-2\epsilon}} \; \frac{\gamma^\mu(\not{p} - \not{q} + m)\gamma_\mu}{[\tilde{q}^2 - \Lambda^2 + i\epsilon]^2} \tag{28}$$

where

$$\tilde{q}^\mu = q^\mu - \alpha p^\mu \tag{29}$$

and

$$\Lambda^2 = \alpha m^2 - \alpha(1 - \alpha)p^2. \tag{30}$$

The numerator becomes

$$\gamma^\mu((1 - \alpha)\not{p} - \not{\tilde{q}} + m)\gamma_\mu \tag{31}$$

We can throw away the $\tilde{q}$ term since it integrates to zero. For the other terms, we use our $\gamma^\mu \Gamma \gamma_\mu$ rules, giving

$$-2(1-\epsilon)(1-\alpha)\not{p} + (4-2\epsilon)m \tag{32}$$

Thus

$$-i\Sigma(p) = -e^2 \int_0^1 d\alpha\ \tilde{\mu}^{2\epsilon} \int \frac{d^{4-2\epsilon}q}{(2\pi)^{4-2\epsilon}}\ \frac{-2(1-\epsilon)(1-\alpha)\not{p} + (4-2\epsilon)m}{[\tilde{q}^2 - \Lambda^2 + i\epsilon]^2} \tag{33}$$

Now we can perform the q integration, giving

$$-i\Sigma(p) = \frac{-ie^2}{(4\pi)^2}\Gamma(\epsilon)\int_0^1 d\alpha \left(\frac{4\pi\tilde{\mu}^2}{\Lambda^2}\right)^\epsilon [-2(1-\epsilon)(1-\alpha)\not{p} + (4-2\epsilon)m], \tag{34}$$

or

$$\Sigma(p) = \frac{\alpha_{\rm em}}{4\pi}\Gamma(\epsilon)\int_0^1 d\alpha \left(\frac{4\pi\tilde{\mu}^2}{\Lambda^2}\right)^\epsilon [-2(1-\epsilon)(1-\alpha)\not{p} + (4-2\epsilon)m]\,. \tag{35}$$

To use this result, we add in the counter terms and expand around $\epsilon = 0$:

$$\begin{aligned}
\Sigma(p) &= \frac{\alpha_{\rm em}}{4\pi}\frac{1}{\epsilon}(1-\epsilon\gamma_E+\cdots)\int_0^1 d\alpha \left[1 + \epsilon \ln\left(\frac{4\pi\tilde{\mu}^2}{\Lambda^2}\right) + \cdots\right] \\
&\quad \times [-2(1-\epsilon)(1-\alpha)\not{p} + (4-2\epsilon)m] + \text{C.T.} \\
&= \frac{\alpha_{\rm em}}{4\pi}\frac{1}{\epsilon}\int_0^1 d\alpha\,[-2(1-\alpha)\not{p} + 4m] \\
&\quad + \frac{\alpha_{\rm em}}{4\pi}\int_0^1 d\alpha \ln\left(\frac{4\pi\tilde{\mu}^2}{\Lambda^2}\,e^{-\gamma_E}\right)[-2(1-\alpha)\not{p} + 4m] \\
&\quad + \frac{\alpha_{\rm em}}{4\pi}\int_0^1 d\alpha\,[2(1-\alpha)\not{p} - 2m] + \cdots + \text{C.T.}.
\end{aligned} \tag{36}$$

In the first and last terms, we can perform the α itegration immediately, while we leave the integration undone in the second term. We simplify the second term with the substitution

$$4\pi\tilde{\mu}^2\,e^{-\gamma_E} \equiv \mu^2. \tag{37}$$

Then

$$\begin{aligned}
\Sigma(p) &= \frac{\alpha_{\rm em}}{4\pi}\frac{1}{\epsilon}[-\not{p} + 4m] \\
&\quad + \frac{\alpha_{\rm em}}{4\pi}\int_0^1 d\alpha \ln\left(\frac{\alpha m^2 - \alpha(1-\alpha)p^2}{\mu^2}\right)[2(1-\alpha)\not{p} - 4m] \\
&\quad + \frac{\alpha_{\rm em}}{4\pi}[\not{p} - 2m] + \cdots \\
&\quad -(Z_\psi - 1)\not{p} + (Z_\psi Z_m - 1)m.
\end{aligned} \tag{38}$$

Now we get to choose what $(Z_\psi - 1)$ and $(Z_\psi Z_m - 1)$ are. These quantities will have expansions in powers of $\alpha_{\rm em}$. At our current order of perturmbation theory, we are only concerned with the $\alpha_{\rm em}^1$ contributions.

We can choose the "modified minimal subtraction" prescription, usually denoted $\overline{\rm MS}$. In this prescription, we choose the counter terms to cancel the poles and no more:

$$\begin{aligned}(Z_\psi - 1) &= -\frac{\alpha_{\rm em}}{4\pi}\frac{1}{\epsilon} + \mathcal{O}(\alpha_{\rm em}^2) \\ (Z_\psi Z_m - 1) &= -\frac{\alpha_{\rm em}}{\pi}\frac{1}{\epsilon} + \mathcal{O}(\alpha_{\rm em}^2).\end{aligned} \tag{39}$$

With the $\overline{\rm MS}$ prescription we have

$$\begin{aligned}\Sigma(p) &= \frac{\alpha_{\rm em}}{4\pi}\int_0^1 d\alpha \ln\left(\frac{\alpha m^2 - \alpha(1-\alpha)p^2}{\mu^2}\right)[2(1-\alpha)\not{p} - 4m] \\ &\quad + \frac{\alpha_{\rm em}}{4\pi}[\not{p} - 2m] + \cdots .\end{aligned} \tag{40}$$

This is an analytic function of p^2 for spacelike p^μ, that is $p^2 < 0$. It has a branch point at $p^2 = m^2$, corresponding to an electron splitting into an electron and a photon. We may consider that the function has a branch cut from $p^2 = m^2$ to $p^2 = \infty$. The physical side of the branch cut is ${\rm Im}(p^2) > 0$.

This may seem very strange. We seem to be hiding something infinite with the $(Z-1)$ trick while all the time we are expanding in a small quantity $\alpha_{\rm em}$. How can we expand in powers of $\alpha_{\rm em} \times \infty$? A way to think of this is that our integralions really end at $q^2 \sim M^2$, where M^2 is a momentum scale beyond our experimental reach. For instance $M = 10$ TeV. Beyond the scale M^2 there is some new physics that we don't know about. We may suppose that below M^2 we have Standard Model physics, for which we have just calculated the most important diagram. The "bare" quantities refer to the effective action that applies at scale M^2 and below.

Now if we perform the integrations with a cutoff around M^2, we will get $\alpha_0 \ln(p^2/M^2)$ factors. (There is a log because the integration is logarithmically divergent.) Then we use the $(Z-1)$ trick to change to "renormalized" quantities with

$$(Z-1) \sim (const)\frac{\alpha_{\rm em}}{\pi}\ln(M^2/\mu^2) + \mathcal{O}(\alpha_{\rm em}^2) \tag{41}$$

This cancels the $\ln(M^2)$ and replacees it by $\ln(\mu^2)$, where μ can be of the order of the electron mass. Thus the renormalized expansion has powers of

$\alpha_{\rm em}$ multiplied by quantities of order 1. This compares to the unrenormalized perturbation theory, which has powers of $\alpha_0 \ln(M^2/\text{“}m_e^2\text{”})$ and hence does not converge so well. As a technical trick, we don't just cut off the integrations, but rather regulate them with dimensional regulation. Thus we should understand $\mu^{2\epsilon}/\epsilon$ as a stand in for $\ln(\mu^2/M^2)$.

Exercise: Calculate the photon self energy $\Pi^{\mu\nu}$ at one loop order to obtain a formula analogous to Eq. (35), that is $\Pi^{\mu\nu}$ in $4-2\epsilon$ dimensions before subtracting the counter term.

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4 Structure of the propagator

We will want to investigate on-shell renormalization. Before starting this, let's look at the complete propagator $iD(p)$. Let $-i\Sigma(p)$ be the one particle irreducible two point function for the electron, summed to all orders of perturbation theory. Then we have

$$D(p) = \frac{1}{\not p - m}\left\{1 + \Sigma(p)\frac{1}{\not p - m} + \left(\Sigma(p)\frac{1}{\not p - m}\right)^2 + \cdots\right\}. \tag{42}$$

This sums to

$$D(p) = \frac{1}{\not p - m}\left[1 - \Sigma(p)\frac{1}{\not p - m}\right]^{-1}. \tag{43}$$

We can simplify this, being careful of the matrix ordering: If we let $E = \not p - m$, we have

$$D = E^{-1}(1-\Sigma E^{-1})^{-1} \tag{44}$$

so

$$D(1-\Sigma E^{-1}) = E^{-1} \tag{45}$$

so

$$D(E-\Sigma) = 1 \tag{46}$$

or

$$D = (E-\Sigma)^{-1}. \tag{47}$$

That is

$$D(p) = [\not p - m - \Sigma(p)]^{-1} \tag{48}$$

Thus if we knew $\Sigma(p)$ exactly, we would have the inverse of the propagator.

We can carry this a little further by writing

$$\Sigma(p) = A(p^2)[\not{p} - m] + B(p^2)m. \tag{49}$$

Then

$$D = \frac{1}{1 - A(p^2)} \frac{\not{p} + (1 + C(p^2))\, m}{p^2 - (1 + C(p^2))^2\, m^2} \tag{50}$$

where

$$C(p^2) = \frac{B(p^2)}{1 - A(p^2)}. \tag{51}$$

We see that (if C is close to 0) D has a pole at some value of p^2 which may not be m^2. The residue of this pole may not be 1. Note that we have to sum the series (42) in order to see that the pole moves. We do not, however, have to sum all of the contributions to Σ to see this.

