The Lorentz Group,
Relativistic Particles, and Quantum Mechanics[1]

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In these notes, I discuss the relation of relativistic invariance to quantum mechanics. First, I discuss the rotation group and its representations. Then I outline the structure of Lorentz group and build up the finite dimensional representations of the Lorentz Group. Finally, I add the space and time translations to get the Poincaré group. This allows us to find how to treat the spin of particles in a relativistically covariant way.

1 Rotations

We can specify a rotation by giving a 3×3 real matrix R: if p^i are the components of a particle momentum (or any other vector) and p'^i are the components of the rotated momentum then

$$p'^i = R^{ij} p^j. \tag{1}$$

(A sum over the repeated index, $j = 1, 2, 3$ is implied here.) Note that I consider that we rotate the system, keeping the coordinate axes fixed. With this convention, if I have a vector field, say the electric field with components E^i, then the rotated system has an electric field E' given by

$$E'^i(x') = R^{ij} E^j(x) \quad \text{with} \quad x^i = (R^{-1})^{ij}\, x'^j. \tag{2}$$

A rotation matrix must be "orthogonal":

$$R^T = R^{-1}. \tag{3}$$

Exercise: Show that the fact that dot products between vectors are preserved under rotations implies that the rotation matrices obey $R^T = R^{-1}$.

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If we apply one rotation, $p'^{\,i} = R_1^{ij} p^j$, and then we apply another, $p''^{\,i} = R_1^{ij} p'^{\,j}$, the net result is applying a rotation $R_{\rm net}$ with

$$R_{\rm net}^{ik} = R_2^{ij} R_1^{jk}. \tag{4}$$

Thus the rotations form a *group*, usually called $O(3)$: 1) for any two R's in $O(3)$ their product is in $O(3)$, 2) $(R_3\,R_2)R_1 = R_3(R_2\,R_1)$, 3) $O(3)$ has a unit element "1" with $1\,R = R\,1 = R$, 4) for every R in $O(3)$ there is an inverse rotation R^{-1} with $R\,R^{-1} = R^{-1}\,R = 1$.

There is a technical point. The relation $R^T = R^{-1}$ implies that $\det R = \pm 1$. Since $\det(R_2 R_1) = \det R_2\ \det R_1$, we see that the set of matrices with $R^T = R^{-1}$ and $\det R = +1$ is itself a group. This is the group that we will denote by $O(3)$. The larger group that is specified by demanding only that $R^T = R^{-1}$ consists of all $O(3)$ matrices R together with the matrix $-R$ for every $R \in O(3)$. That is, we add the matrix -1 and all of its products with $O(3)$ matrices. The matrix -1 represents a parity transformation.

Exercise: Show that $R^T = R^{-1}$ implies that $\det R = \pm 1$.

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2 Rotations and states

In quantum mechanics, a rotation R should map each possible state $|\psi\rangle$ onto a new state that we shall denote by

$$|\psi'\rangle = U(R)|\psi\rangle. \tag{5}$$

In order to preserve the linear superposition principles of quantum mechanics, we can demand that $U(R)$ be a linear operator on $\mathcal{H}$. Also, in order to preserve the probabilistic interpretation of quantum mechanics, we can demand that if $|\psi\rangle$ and $|\phi\rangle$ are any two states in $\mathcal{H}$ and $|\psi'\rangle = U(R)|\psi\rangle$ and $|\phi'\rangle = U(R)|\phi\rangle$ then

$$\langle\phi'|\psi'\rangle = \langle\phi|\psi\rangle. \tag{6}$$

Then $U(R)$ must be a unitary operator:

$$U(R)^\dagger = U(R)^{-1}. \tag{7}$$

Exercise: Suppose that if $|\psi\rangle$ and $|\phi\rangle$ are any two states in $\mathcal{H}$ and $|\psi'\rangle = U(R)|\psi\rangle$ and $|\phi'\rangle = U(R)|\phi\rangle$ then $\langle\phi'|\psi'\rangle = \langle\phi|\psi\rangle$. Show that this implies that $U(R)^\dagger = U(R)^{-1}$.

(A technical note: a little thought is required to eliminate the possibility that $\langle\phi'|\psi'\rangle = \langle\phi|\psi\rangle^*$, which would correspond to an "antiunitary" operator. Basically it is continuity that rules this out. But in the case of time reversal symmetry one does run into an antiunitary time reversal operator.)

Now if we consider applying a rotation R_1 followed by a rotation R_2, we conclude that we need

$$U(R_2)\,U(R_1) = U(R_2\,R_1). \tag{8}$$

We say that the operators U provide a "representation" of $O(3)$.

Well, actually we shouldn't conclude anything too fast. It would suffice to have $U(R_2)\,U(R_1) = \exp(i\phi(R_2,R_1))\,U(R_2\,R_1)$ since only relative phases matter in quantum mechanics. We will return to this point later.

3 Infinitesimal rotations and finite rotations

In order to study unitary representations of the rotation group, we consider infinitesimal rotations,

$$R = 1 + \delta R + \cdots. \tag{9}$$

The infinitesimal matrix δR should be real and, since R obeys $R^T = R^{-1}$, δR should obey

$$\delta R^T = -\delta R. \tag{10}$$

We can construct any such δR as a linear combination of three matrices

$$\delta R_{ij} = \delta\theta_k\; r^k_{ij} = -\delta\theta_k\;\epsilon_{ijk} \tag{11}$$

where ϵ_{ijk} is the completely antisymmetric tensor with $\epsilon_{123} = +1$. This is conventionally written as

$$\delta R_{ij} = -i\,\delta\theta_k\;\mathcal{J}^k_{ij} \;\;\text{with}\;\; \mathcal{J}^k_{ij} = -i\,\epsilon_{ijk}. \tag{12}$$

The matrix $R = 1+\delta R$ as given above corresponds to an infinitesimal rotation through angle $|\vec{\delta\theta}|$ about the axis $\vec{\delta\theta}/|\vec{\delta\theta}|$. (Note that in this context it doesn't matter if the index k on $\mathcal{J}^k$ is a superscript or a subscript. But when we get to 4-vectors instead of 3-vectors, it will matter whether we use an upper index or a lower index.)