5 On-shell renormalization

We can adjust the $Z - 1$ factors to make m the physical electron mass, that is, the location of the pole. Evidently, what we need is

$$B(m^2) = 0, \tag{52}$$

so that $C(m^2) = 0$. We can also adjust the $Z - 1$ factors to make the residue of the pole at $p^2 = m^2$ equal to $\not{p} + m$:

$$D(p) \sim \frac{\not{p} + m}{p^2 - m^2} \qquad \text{as} \quad p^2 \to m^2. \tag{53}$$

With a little algebra, we find that the required condition is

$$A(m^2) = 2m^2 B'(m^2). \tag{54}$$

We can call Eqs. (52) and (54) the on-shell renormalization conditions.

Let's work this out. But first, we give the photon a small mass m_γ in order to control certain infrared divergences that we don't want to worry about yet. If we give a mass to the photon, our previous analysis works except that the denominator function becomes

$$\Lambda^2 = \alpha m^2 + (1 - \alpha) m_\gamma^2 - \alpha(1 - \alpha) p^2. \tag{55}$$

Then let us imagine that we write the counter terms as $a[\not{p}-m]+bm$ where

$$\begin{aligned} a &= \frac{a_{\rm pole}}{\epsilon}+a' \\ b &= \frac{b_{\rm pole}}{\epsilon}+b' \end{aligned} \tag{56}$$

where the first parts are the counterterms in the $\overline{\rm MS}$ prescription and a' and b' are additional pieces that we now add to satisfy the on-shell renormalization conditions. Once we have subtracted the pole terms, we can take $\epsilon \to 0$ (as long as $m_\gamma \neq 0$). (Peskin and Schroeder do this with $\epsilon \neq 0$ but then it is a little more messy.) We find

$$\begin{aligned} \Sigma(p) &= \frac{\alpha_{\rm em}}{4\pi}\int_0^1 d\alpha\ \ln\!\left(\frac{\alpha m^2+(1-\alpha)m_\gamma^2-\alpha(1-\alpha)p^2}{\mu^2}\right) \\ &\quad\times[2(1-\alpha)(\not{p}-m)-2(1+\alpha)m] \\ &\quad+\frac{\alpha_{\rm em}}{4\pi}[(\not{p}-m)-m]+a'(\not{p}-m)+b'm \end{aligned} \tag{57}$$

The functions A and B have the form

$$\begin{aligned} A(p^2) &= A_1(p^2)+a' \\ B(p^2) &= B_1(p^2)+b', \end{aligned} \tag{58}$$

where $A_1(p^2)$ and $B_1(p^2)$ are the one-loop functions that we read off from Eq. (57) and a' aned b' are the corresponding counterterms, which are independent of p^2. The solution of the on-shell renormalization conditions (52) and (54) is

$$\begin{aligned} a' &= -A_1(m^2)+2m^2\,B_1'(m^2) \\ b' &= -B_1(m^2). \end{aligned} \tag{59}$$

We have

$$\begin{aligned} A_1(p^2) &= \frac{\alpha_{\rm em}}{4\pi}\int_0^1 d\alpha\left\{\ln\!\left(\frac{\alpha m^2+(1-\alpha)m_\gamma^2-\alpha(1-\alpha)p^2}{\mu^2}\right)2(1-\alpha)+1\right\} \\ B_1(p^2) &= \frac{\alpha_{\rm em}}{4\pi}\int_0^1 d\alpha \\ &\quad\times\left\{-\ln\!\left(\frac{\alpha m^2+(1-\alpha)m_\gamma^2-\alpha(1-\alpha)p^2}{\mu^2}\right)2(1+\alpha)-1\right\}. \end{aligned} \tag{60}$$

Then

$$B_1'(p^2) \;=\; \frac{\alpha_{\rm em}}{4\pi}\int_0^1 d\alpha\left\{\frac{2\alpha(1-\alpha^2)}{\alpha m^2+(1-\alpha)m_\gamma^2-\alpha(1-\alpha)p^2}\right\}. \tag{61}$$

Thus

$$\begin{aligned} a' \;&=\; \frac{\alpha_{\rm em}}{4\pi}\int_0^1 d\alpha\Big\{-\ln\left(\frac{\alpha^2 m^2+(1-\alpha)m_\gamma^2}{\mu^2}\right)2(1-\alpha)-1 \\ &\qquad -\frac{4\alpha(1-\alpha^2)}{\alpha^2+(1-\alpha)m_\gamma^2/m^2}\Big\} \\ b' \;&=\; \frac{\alpha_{\rm em}}{4\pi}\int_0^1 d\alpha\Big\{\ln\left(\frac{\alpha^2 m^2+(1-\alpha)m_\gamma^2}{\mu^2}\right)2(1+\alpha)+1\Big\}. \end{aligned} \tag{62}$$

Putting this together, we have

$$\begin{aligned} \Sigma(p) \;&=\; \frac{\alpha_{\rm em}}{4\pi}\int_0^1 d\alpha\Big\{\ln\left(\frac{\alpha m^2+(1-\alpha)m_\gamma^2-\alpha(1-\alpha)p^2}{\alpha^2 m^2+(1-\alpha)m_\gamma^2}\right) \\ &\qquad\times[2(1-\alpha)(\not p-m)-2(1+\alpha)m] \\ &\qquad -\frac{4\alpha(1-\alpha^2)(\not p-m)}{\alpha^2+(1-\alpha)m_\gamma^2/m^2}\Big\}. \end{aligned} \tag{63}$$

The details of this result are not so important. The most important feature are that the dependence on μ has disappeared. It didn't really matter how we regulated the integral. Also, the on-shell renormalization is pretty much like the $\overline{\rm MS}$ renormalization with the scale μ of the order of the electron mass. Finally, note that if we set m_γ to zero we will have a logarithmic divergence from the $\alpha\to 0$ region in the integral.

The logarithmic divergence when $m_\gamma = 0$ arises because, if $m_\gamma = 0$, the electron propagator doesn't really have a pole. We can illustrate this as follows. If we start from Eq. (57) with $m_\gamma = 0$ and use the on-shell value for b' so that the location of the singularity in the propagator is at $p^2 = m^2$ (so that $m = m_e$) but use the $\overline{\rm MS}$ value for a', namely $a' = 0$, we get

$$\begin{aligned} \Sigma(p) \;&=\; \frac{\alpha_{\rm em}}{4\pi}\int_0^1 d\alpha \\ &\quad\times\Big\{\ln\Big(\frac{\alpha m_e^2-\alpha(1-\alpha)p^2}{\mu^2}\Big)2(1-\alpha)+1\Big\}(\not p-m_e) \\ &\quad -\frac{\alpha_{\rm em}}{4\pi}\int_0^1 d\alpha\;\ln\Big(\frac{\alpha m_e^2-\alpha(1-\alpha)p^2}{\alpha^2 m_e^2}\Big)2(1+\alpha)m_e. \end{aligned} \tag{64}$$

This is a finite result for $p^2 < m_e^2$. The coefficient $B(p^2)$ of m vanishes at $p^2 = m_e^2$, so that $p^2 = m_e^2$ is where the singularity of the propagator is located. However

$$B(p^2) = -\frac{\alpha_{\rm em}}{4\pi}\int_0^1 d\alpha\ \ln\left(1 - \frac{1-\alpha}{\alpha}\,\frac{p^2 - m_e^2}{m_e^2}\right) 2(1+\alpha). \tag{65}$$

As $p^2 \to m_e^2$, this function does not behave like $p^2 - m_e^2$, but rather like

$$(p^2 - m_e^2)\,\ln(p^2 - m_e^2) \tag{66}$$

This term will dominate the denominator in Eq. (50), so that instead of having a pole, $D(p^2)$ behaves like

$$\frac{const.}{(p^2 - m_e^2)\,\ln(p^2 - m_e^2)}. \tag{67}$$

The consequence is that, with the on-shell renormalization prescription, the renormalized Green functions depend on an artificial cutoff parameter, m_γ. Only when we calculate a physical observable will the m_γ dependence disappear in the limit $m_\gamma \to 0$.

Compare this with the $\overline{\rm MS}$ prescription, in which the renormalized electron propagator is perfectly finite in the physical case $m_\gamma = 0$ as long as we stay away from $p^2 = m_e^2$.

If we calculate electron-electron scattering at one loop order, then the on-shell renormalization prescription is the most convenient. But, however we do it, we will have to deal with an infrared divergence that can be regulated by taking $m_\gamma \neq 0$. The problem is that on-shell electron-electron scattering is not physically observable. Real electrons are always accompanied by soft photons – photons that have arbitrarily small momenta. We have to build into the calculation the fact that some of the photons are too soft to be detected before we can get an answer for electron-electron scattering that is finite when $m_\gamma \to 0$.

Another way to regulate the soft photon divergence is by staying in $4 - 2\epsilon$ dimensions. In this style of calculation, with on-shell renormalization, the renormalized electron propagator has some $1/\epsilon$ terms. The renormalization subtraction gets rid of the $1/\epsilon$ terms that come from ultraviolet momenta in the loop. But then renormalization subtraction adds $1/\epsilon$ terms associated with infrared momenta and arise from the unphysical renormalization condition.

6 The photon propagator

The 1PI graph is generally called $-i\Pi^{\mu\nu}$ (or $+i\Pi^{\mu\nu}$ in the notation of Peskin and Schroeder). At one loop we have

$$-i\Pi^{\mu\nu}(q) = -e^2\tilde{\mu}^{2\epsilon}\int\frac{d^{4-2\epsilon}k}{(2\pi)^{4-2\epsilon}}\;\frac{\mathrm{Tr}\,[\gamma^\mu(\not{k}-\not{q}+m)\gamma^\nu(\not{k}+m)]}{[(k-q)^2-m^2+i\epsilon][k^2-m^2+i\epsilon]}+\mathrm{C.T.} \tag{68}$$

We already know how to do this kind of calculation, so I will leave out the intermediate steps, except to point out that in one of the terms we get a factor

$$\frac{-1+\epsilon}{-2+\epsilon}\,[-\Gamma(-1+\epsilon)+\Gamma(\epsilon)] = \frac{1}{-2+\epsilon}\,[-\Gamma(\epsilon)+(-1+\epsilon)\Gamma(\epsilon)] = \Gamma(\epsilon). \tag{69}$$

The result is

$$\begin{aligned}\Pi^{\mu\nu}(q) \;&=\; 2\frac{\alpha_{\rm em}}{\pi}\,\Gamma(\epsilon)\,(g^{\mu\nu}q^2-q^\mu q^\nu)\int_0^1 d\alpha\;\alpha(1-\alpha)\left(\frac{m^2-\alpha(1-\alpha)q^2}{4\pi\tilde{\mu}^2}\right)^{-\epsilon}\\ &\quad+\frac{\alpha_{\rm em}}{\pi}(g^{\mu\nu}q^2-q^\mu q^\nu)\left(-\frac{1}{3\epsilon}+\delta c\right).\end{aligned} \tag{70}$$

The second line is the counter term, which, according to the Feynman rules, must be proportional to $(g^{\mu\nu}q^2-q^\mu q^\nu)$. If we choose $\overline{\rm MS}$ renormalization, then the counter term is simply the pole term and the extra constant that I have called δc is zero. That is

$$(Z_A-1) = -\frac{\alpha_{\rm em}}{3\pi}\frac{1}{\epsilon}+\mathcal{O}(\alpha_{\rm em}^2). \tag{71}$$

Since the pole term gets rid of the divergence, we can take the limit $\epsilon\to 0$ to get (with any renormalization prescription)

$$\begin{aligned}\Pi^{\mu\nu}(q) \;&=\; -2\frac{\alpha_{\rm em}}{\pi}\,(g^{\mu\nu}q^2-q^\mu q^\nu)\int_0^1 d\alpha\;\alpha(1-\alpha)\ln\!\left(\frac{m^2-\alpha(1-\alpha)q^2}{\mu^2}\right)\\ &\quad+\frac{\alpha_{\rm em}}{\pi}(g^{\mu\nu}q^2-q^\mu q^\nu)\delta c.\end{aligned} \tag{72}$$

If we would like to use on-shell renormalization, we need a little analysis of the photon propagator. (We stick to Feynman gauge, for simplicity.) Define

$$C(q)^\mu_\nu = g^\mu_\nu - \frac{q^\mu q_\nu}{q^2}. \tag{73}$$

Note the properties

$$C(q)^\mu_\nu q^\nu = 0, \qquad C(q)^\mu_\lambda C(q)^\lambda_\nu = C(q)^\mu_\nu. \tag{74}$$

We can define $\Pi(q^2)$ by

$$\Pi^{\mu\nu}(q) = -q^2 C(q)^\mu_\nu \Pi(q^2). \tag{75}$$

Then the complete photon propagator is

$$D^{\mu\nu} = \frac{-g^{\mu\nu}}{q^2} + \frac{-g^{\mu\nu}}{q^2}(-q^2 C(q)_{\alpha\beta}\Pi(q^2))\frac{-g^{\alpha\nu}}{q^2} + \cdots \tag{76}$$

This is

$$D^{\mu\nu} = -\frac{1}{q^2}\frac{q^\mu q^\nu}{q^2} - \frac{C(q)^{\mu\nu}}{q^2}\left[1 + \Pi(q^2) + \Pi(q^2)^2 + \cdots\right]. \tag{77}$$

There is an orphan $q^\mu q^\nu$ term, but then a geometric series. Summing the series, we have

$$D^{\mu\nu} = -\frac{1}{q^2}\frac{q^\mu q^\nu}{q^2} - \frac{C(q)^{\mu\nu}}{q^2[1-\Pi(q^2)]}. \tag{78}$$

There is some real physics content here. Because $\Pi^{\mu\nu}$ has the structure given in Eq. (75), the pole in the photon propagator remains at $q^2 = 0$.

Now to implement on-shell renormalization, we adjust the counter-terms so that

$$\Pi(0) = 0. \tag{79}$$

Then

$$D^{\mu\nu} \sim -\frac{1}{q^2}\frac{q^\mu q^\nu}{q^2} - \frac{C(q)^{\mu\nu}}{q^2} = -\frac{g^{\mu\nu}}{q^2}. \tag{80}$$

as $q^2 \to 0$.

With on-shell renormalization, we have

$$\Pi^{\mu\nu}(q) = -2\frac{\alpha_{\rm em}}{\pi}\left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)\int_0^1 d\alpha\ \alpha(1-\alpha)\ln\left(\frac{m^2-\alpha(1-\alpha)q^2}{m^2}\right) \tag{81}$$

Recall that the counter term that adjusts $\Pi^{\mu\nu}$ is

$$(Z_A - 1)(g^{\mu\nu}q^2 - q^\mu q^\nu). \tag{82}$$

Our next exercise will be an investigation of the renormalization of the 1PI electron-photon vertex. We will find then that $Z_e = 1/\sqrt{Z_A}$. Thus when we decide on the renormalization prescription for the photon propagator, we are really deciding how to define the unit of electric charge squared, $e^2/(4\pi) = \alpha_{\rm em}$. With the $\overline{\rm MS}$ definition, we get a definition that depends on the scale μ: $\alpha_{\rm em}(\mu^2)$. With the on-shell definition, we get the standard coupling that is usually called simply $\alpha_{\rm em}$ and is approximately 1/137. We have seen that to one-loop accuracy, and if electron are the only charged fermions, $\alpha_{\rm em} = \alpha_{\rm em}(m_e^2)$.

7 Renormalization of the vertex function

We now examine the $ee\gamma$ vertex function, Γ^μ. our aim is to see what counter term we have to include in Γ^μ to make it finite. The calculation starts as in the calculation of the electron anomalous magnetic moment, except that we do not necessarily take the incoming and outgoing electrons to be on-shell. We have

$$\begin{aligned}\Gamma^\mu &= -ie^2\tilde{\mu}^{2\epsilon}\int\frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}}\frac{\gamma^\alpha(\not{k}'-\not{l}+m)\gamma^\mu(\not{k}-\not{l}+m)\gamma_\alpha}{[(k'-l)^2-m^2+i\epsilon][(k-l)^2-m^2+i\epsilon][l^2+i\epsilon]}\\ &\quad +\text{C.T.}.\end{aligned} \tag{83}$$

For our limited purposes, we do not need to introduce Feynman parameters, but let's do so so that we can follow along with the complete calculation of Γ^μ for awhile. With Feynman parameters, we have

$$\begin{aligned}\Gamma^\mu &= -2ie^2\int_0^1 d\alpha\int_0^{1-\alpha}d\beta\ \tilde{\mu}^{2\epsilon}\int\frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}}\\ &\quad\times\frac{\gamma^\alpha(\not{k}'-\not{l}+m)\gamma^\mu(\not{k}-\not{l}+m)\gamma_\alpha}{[D^2+i\epsilon]^3}\\ &\quad+\text{C.T.},\end{aligned} \tag{84}$$

where

$$\begin{aligned}D &= \alpha[(k'-l)^2-m^2]+\beta[(k-l)^2-m^2]+(1-\alpha-\beta)l^2\\ &= l^2-2\alpha k'\cdot l-2\beta k\cdot l+\alpha(k'^2-m^2)+\beta(k^2-m^2)\\ &= (l-\alpha k'-\beta k)^2+\alpha(1-\alpha)k'^2+\beta(1-\beta)k^2+2\alpha\beta k\cdot k'-(\alpha+\beta)m^2\\ &= \tilde{l}^2-\Lambda^2.\end{aligned} \tag{85}$$

Here the shifted loop momentum is

$$\tilde{l}^{\mu} = l^{\mu} - \alpha k'^{\mu} - \beta k^{\mu}. \tag{86}$$

If we use $\gamma = 1 - \alpha - \beta$ and $q^{\mu} = k'^{\mu} - k^{\mu}$, we can write Λ^2 in a nice form:

$$\Lambda^2 = -\alpha\gamma k'^2 - \beta\gamma k^2 - \alpha\beta q^2 + (\alpha+\beta)m^2 \tag{87}$$

Thus

$$\begin{aligned}\Gamma^{\mu} &= -2ie^2 \int_0^1 d\alpha \int_0^{1-\alpha} d\beta\ \tilde{\mu}^{2\epsilon} \int \frac{d^{4-2\epsilon}\tilde{l}}{(2\pi)^{4-2\epsilon}} \\ &\quad \times \frac{\gamma^{\alpha}((1-\alpha)\not{k}' - \beta\not{k} - \tilde{\not{l}} + m)\gamma^{\mu}(-\alpha\not{k}' + (1-\beta)\not{k} - \tilde{\not{l}} + m)\gamma_{\alpha}}{[\tilde{l}^2 - \Lambda^2 + i\epsilon]^3} \\ &\quad + \text{C.T.},\end{aligned} \tag{88}$$

So far, we have a complete calculation. Now we concentrate on pieces that can give a $1/\epsilon$, which are those arising from having two factors of $\tilde{l}^{\nu}$ in the numerator. We have

$$\begin{aligned}\gamma^{\alpha}\tilde{\not{l}}\gamma^{\mu}\tilde{\not{l}}\gamma_{\alpha} &= -2(1-\epsilon)\tilde{\not{l}}\gamma^{\mu}\tilde{\not{l}} \\ &\to -\frac{(1-\epsilon)}{2(1-\epsilon/2)}\ \tilde{l}^2\ \gamma_{\beta}\gamma^{\mu}\gamma^{\beta} \\ &= -\frac{(1-\epsilon)^2}{(1-\epsilon/2)}\ \tilde{l}^2\ \gamma^{\mu} \\ &= -\frac{(1-\epsilon)^2}{(1-\epsilon/2)}\ (\tilde{l}^2 - \Lambda^2)\ \gamma^{\mu} - \frac{(1-\epsilon)^2}{(1-\epsilon/2)}\ \Lambda^2\, \gamma^{\mu}.\end{aligned} \tag{89}$$

In the step indicated by an arrow, we have used the fact that $\tilde{l}_{\alpha}\tilde{l}_{\beta}$ and $\tilde{l}^2 g_{\alpha\beta}/(4-2\epsilon)$ have the same integral.

The second term of this result will not give a divergence as $\epsilon \to 0$. And in the first term, we can set $\epsilon \to 0$ in the coefficient if all we want is the coefficient of $1/\epsilon$. Then

$$\begin{aligned}\Gamma^{\mu} &= -2ie^2 \int_0^1 d\alpha \int_0^{1-\alpha} d\beta\ \tilde{\mu}^{2\epsilon} \int \frac{d^{4-2\epsilon}\tilde{l}}{(2\pi)^{4-2\epsilon}} \frac{\gamma^{\mu}}{[\tilde{l}^2 - \Lambda^2 + i\epsilon]^2} \\ &\quad + \text{C.T.} + \text{finite}.\end{aligned} \tag{90}$$

We perform the $\tilde{l}^\nu$ integration to get

$$\begin{aligned}\Gamma^\mu &= -2ie^2\gamma^\mu \int_0^1 d\alpha \int_0^{1-\alpha} d\beta\ i[4\pi]^{-(2-\epsilon)}\Gamma(\epsilon)[\Lambda^2/\tilde{\mu}^2]^{-\epsilon} \\ &\quad +\text{C.T.} + \text{finite} \\ &= \frac{\alpha_{\rm em}}{2\pi}\gamma^\mu \int_0^1 d\alpha \int_0^{1-\alpha} d\beta\ \Gamma(\epsilon)\left[\frac{\Lambda^2}{4\pi\tilde{\mu}^2}\right]^{-\epsilon} \\ &\quad +\text{C.T.} + \text{finite} \\ &= \frac{\alpha_{\rm em}}{4\pi}\gamma^\mu \frac{1}{\epsilon} \\ &\quad +\text{C.T.} + \text{finite}.\end{aligned} \tag{91}$$

It should be evident from this derivation how we could keep all of the finite terms. But if we want just the $\overline{\text{MS}}$ counter term, it is C.T. $= (Z_\psi\sqrt{Z}_A Z_e - 1)$ with

$$(Z_\psi\sqrt{Z}_A Z_e - 1) = -\frac{\alpha_{\rm em}}{4\pi}\frac{1}{\epsilon} + \mathcal{O}(\alpha_{\rm em}^2). \tag{92}$$

Recall that we found (with the $\overline{\text{MS}}$ prescription)

$$\begin{aligned}(Z_\psi - 1) &= -\frac{\alpha_{\rm em}}{4\pi}\frac{1}{\epsilon} + \mathcal{O}(\alpha_{\rm em}^2) \\ (Z_\psi Z_m - 1) &= -\frac{\alpha_{\rm em}}{\pi}\frac{1}{\epsilon} + \mathcal{O}(\alpha_{\rm em}^2).\end{aligned} \tag{93}$$

and

$$(Z_A - 1) = -\frac{\alpha_{\rm em}}{3\pi}\frac{1}{\epsilon} + \mathcal{O}(\alpha_{\rm em}^2). \tag{94}$$

Thus

$$\begin{aligned}Z_\psi &= 1 - \frac{1}{4}\frac{\alpha_{\rm em}}{\pi}\frac{1}{\epsilon} + \mathcal{O}(\alpha_{\rm em}^2) \\ Z_A &= 1 - \frac{1}{3}\frac{\alpha_{\rm em}}{\pi}\frac{1}{\epsilon} + \mathcal{O}(\alpha_{\rm em}^2) \\ Z_m &= 1 - \frac{3}{4}\frac{\alpha_{\rm em}}{\pi}\frac{1}{\epsilon} + \mathcal{O}(\alpha_{\rm em}^2) \\ Z_e &= 1 + \frac{1}{6}\frac{\alpha_{\rm em}}{\pi}\frac{1}{\epsilon} + \mathcal{O}(\alpha_{\rm em}^2).\end{aligned} \tag{95}$$

If we want to use on-shell renormalization, we have to consider

$$\bar{\mathcal{U}}(k', s')\Gamma^\mu\mathcal{U}(k, s) \tag{96}$$

with $k'^2 = m^2$ and $k^2 = m^2$. We will get an infrared divergence, so we give the photon a tiny mass m_γ. Recall that between on shell spinors, Γ^μ will have the structure

$$\begin{aligned} & Qe\bar{\mathcal{U}}(k's')\Gamma^\mu(k',k)\mathcal{U}(k,s) \\ = \;& \bar{\mathcal{U}}(k's')\left\{\gamma^\mu Q e F_1(q^2) + i\frac{Qe}{2m}\,\sigma^{\mu\nu}q_\nu\,\kappa\,F_2(q^2)\right\}\mathcal{U}(k,s) \end{aligned} \tag{97}$$

We take the limit $q^2 \to 0$. Then the $\sigma^{\mu\nu}q_\nu$ goes away. We add a little extra (besides the $1/\epsilon$ part) to the γ^μ term so that the one loop contribution to F_1 vanishes. That is, the on-shell renormalization condition is

$$F_1(0) = 1. \tag{98}$$

The result is that $(Z_\psi\sqrt{Z}_A Z_e - 1)$ is the same as $(Z_\psi - 1)$. That is, $\sqrt{Z}_A Z_e = 1$, just as with the $\overline{\mathrm{MS}}$. We could have discovered this with less calculation by using the "current conservation identity" or "Ward identity." I won't go into this, but the identity arises by considering $q_\mu\Gamma^\mu$.

Exercise: Calculate the renormalized two point function and four point function at one loop in ϕ^4 theory using the $\overline{\mathrm{MS}}$ prescription.

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8 The role of μ

If we use $\overline{\mathrm{MS}}$ renormalization, the Green functions, calculated at a fixed order of perturbation theory, depend on μ. Suppose the σ is some observable that can be calculated in perturbation theory. Then

$$\sigma = \sum_{n=0}^{\infty} e^n\,\sigma_n \tag{99}$$

The perturbative coefficients σ_n will depend on m and μ. However, physical predictions cannot depend on μ, at least if we calculate to infinite order. The only way this can happen is if e and m also depend on μ. That is, σ must have the form

$$\sigma(e(\mu), m(\mu), \mu) = \sum_{n=0}^{\infty} e(\mu)^n\,\sigma_n(m(\mu),\mu) \tag{100}$$

and it must be the case that $\sigma(g(\mu), m(\mu), \mu)$ is independent of μ. We will see in the next section how this can happen and what the form of the dependence of $e(\mu)$ and $m(\mu)$ is.

Before we go on to this, we can answer one question: if μ is arbitrary, what value should we choose for it? The (approximate) answer is easy. We have already seen that in a perturbative calculation we will get logarithms of p^2/μ^2 where p is a momentum in the problem. The logarithms arise from the loop integrations, and typically we can get one logarithm per loop. Thus we really have an expansion in powers of $\alpha \ln(p^2/\mu^2)$. If this is going to be a useful expansion, we would like $\alpha \ln(p^2/\mu^2)$ to be small. This means that $\ln(p^2/\mu^2)$ should not be large. That is, μ should be the same rough size as the typical momenta p in the problem.

9 The renormalization group

The action for the bare theory is

$$S = \int d^d x \left[\bar{\psi}_0(x)\{ i\not\partial - Qe_0 \not{A}(x) - m_0 \} \psi_0(x) - \tfrac{1}{4} F_0^{\mu\nu}(x) F_{0,\mu\nu}(x) \right]. \qquad (101)$$

Our definition is that none of the quantities appearing here depend on the scale μ. Then Green functions for the bare theory do not depend on μ either. (Here we should take Fourier transforms without a factor of $\tilde{\mu}^{-2\epsilon}$ in order to avoid introducing any μ dependence if we want to calculate Green functions for the bare theory in momentum space.)

Now we define renormalized quantities by

$$\begin{aligned} \psi_0 &= \tilde{\mu}^{-\epsilon} \sqrt{Z_\psi}\, \psi \\ A_0^\mu &= \tilde{\mu}^{-\epsilon} \sqrt{Z_A}\, A^\mu \\ m_0 &= Z_m\, m \\ e_0 &= \tilde{\mu}^{\epsilon} Z_e\, e, \end{aligned} \qquad (102)$$

so that

$$\begin{aligned} S = \tilde{\mu}^{-2\epsilon} \int d^d x \Big[& \bar{\psi}(x)\{ Z_\psi i\not\partial - Z_\psi \sqrt{Z_A} Z_e Q e \not{A}(x) - Z_\psi Z_m m \} \psi(x) \\ & - \tfrac{1}{4} Z_A F^{\mu\nu}(x) F_{\mu\nu}(x) \Big]. \end{aligned} \qquad (103)$$

Consider the dimensionality of the various quantities that appear in the action. The quantities $\{\psi_0, A_0^\mu, m_0, e_0\}$ have dimensions

$$\{(mass)^{3/2-\epsilon}, (mass)^{1-\epsilon}, (mass)^1, (mass)^\epsilon\}. \tag{104}$$

The functions Z are, by defintion, dimensionless. Thus we have defined the renormalized quantities $\{\psi, A^\mu, m, e\}$ so that they have dimension

$$\{(mass)^{3/2}, (mass)^1, (mass)^1, (mass)^0\}. \tag{105}$$

With $\overline{\rm MS}$ renormalization, the Zs depend only on the the renormalized coupling g but not separately on μ/m.(We have seen this at least at first order). But g_0 does not depend on μ. Thus the only way that $e_0 = \tilde\mu^\epsilon Z_g\, e$, can hold is that e depends on μ. Similarly m must depend on μ. I argued in the preceeding section that the requirement that physical observables must be independent of μ requires that e and m must depend on μ. The easy way to see how e and m depend on μ is to use the fact that the bare theory does not have any μ dependence and use "the bare quantities are independent of μ" instead of "physical observables are independent of μ."

10 The beta function

Consider, then, the coupling e. It is a function of the renormalization scale μ. If we have not yet taken $\epsilon \to 0$, it can also depend on ϵ. For a given ϵ, it also depends on what theory we have, as represented, say, by the value obtained in some physical measurement. Leaving this last quantity implicit, we may say define the "beta function" by

$$\beta(e,\epsilon) = \frac{\partial e(\mu,\epsilon)}{\partial \ln\mu}. \tag{106}$$

Normally, we work at $\epsilon = 0$. Thus we define

$$\beta(e) = \frac{\partial e(\mu)}{\partial \ln\mu}. \tag{107}$$

where

$$\beta(e) = \beta(e,0). \tag{108}$$

It must be the case that $\beta(e,\epsilon)$ has a finite $\epsilon \to 0$ limit.

Now we calculate $\beta(e,\epsilon)$ by differentiating e_0 with respect to μ:

$$0 = \frac{\partial \ln e_0}{\partial \ln \mu} = \epsilon + \frac{\partial \ln Z_e(e,\epsilon)}{\partial e}\beta(e,\epsilon) + \frac{1}{e}\beta(e,\epsilon). \tag{109}$$

Thus

$$\beta(e,\epsilon) = -\frac{\epsilon\, e}{1 + e\,\partial \ln Z_e(e,\epsilon)/\partial e} \tag{110}$$

With a little manipulation, we can see that the beta function takes a remarkably simple form. First, expend Z_e in powers of e:

$$Z_e \sim 1 + \sum_{n=1}^{\infty} Z_n(\epsilon) e^{2n}. \tag{111}$$

Now apply the definition of $\overline{\mathrm{MS}}$ renormalization: the coefficients $Z_n(\epsilon)$ contain *only* powers of $1/\epsilon$. We have seen that $Z_1(\epsilon)$ has a $1/\epsilon$ term only. With a little analysis of how loop integrals work, we can see that $Z_n(\epsilon)$ needs terms with $1/\epsilon$, $1/\epsilon^2$, ..., $1/\epsilon^n$. Thus

$$Z_e(e,\epsilon) \sim 1 + \sum_{n=1}^{\infty}\sum_{j=1}^{n} Z_{n,j}\,\frac{e^{2n}}{\epsilon^j}. \tag{112}$$

Thus

$$\ln Z \sim \sum_{n=1}^{\infty} e^{2n}\{Z_{n,1}\,\frac{1}{\epsilon} + \mathcal{O}(1/\epsilon^2)\} \tag{113}$$

where I have omitted writing the terms with higher powers of $1/\epsilon$. Then

$$1 + e\frac{\partial \ln Z}{\partial e} \sim 1 + \sum_{n=1}^{\infty} 2n\, e^{2n}\{Z_{n,1}\,\frac{1}{\epsilon} + \mathcal{O}(1/\epsilon^2)\}. \tag{114}$$

Then the perturbative expansion of β has the form

$$\beta(e,\epsilon) \sim -\epsilon e\left[1 - \sum_{n=1}^{\infty} 2n\, e^{2n}\{Z_{n,1}\,\frac{1}{\epsilon} + \mathcal{O}(1/\epsilon^2)\}.\right]. \tag{115}$$

That is,

$$\beta(e,\epsilon) \sim -\epsilon e + \sum_{n=1}^{\infty} 2n\, e^{2n+1}\{Z_{n,1} + \mathcal{O}(1/\epsilon)\}. \tag{116}$$

If $\beta(e,\epsilon)$ is to have a finite $\epsilon \to 0$ limit, the $1/\epsilon$, $1/\epsilon^2$, ... terms here must cancel. Thus

$$\beta(e,\epsilon) \sim -\epsilon e + \sum_{n=1}^{\infty} 2n\, e^{2n+1} Z_{n,1}. \tag{117}$$

and

$$\beta(e) \sim \sum_{n=1}^{\infty} 2n\, e^{2n+1} Z_{n,1}. \tag{118}$$

That is to say, the coefficient of e^{2n+1} in the expansion of $\beta(e)$ is obtained by calculating the $1/\epsilon$ term in $Z_n(\epsilon)$ and multiplying by $2n$.

11 The running coupling

Before going on to more about the renormalization group, let's look at the behavior of the running coupling.

Consider first the case $\epsilon > 0$. Then for $e^2 \ll 1$ we have the approximate equation

$$\frac{\partial e(\mu, \epsilon)}{\partial \ln \mu} = -\epsilon e. \tag{119}$$

The solution of this with a boundary condition at μ_0 is

$$e(\mu, \epsilon) = e(\mu_0, \epsilon) \exp(-\epsilon \ln(\mu/\mu_0)). \tag{120}$$

That is

$$e(\mu, \epsilon) = e(\mu_0, \epsilon)(\mu/\mu_0)^{-\epsilon}. \tag{121}$$

As the scale μ increases, the coupling decreases like a power of μ. We say that the renormalization group has an ultraviolet-stable fixed point at $e = 0$.

A similar situation could occur at $\epsilon = 0$ if the beta function has a zero at some point $e_\star$. That is, suppose that $\beta(e_\star) = 0$ and

$$\beta(e) = \beta_0\, (e - e_\star) + \mathcal{O}((e - e_\star)^2) \tag{122}$$

Then if $e(\mu) \approx e_\star$ we have, approximately,

$$\frac{\partial e(\mu)}{\partial \ln \mu} = \beta_0\, (e(\mu) - e_\star). \tag{123}$$

The solution of this is

$$e(\mu) = e_\star + [e(\mu_0) - e_\star] \left(\frac{\mu}{\mu_0}\right)^{\beta_0}. \tag{124}$$

If $\beta_0 > 0$, we have an ultraviolet-stable fixed point at $e = e_\star$. The coupling is driven to $e_\star$ as μ increases. It approaches $e_\star$ like a power of μ. If $\beta_0 < 0$, then the coupling is driven *away* from $e_\star$ as μ increases and as soon as it gets far away we need to include a more accurate representation of the beta function. In this case, we have in infrared stable fixed point. The coupling is driven to $e_\star$ as μ increases. It approaches $e_\star$ like a power of $1/\mu$.

Now, let's consider the real case at $\epsilon = 0$ in quantum electrodynamics. There

$$\beta(e) = 2e^3 Z_{1,1}^{(e)} + \cdots = +\frac{1}{3}\frac{e^3}{4\pi^2} + \mathcal{O}(e^5) \tag{125}$$

We have the approximate renormalization group equation

$$\frac{de(\mu)}{d\ln\mu} = \beta_1\, e(\mu)^3. \tag{126}$$

with $\beta_1 = 1/(12\pi^2)$. The solution of this is

$$e(\mu)^2 = \frac{e(\mu_0)^2}{1 - \beta_1\, e(\mu_0)^2\, \ln(\mu^2/\mu_0^2)}. \tag{127}$$

We might rewrite this as

$$\alpha(\mu)^2 = \frac{\alpha(\mu_0)}{1 - (\alpha(\mu_0)/(3\pi))\, \ln(\mu^2/\mu_0^2)}. \tag{128}$$

We have an infrared stable fixed point at $e = 0$. As μ decreases, $e(\mu)$ goes to zero. It goes to zero slowly, proportionally to $1/\ln(\mu^2)$.

As μ increases, $e(\mu)$ increases. If μ gets to be very large, we need more terms in the beta function and the form of the solution given above is no longer valid. In particular, there is an apparent pole in the solution above if we go to very large μ, but the existence of such a pole is not to be believed because it arises from solving an approximation to the renormalization group equation outside the region of validity of the approximation.

It is worth noting that in quantum chromodynamics we have the same situation, except that $\beta_1 < 0$. Thus zero coupling is an ultraviolet stable fixed point. The effective coupling $\alpha_s(\mu)$ is large for $\mu \sim 100$ MeV. But the $\alpha_s(\mu)$ becomes small for large μ, with $\alpha_s(100 \text{ GeV}) \approx 0.1$.

12 Mass and field strength renormalization

The mass parameter in the lagrangian also depends on the scale μ. We have

$$m(\mu, \epsilon) = Z_m(e(\mu, \epsilon), \epsilon)^{-1}\, m_0 \tag{129}$$

so

$$\frac{\partial \ln m(\mu,\epsilon)}{\partial \ln \mu} = \gamma_m(e,\epsilon) \tag{130}$$

where

$$\gamma_m(e,\epsilon) = -\frac{\partial \ln Z_m(e(\mu),\epsilon)}{\partial e}\,\beta(e,\epsilon). \tag{131}$$

Let

$$Z_m(e,\epsilon) \sim 1 + \sum_{n=1}^{\infty}\sum_{j=1}^{n} Z_{n,j}^{(m)}\,\frac{e^{2n}}{\epsilon^j}. \tag{132}$$

Then

$$\ln Z_m \sim \sum_{n=1}^{\infty} e^{2n}\{Z_{n,1}^{(m)}\,\frac{1}{\epsilon} + \mathcal{O}(1/\epsilon^2)\} \tag{133}$$

and

$$\begin{aligned}\gamma_m(e,\epsilon) &= -\sum_{n=1}^{\infty} 2n\,e^{2n-1}\{Z_{n,1}^{(m)}\,\frac{1}{\epsilon} + \mathcal{O}(1/\epsilon^2)\}\,[-\epsilon e + \beta(e)]. \\ &= \sum_{n=1}^{\infty} 2n\,e^{2n}\{Z_{n,1}^{(m)} + \mathcal{O}(1/\epsilon)\} \end{aligned}\tag{134}$$

The function $\gamma_m(e,\epsilon)$ has to have a finite $\epsilon \to 0$ limit, so the $1/\epsilon$ terms must cancel. Thus $\gamma_m(e,\epsilon)$ is independent of ϵ and is given at each order of perturbation theory by the $1/\epsilon$ term in Z_m:

$$\gamma_m(e,\epsilon) = \gamma_m(e) = \sum_{n=1}^{\infty} 2n\,e^{2n}\,Z_{n,1}^{(m)}. \tag{135}$$

Let's also look at the field strength renormalization at this time. We have

$$\psi = \tilde{\mu}^{\epsilon} Z_{\psi}(e(\mu,\epsilon),\epsilon)^{-1/2}\,\psi_0 \tag{136}$$

with a similar equation for the field A^{μ}. To find out how anything with a factor of ψ changes when we change μ, we will need

$$\gamma_{\psi}(e,\epsilon) \equiv -\frac{\partial}{\partial \ln \mu}\,\ln\left(\mu^{\epsilon} Z_{\psi}(e(\mu,\epsilon),\epsilon)^{-1/2}\right) \tag{137}$$

That is

$$\gamma_{\psi}(e,\epsilon) = -\epsilon + \frac{1}{2}\frac{\partial \ln Z_{\psi}(e(\mu),\epsilon)}{\partial e}\,\beta(e,\epsilon). \tag{138}$$

As before, we let

$$Z_\psi(e,\epsilon) \sim 1 + \sum_{n=1}^{\infty}\sum_{j=1}^{n} Z_{n,j}^{(\psi)}\,\frac{e^{2n}}{\epsilon^j}. \tag{139}$$

Then

$$\begin{aligned}\gamma_\psi(e,\epsilon) &= -\epsilon + \frac{1}{2}\sum_{n=1}^{\infty} 2n\,e^{2n-1}\{Z_{n,1}^{(\psi)}\,\frac{1}{\epsilon} + \mathcal{O}(1/\epsilon^2)\}\,[-\epsilon e + \beta(e)]. \\ &= -\epsilon - \sum_{n=1}^{\infty} n\,e^{2n}\{Z_{n,1}^{(\psi)} + \mathcal{O}(1/\epsilon)\}\end{aligned} \tag{140}$$

The function $\gamma_\psi(e,\epsilon)$ has to have a finite $\epsilon \to 0$ limit, so the $1/\epsilon$ terms must cancel. Thus $\gamma_\psi(e,\epsilon)$ has a term $+\epsilon$ plus an ϵ independent part:

$$\gamma_\psi(e,\epsilon) = -\epsilon + \gamma_\psi(e) \tag{141}$$

The ϵ independent $\gamma_\psi(e)$ is given at each order of perturbation theory by the $1/\epsilon$ term in Z_ψ:

$$\gamma_\psi(e) = -\sum_{n=1}^{\infty} n\,e^{2n}\,Z_{n,1}^{(\psi)}. \tag{142}$$

Similarly $\gamma_A(e,\epsilon) = \epsilon + \gamma_A(e)$ with

$$\gamma_A(e) = -\sum_{n=1}^{\infty} n\,e^{2n}\,Z_{n,1}^{(A)}. \tag{143}$$

13 The running mass

We have seen that

$$\frac{\partial \ln m(\mu)}{\partial \ln \mu} = \gamma_m(e(\mu)). \tag{144}$$

Let's see what would happen if the coupling has an ultraviolet fixed point at $e = e_\star$. Then as μ increases, $e(\mu)$ approaches $e_\star$ quite quickly, like a power. That means that (as soon as e is sufficiently near to $e_\star$, the equation for the evolution of m can be approximated by

$$\frac{\partial \ln m(\mu)}{\partial \ln \mu} = \gamma_m(e_\star). \tag{145}$$

The solution of this is

$$m(\mu) = m(\mu_0)\left(\frac{\mu}{\mu_0}\right)^{\gamma_m}. \tag{146}$$

Thus m approaches either zero or infinity (depending on the sign of γ_m) with a power behavior.

In the real case of quantum electrodynamics, the physical coupling is near to an infrared stable fixed point at $e = 0$. In this case, the behavior of $m(\mu)$ is quite different because γ_m approaches zero as e approaches the fixed point. We have

$$\begin{aligned} \frac{\partial \ln m(\mu)}{\partial \ln \mu} &= \gamma_m(e(\mu)). \\ &\sim \gamma_m^{(1)} e(\mu)^2 \\ &\sim \frac{\gamma_m^{(1)}}{(1/e(\mu_0)^2) - 2\beta_1 \ln(\mu/\mu_0)} \end{aligned} \tag{147}$$

Our solution for the running coupling can be rewritten so that the equation becomes

$$\frac{\partial \ln m(\mu)}{\partial \ln \mu} \sim -\frac{\gamma_m^{(1)}}{2\beta_1 \ln(\mu/\Lambda)} \tag{148}$$

This step defines Λ as a measure of the size of the coupling. For quantum electrodynamics, Λ is a very large mass scale, much bigger than any μ of practical interest. Thus $\ln(\mu/\Lambda)$ is negative. The solution of this is

$$\ln\left(\frac{m(\mu)}{m(\mu_0)}\right) = -\frac{\gamma_m^{(1)}}{2\beta_1}\,\ln\left(\frac{\ln(\mu/\Lambda)}{\ln(\mu_0/\Lambda)}\right). \tag{149}$$

or

$$m(\mu) = m(\mu_0)\left(\frac{\ln(\mu/\Lambda)}{\ln(\mu_0/\Lambda)}\right)^{-\gamma_m^{(1)}/(2\beta_1)}. \tag{150}$$

Another way to write this is

$$m(\mu) = m(\mu_0)\left(\frac{e(\mu_0)}{e(\mu)}\right)^{-\gamma_m^{(1)}/\beta_1}. \tag{151}$$

Thus $m(\mu)$ varies slowly as μ changes, like a power of $e(\mu)$, which itself doesn't change very quickly.

Recall that e doesn't change quickly because the derivative of e with respect to $\ln\mu$ is proportional to $e \times e^2$ and e^2 is small. It is instructive to compare to the case of quantum chromodynamics. There β_1 is negative, so that the coupling gets smaller as we go toward larger μ. But also, the coupling is larger than in quantum electrodynamics. In fact, Λ is on the order of 100 MeV. Here we can more clearly see the distinction between a fixed point at a non-zero coupling and a fixed point a zero coupling. In the first case, m is proportional to a power of μ while in the second case m is proportional to a power of $\ln\mu$.

14 Renormalization group for Green functions

Consider a Green function written in coordinate space. We will try a three point $e\bar{e}\gamma$ Green function just for definiteness:

$$G(x_1, x_2, x_3; e(\mu), m(\mu), \mu) = \langle\Omega|\psi(x_1)\bar{\psi}(x_2)A^\mu(x_3)|\Omega\rangle \tag{152}$$

How does this depend on μ? We have

$$G_0(x_1, x_2, x_3; e_0, m_0) = \tilde{\mu}^{-3\epsilon} Z_\psi(e,\epsilon) Z_A(e,\epsilon)^{1/2} G(x_1, x_2, x_3; e(\mu), m(\mu), \mu) \tag{153}$$

The bare Green function does not depend on μ. Therefore

$$0 = \frac{d}{d\ln\mu}\tilde{\mu}^{-3\epsilon} Z_\psi(e,\epsilon) Z_A(e,\epsilon)^{1/2} G(x_1, x_2, x_3; e(\mu), m(\mu), \mu) \tag{154}$$

In taking the derivative, e_0, m_0, and ϵ are held fixed, but then e and m vary and we must differentiate the explicit μ dependence of G:

$$\begin{aligned} 0 &= \left\{+2\gamma_\psi(e,\epsilon) + \gamma_A(e,\epsilon) + \beta(e,\epsilon)\frac{\partial}{\partial e} + \gamma_m(e) m\,\frac{\partial}{\partial m} + \frac{\partial}{\partial\ln\mu}\right\} \\ &\quad \times G(x_1, x_2, x_3; e(\mu), m(\mu), \mu) \end{aligned} \tag{155}$$

Taking the limit $\epsilon \to 0$ this becomes

$$\begin{aligned} 0 &= \left\{2\gamma_\psi(e) + \gamma_A(e) + \beta(e)\frac{\partial}{\partial e} + \gamma_m(e) m\,\frac{\partial}{\partial m} + \frac{\partial}{\partial\ln\mu}\right\} \\ &\quad \times G(x_1, x_2, x_3; e(\mu), m(\mu), \mu). \end{aligned} \tag{156}$$

This was for a three-point function. In general, there is a γ_ψ for each ψ or $\bar{\psi}$ and a γ_A for each A^μ.

Exercise: For an observable quantity σ, we get the same equation without γ_ψ and γ_A. Then the solution of the equation is

$$\sigma(e(\mu), m(\mu), \mu) = \sigma(e(\mu_0), m(\mu_0), \mu_0) \tag{157}$$

What is the solution with nonzero γ_ψ and γ_A? (You can make appropriate approximations).

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15 Solution for amputated Green functions

Putting the renormalization group equation for Green functions into momentum space and applying it to amputated Green functions with n_γ photon legs and n_ψ quark or antiquark legs, we have

$$\begin{aligned} 0 \;&=\; \left\{\mu\,\frac{\partial}{\partial\mu} + \beta(e)\frac{\partial}{\partial e} + \gamma_m(e)m\,\frac{\partial}{\partial m} - n_\psi\gamma_\psi(e) - n_A\gamma_A(e)\right\} \\ &\quad\times G(p_1,\ldots,p_N; e(\mu), m(\mu), \mu). \end{aligned} \tag{158}$$

(The signs of the γ_ψ and γ_A terms gets reversed because we are considering the amputated Green functions. This happens because we have to divide by a two point function for each external leg to amputate it.) The solution is

$$\begin{aligned} G(p_1,\ldots,p_N; e(\mu), m(\mu), \mu) \;&=\; \exp\left(\int_{\mu_0}^{\mu}\frac{d\bar\mu}{\bar\mu}\,\left[n_\psi\gamma_\psi(e(\mu)) + n_A\gamma_A(e(\mu))\right]\right) \\ &\quad\times G(p_1,\ldots,p_N; e(\mu_0), m(\mu_0), \mu_0) \end{aligned} \tag{159}$$

This is without approximation. We can rewrite a factor like

$$\exp\left(\int_{\mu_0}^{\mu}\frac{d\bar\mu}{\bar\mu}\;n\gamma(e(\bar\mu))\right) \tag{160}$$

approximately by expanding the objects involved to lowest order in perturbation theory. Let us adopt the notation

$$\gamma(e) = \gamma_1\alpha + \cdots \tag{161}$$

and

$$\frac{d\alpha(\mu)}{d\ln\mu} \equiv \tilde{\beta}(\alpha) = \beta_1 \alpha^2 + \cdots \tag{162}$$

Then we have

$$\begin{aligned}
\exp\left(\int_{\mu_0}^{\mu} \frac{d\bar{\mu}}{\bar{\mu}}\; n\gamma(e(\mu))\right) &= \exp\left(\int_{\mu_0}^{\mu} \frac{d\bar{\mu}}{\bar{\mu}}\; n\gamma_1\alpha(\mu)\right) \\
&= \exp\left(\int_{\alpha(\mu_0)}^{\alpha(\mu)} \frac{d\bar{\alpha}}{\beta_1\bar{\alpha}^2}\; n\gamma_1\bar{\alpha}\right) \\
&= \exp\left(\frac{\gamma_1}{\beta_1}\int_{\alpha(\mu_0)}^{\alpha(\mu)} \frac{d\bar{\alpha}}{\bar{\alpha}}\right) \\
&= \exp\left(\frac{n\gamma_1}{\beta_1}\ln\left(\frac{\alpha(\mu)}{\alpha(\mu_0)}\right)\right) \\
&= \left(\frac{\alpha(\mu)}{\alpha(\mu_0)}\right)^{n\gamma_1/\beta_1}
\end{aligned} \tag{163}$$

This is essentially the same result that we derived for $m(\mu)$ in a slightly different way. In a theory that has a fixed point at a non-zero value of the coupling, we would have a power of μ. In the case of QED, we get a power of a constant plus $\ln(\mu)$.

16 Application of the R-G equation

Now I would like to apply the renormalization group equation to the following physical problem. We consider an amputated Green function with external momenta $p_1^\nu, \ldots, p_n^\nu$. (We could consier other quantities that are made by multiplying or dividing Green functions by one another, but a simple amputeated Green function will serve as a good example.) We can start with the momenta all of order some reference scale μ_0. (For particle physics, μ_0 might be the mass of the Z boson. For field theory models in condensed matter physics, the reference momentum scale might be an inverse nanometer.) Now we ask what happens when we scale all of the momenta by a large factor, either toward the infrared or the ultraviolet. That is, we take

$$p_i^\nu \sim \mu \tag{164}$$

where $\mu \ll \mu_0$ or else $\mu \gg \mu_0$. We ask how

$$G(p_1, \ldots, p_n; e(\mu_0), m(\mu_0), \mu_0) \tag{165}$$

behaves as we change the scale of the $p_i^{\nu_1}$. Note that we are keeping the theory absolutely fixed here: we have arguments $e(\mu_0), m(\mu_0), \mu_0$ of G.

We can use the renormalization group to help us. We know that we shold not calculate with a renormalization scale μ_0 if the momentum scale is very different. Therefore, let's change the renormalization scale to μ. We have

$$\begin{aligned}
&G(p_1,\dots,p_n;e(\mu_0),m(\mu_0),\mu_0) = \\
&\quad \exp\left(-\int_{\mu_0}^{\mu}\frac{d\bar\mu}{\bar\mu}\;[n_\psi\gamma_\psi(e(\mu)) + n_A\gamma_A(e(\mu))]\right) \\
&\quad \times G(p_1,\dots,p_n;e(\mu),m(\mu),\mu)
\end{aligned} \tag{166}$$

Now we can apply dimensional analysis. I claim that the dimension of G is

$$d_G = 4 - n_A - 3n_\psi/2. \tag{167}$$

We can prove this later. For the moment, let's just apply it. We have

$$\begin{aligned}
&G(p_1,\dots,p_n;e(\mu),m(\mu),\mu) = \\
&\quad \mu^{4-n_A-3n_\psi/2}\;G(p_1/\mu,\dots,p_n/\mu;e(\mu),m(\mu)/\mu,1)
\end{aligned} \tag{168}$$

On the right hand side, we have G a dimensionless function of dimensionless variables times a factor μ^{d_G} that carries the dimension. Then

$$\begin{aligned}
&G(p_1,\dots,p_n;e(\mu_0),m(\mu_0),\mu_0) = \\
&\quad \exp\left(-\int_{\mu_0}^{\mu}\frac{d\bar\mu}{\bar\mu}\;\left\{n_\psi[3/2+\gamma_\psi(e(\mu))] + n_A[1+\gamma_A(e(\mu))]\right\}\right) \\
&\quad \times(\mu/\mu_0)^4\mu_0^{4-n_A-3n_\psi/2}\;G(p_1/\mu,\dots,p_n/\mu;e(\mu),m(\mu)/\mu,1).
\end{aligned} \tag{169}$$

What lessons do we learn. First, there is a factor $(\mu/\mu_0)^{-n_A-3n_\psi/2}$ that comes from the dimensionalities 3/2 and 1 of the fields ψ and A^ν respectively. I have put these into the integral in the exponential. We see that the 3/2 gets changed to $3/2+\gamma_\psi$. For this reason, we call γ_ψ the anomalous dimension of ψ. Similarly the 1 gets changed to $1+\gamma_A$, so that we call γ_A the anomalous dimension of A^ν. Thus the scaling that we might have expected because of the dimensions of the field operators gets modified by loop graphs. Second, a rather trivial scale dependent factor remains $(\mu/\mu_0)^4$ because there is a term "4" in the dimension of G. Third, there is a fixed factor $\mu_0^{4-n_A-3n_\psi/2}$ that carries the dimensions of G. Finally, there is a dimensionless factor

$$G(p_1/\mu,\dots,p_n/\mu;e(\mu),m(\mu)/\mu,1). \tag{170}$$

Typically, this factor does not have any large logarithms in it, so a perturbative calculation should be OK as long as $e(\mu)$ is small. This factor *does*, however, depend on μ even when we hold the p_i^ν/μ fixed. That is because the dimensionless parameters $e(\mu)$ and $m(\mu)/\mu$ change with μ.

Exercise: Prove that the dimension of an amputated Green function in QED with n_ψ electron or positron legs and n_A photon legs is $d_G = 4 - n_A - 3n_\psi/2$. Hint: First consider tree diagrams and just count the dimensions coming from the Feynman rules. Next, show that adding any number of loops does not change the dimensionality of a graph.

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17 Renormalization group flows

We see that aside from the exponential factor, our amputated Green function is controlled by the function

$$G(p_1/\mu, \ldots, p_n/\mu; e(\mu), m(\mu)/\mu, 1). \tag{171}$$

where we are supposing that all of the p_1^ν are of the same order and we have chosen μ to be of this order. What happens if we start out with a big μ (say 100 GeV) and move toward smaller values of μ? Then $e(\mu)$ and $m(\mu)/\mu$ change according to

$$\begin{aligned} \Delta e &= -\beta(e)\ \Delta \ln(1/\mu) \\ \Delta\left(\frac{m}{\mu}\right) &= (1-\gamma_m(e))\ \left(\frac{m}{\mu}\right) \Delta \ln(1/\mu). \end{aligned} \tag{172}$$

A convenient way to think about this is to make a plot of m/μ versus e and, for each point, to draw a little arrow indicating how m/μ and e change as μ decreases. The arrows are tangent to the actual paths $\{m(\mu)/\mu, e(\mu)\}$. There could be different paths with the arrows as their tangents that correspond to different versions of QED. (That is, different paths correspond to different values of the physical electron mass and fine structure constant.)

What this picture shows is that, at least in the small e region, e decreases (slowly) as we move toward the infrared, while m/μ increases. As long as m/μ is small, its rate of increase is small, but as it becomes larger its rate of increase becomes larger. It is the "1" in $(1 - \gamma_m(e))$ that is important here. The 1 corresponds to the $1/\mu$ in m/μ. We are learning that for large μ, the dimensionless parameter m/μ is small. In this case we can replace it by 0 in Eq. (171). But if we decrease μ until it is of order m, then we can't neglect m and the theory changes character. For $m(\mu)/\mu \ll 1$ we are in a scaling regime in which the exponential factor in Eq. (169) is controlling the behavior of the Green function on the left hand side while $G(p_1/\mu, \ldots, p_n/\mu; e(\mu), m(\mu)/\mu, 1)$ changes only slowly. But when we reach $m(\mu)/\mu \sim 1$, this behavior changes completely. (There are analogies in condensed matter physics. See Peskin and Schroeder.)

In order to see what happens in a more complicated situation, consider the following problem.

Exercise: Suppose that we add to the lagrangian for QED a term

$$\Delta\mathcal{L} = \frac{\lambda}{M^2}(\bar{\psi}\psi)^2. \tag{173}$$

I write the coupling as λ/M^2 in order to emphasize that it has dimension $1/mass^2$. There is a contribution to the $e\bar{e}\gamma$ amputated Green function at one loop, order $\lambda e/M^2$. Calculate (using dimensional regularization) the divergent part of this graph and thus determine the nature of the counter-terms necessary to renormalize it.

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18 Adding more operators to the lagrangian

Let's see what happens if we add more terms to the lagrangian. In the bare theory, let the added terms have the form

$$\Delta\mathcal{L} = \sum_i \frac{\lambda_i^{(0)}}{M^{d_i}}\, \mathcal{O}_i^{(0)} \tag{174}$$

where the operators are made from products of the fields and their derivatives or factors of m. For example

$$(\bar{\psi}\psi)^2, \quad m\,\bar{\psi}[\gamma^\mu, \gamma^\nu]\psi F_{\mu\nu}, \quad \bar{\psi}\gamma_\mu\psi\,(\partial_\nu F^{\mu\nu}) \tag{175}$$

(We count all factors of m as part of the operators as a technical trick to aid in counting dimensions in the analysis that follows.) In the starting version, Eq. (174), we should take all of the operators to be the bare versions. We write the couplings in the form $\lambda_i^{(0)}/M^{d_i}$ in order to display the dimensionality:

$$d_i = \frac{3}{2} n_i(\psi) + n_i(A) - 4 \tag{176}$$

where $n_i(\psi)$ is the number of ψ or $\bar{\psi}$ fields in $\mathcal{O}_i$ and $n_i(A)$ the number of A fields. This makes the $\lambda_i^{(0)}$ dimensionless in 4 space-time dimensions. Then M is some convenient reference mass.

Now we need to renormalize. For this purpose, let

$$\mathcal{O}_i^{(0)} = (\tilde{\mu})^{-[n_i(\psi)+n_i(A)]\epsilon} Z_\psi^{n_i(\psi)/2} \, Z_A^{n_i(A)/2} \mathcal{O}_i. \tag{177}$$

and let

$$\lambda_i^{(0)} = (\tilde{\mu})^{n_i \epsilon} \; [\lambda_i + \Delta_i(e,\lambda)] \tag{178}$$

where

$$n_i = [n_i(\psi) + n_i(A) - 2] \tag{179}$$

and where the functions $\Delta_i(e,\lambda)$ depend on e and all of the λ_j. Then

$$\begin{aligned} \int d^{4-2\epsilon}x \; \Delta\mathcal{L} \;\; &= \;\; \tilde{\mu}^{2\epsilon} \int d^{4-2\epsilon}x \Big\{ \sum_i \frac{\lambda_i}{M^{d_i}} \; \mathcal{O}_i \\ &\qquad + \sum_i \frac{\lambda_i}{M^{d_i}} \; [Z_\psi^{n_i(\psi)/2} \, Z_A^{n_i(A)/2} - 1] \mathcal{O}_i \\ &\qquad + \sum_i \frac{\Delta_i}{M^{d_i}} \; Z_\psi^{n_i(\psi)/2} \, Z_A^{n_i(A)/2} \mathcal{O}_i \Big\}. \end{aligned} \tag{180}$$

This gives us plenty of counter-terms, which we will define following the $\overline{\mathrm{MS}}$ prescription.

The renormalization of the QED coupling e can be cast in this same scheme,

$$e^{(0)} = (\tilde{\mu})^{n_e \epsilon} \; [e + \Delta_e(e,\lambda)] \tag{181}$$

where $n_e = 1$. Then we have counterterms

$$\Big\{ e[Z_\psi \, Z_A^{1/2} - 1] + \Delta_e \; Z_\psi \, Z_A^{1/2} \Big\} \, \bar{\psi}\gamma^\mu \psi A_\mu. \tag{182}$$

The mass renormalization can remain in the same form as in our earlier analysis, with

$$m_0 = Z_m m \tag{183}$$

leading to counterterms

$$[Z_\psi\, Z_A^{1/2} Z_m - 1]\; m\, \bar\psi\psi. \tag{184}$$

Of particular interest is how the new couplings λ_i change when we change the scale. We just modify our previous derivation for how e changes. We define

$$\begin{aligned} \beta(e,\lambda,\epsilon) &= \frac{de(\mu)}{d\ln\mu} \\ \beta_i(e,\lambda,\epsilon) &= \frac{d\lambda_i(\mu)}{d\ln\mu} \end{aligned} \tag{185}$$

Then

$$\begin{aligned} 0 &= \frac{\partial \ln \lambda_i^{(0)}}{\partial \ln \mu} \\ &= n_i\epsilon + \frac{1}{[\lambda_i + \Delta_i(e,\lambda)]} \\ &\quad \times \left(\beta_i(e,\lambda,\epsilon) + \frac{\partial \Delta_i(e,\lambda)}{\partial e}\,\beta(e,\lambda,\epsilon) + \sum_j \frac{\partial \Delta_i(e,\lambda)}{\partial \lambda_j}\beta_j(e,\lambda,\epsilon)\right) \end{aligned} \tag{186}$$

That is

$$-n_i\epsilon\,[\lambda_i + \Delta_i(e,\lambda)] = \beta_i(e,\lambda,\epsilon) + \frac{\partial \Delta_i(e,\lambda)}{\partial e}\,\beta(e,\lambda,\epsilon) + \sum_j \frac{\partial \Delta_i(e,\lambda)}{\partial \lambda_j}\beta_j(e,\lambda,\epsilon) \tag{187}$$

There is an equatin for the renormalization of e of the same structure. Just substitute $\lambda_0 = e$. The corresponding dimension is $n_0 = 1$.

Now we put in the structure of the counter terms:

$$\Delta_i(e,\lambda) = \frac{1}{\epsilon}\Delta_i^{(1)}(e,\lambda) + \cdots \tag{188}$$

where the term displayed collects all of the terms in the perturbative expansion of $\Delta_i(e,\lambda)$ that have factors of $1/\epsilon$ while the $+\cdots$ indicates the other terms, which have factors $1/\epsilon^2$, $1/\epsilon^3$, *etc.* We insist that the β_i must have a finite limit as $\epsilon \to 0$. Then the equation works if $\beta_i(e,\lambda,\epsilon)$ has a single term proportional to ϵ plus an ϵ independent piece:

$$\begin{aligned} \beta_i(e,\lambda,\epsilon) &= -n_i\lambda_i\epsilon + \beta_i(e,\lambda). \\ \beta(e,\lambda,\epsilon) &= -e\epsilon + \beta(e,\lambda). \end{aligned} \tag{189}$$

With this ansatz, the terms proportional to ϵ in Eq. (187) match and the terms proportional to ϵ^0 are

$$-n_i\Delta_i^{(1)}(e,\lambda) = \beta_i(e,\lambda) - \frac{\partial\Delta_i^{(1)}(e,\lambda)}{\partial e}\,e - \sum_j \frac{\partial\Delta_i^{(1)}(e,\lambda)}{\partial\lambda_j}n_j\lambda_j \tag{190}$$

That is

$$\beta_i(e,\lambda) = -n_i\Delta_i^{(1)}(e,\lambda) + e\,\frac{\partial\Delta_i^{(1)}(e,\lambda)}{\partial e} + \sum_j n_j\lambda_j\,\frac{\partial\Delta_i^{(1)}(e,\lambda)}{\partial\lambda_j}\ . \tag{191}$$

The equation for the beta function that controls the evolution of e is, similarly,

$$\beta(e,\lambda) = -\Delta_e^{(1)}(e,\lambda) + e\,\frac{\partial\Delta_e^{(1)}(e,\lambda)}{\partial e} + \sum_j n_j\lambda_j\,\frac{\partial\Delta_e^{(1)}(e,\lambda)}{\partial\lambda_j}\ . \tag{192}$$

If we leave out the third term on the right hand side, this is just the result we had before, simply written in a different way.

For the evolution of the mass, we have an equation that is the same as in our previous analysis except that one can allow, to begin with, that the anomalous dimension might depend on more variables:

$$\frac{\partial\ln m(\mu)}{\partial\ln\mu} = \gamma_m(e,\lambda) \tag{193}$$

We get the anomalous dimension from the $1/\epsilon$ parts of the counterterms using

$$\gamma_m(e,\lambda) = e\,\frac{\partial Z_m^{(1)}(e,\lambda)}{\partial e} + \sum_j n_j\lambda_j\,\frac{Z_m^{(1)}(e,\lambda)}{\partial\lambda_j}\ . \tag{194}$$

Now let's consider what terms can occur. In general $\Delta_i^{(1)}(e,\lambda)$ can have a non-zero perturbative coefficient of

$$e^N\,\lambda_1^{n_1}\,\lambda_2^{n_2}\cdots \tag{195}$$

if

$$d_i = n_1\,d_1 + n_2\,d_2 + \cdots . \tag{196}$$

Otherwise the powers of M won't match to create a counterterm for $\mathcal{O}_i$. For the renormalization of e, we have $d_e = 0$, so the only contributions are those with $n_1 = n_2 = \cdots = 0$. That is, the new terms don't affect the renormalization of e:

$$\beta(e, \lambda) = \beta(e). \tag{197}$$

Similarly, if we expand $Z_m(e, \lambda)$ in powers of e and the λ_j, we can get any power of e, but we cannot have powers of the λ_j. That is

$$\gamma_m(e, \lambda) = \gamma_m(e). \tag{198}$$

In a way, this result is an accounting trick. A graph containing a high dimension term in $\Delta\mathcal{L}$ can lead, for instance to the need for a counter term of the form $(m^3/M^2)\bar{\psi}\psi$, but we count this as renormalizing the coefficient of the opoerator $m^3\bar{\psi}\psi$ rather than as renormalizing the mass parameter.

Thus the new, high dimension terms do not change the renormalization of the conventional QED terms. However, the new terms do affect the renormalization of each other. And if we put just one new term with $d_i > 0$, we will need an infinite number of terms. This makes the theory *nonrenormalizable.* We see that a nonrenormalizable theory has an infinite number of parameters λ_i. Thus people used to reject the possibility of having a nonrenormalizable theory on the grounds that it has no predictive power.

However, a nonrenormalizable theory can be just fine if it is an effective theory that applies approximately at a momentum scale far below the scale M of some ultimate theory. Then all of the λ_i can be not too large and the λ_i/M^{d_i} are very small.

If we draw pictures of renormalization group flows, then we use dimensionless couplings and find, in the language of K. Wilson,

- $m(\mu)/\mu$ gets big as μ decreases. We call the corresponding operator, $\bar{\psi}\psi$ a *relevant* operator.

- $e(\mu)$ changes slowly as μ decreases. We call the corresponding operator, $\bar{\psi}\gamma^\mu\psi A_\mu$, a *marginal* operator. In the case of quantum electrodynamics, $e(\mu)$ gets slowly smaller as μ decreases. In the case of quantum electrodynamics, the corresponding coupling $g(\mu)$ gets slowly larger as μ decreases.

- $\lambda_i(\mu)(\mu/M)^{d_i}$ gets small as μ decreases. We call the corresponding operator, $\mathcal{O}_i$, an *irrelevant* operator.