Exercise: Show explicitly that if $\delta\theta_k = \delta\phi\,\delta_{k3}$ then $R = 1 - i\,\delta\theta_k\,\mathcal{J}^k_{ij}$ with $\mathcal{J}^k_{ij} = -i\,\epsilon_{ijk}$ represents an infinitesimal rotation through angle $\delta\phi$ about the 3-axis.

Now how about a finite rotation through angle $|\vec{\theta}|$ about an axis $\vec{\theta}/|\vec{\theta}|$? That's easy. Call the rotation matrix $R(\vec{\theta})$. Evidently, $R(\vec{0}) = 1$. Also

$$R(\lambda\vec{\theta} + \delta\lambda\,\vec{\theta}) = R(\delta\lambda\,\vec{\theta})\;R(\lambda\vec{\theta}) = [1 - i\delta\lambda\theta_k\mathcal{J}_k + \cdots]\;R(\lambda\vec{\theta}) \tag{13}$$

so

$$\frac{\partial}{\partial\lambda}R(\lambda\vec{\theta}) = -i\theta_k\mathcal{J}_k\;R(\lambda\vec{\theta}) \tag{14}$$

so

$$R(\vec{\theta}) = \exp(-i\theta_k\mathcal{J}_k). \tag{15}$$

For our study of representations of $O(3)$ we will need to consider products of rotations. It suffices to study the simple example

$$R(\vec{\phi})\;R(\vec{\theta})\;R(-\vec{\phi}) \equiv R(\vec{\theta}'(\vec{\phi})). \tag{16}$$

It is easy to see by a geometrical argument that the new vector $\vec{\theta}'$ is the old vector $\vec{\theta}$ rotated by $R(\vec{\phi})$. For our purposes, we need this relation only for infinitesimal angles, and we can get this relation by algebra. We write

$$[1 - i\phi_i\mathcal{J}_i + \cdots]\,[1 - i\theta_j\mathcal{J}_j + \cdots]\,[1 + i\phi_i\mathcal{J}_i + \cdots]\; = 1 - i\theta'_k\mathcal{J}_k + \cdots \tag{17}$$

so

$$1 - i\theta_j\mathcal{J}_j - \phi_i\theta_j[\mathcal{J}_i, \mathcal{J}_j] + \cdots = 1 - i\theta'_k\mathcal{J}_k + \cdots \tag{18}$$

where the omitted terms on the left hand side have either two factors of θ or two factors of ϕ. We compute

$$[\mathcal{J}_i, \mathcal{J}_j] = i\epsilon_{ijk}\,\mathcal{J}_k \tag{19}$$

so

$$1 - i\theta_k\mathcal{J}_k - i\epsilon_{ijk}\,\phi_i\theta_j\mathcal{J}_k + \cdots = 1 - i\theta'_k\mathcal{J}_k + \cdots. \tag{20}$$

We conclude that

$$\theta'_k = \theta_k + \epsilon_{ijk}\,\phi_i\theta_j + \cdots \tag{21}$$

where the omitted terms have either two factors of θ or two factors of ϕ. (Actually θ' is exactly linear in θ but our simple derivation didn't show that.)

4 Representations of $O(3)$

Recall that we wanted to study unitary representations of $O(3)$, $R \to U(R)$. For $R(\vec{\theta}) = \exp(-i\theta_k \mathcal{J}_k)$ denote $U(R)$ by $U(\vec{\theta})$. Define

$$J_k = i \left[\frac{\partial}{\partial \theta_k} U(\vec{\theta}) \right]_{\vec{\theta}=0} \tag{22}$$

These operators are hermitian ($J^\dagger = J$) since U is unitary. The J_k are called the infinitesimal generators of the representation. Then we can express $U(\vec{\theta})$ for a finite $\vec{\theta}$ by repeating an argument given earlier. Consider $U(\vec{\theta} + \delta\lambda\,\vec{\theta})$. This is $U(R_2 R_1)$ where $R_2 = R(\delta\lambda\,\vec{\theta})$ and $R_1 = R(\vec{\theta})$. Thus it is $U(R_2)U(R_1)$. Then

$$U(\vec{\theta} + \delta\lambda\,\vec{\theta}) = U(\delta\lambda\,\vec{\theta})\; U(\vec{\theta}) = [1 - i\delta\lambda\theta_k J_k + \cdots]\; U(\vec{\theta}) \tag{23}$$

so

$$\frac{\partial}{\partial\lambda} U(\lambda\vec{\theta}) = -i\theta_k J_k\; U(\vec{\theta}) \tag{24}$$

so

$$U(\vec{\theta}) = \exp(-i\theta_k J_k). \tag{25}$$

Now consider the product $U(\vec{\phi})\; U(\vec{\theta})\; U(-\vec{\phi})$. Again using the fact that we have a representation of the group, we have

$$U(\vec{\phi})\; U(\vec{\theta})\; U(-\vec{\phi}) \equiv U(\vec{\theta}'(\vec{\phi})). \tag{26}$$

Expanding, we have

$$[1 - i\phi_i J_i + \cdots]\,[1 - i\theta_j J_j + \cdots]\,[1 + i\phi_i J_i + \cdots] \;= 1 - i\theta'_k J_k + \cdots \tag{27}$$

or

$$1 - i\theta_j J_j - \phi_i\theta_j [J_i, J_j] + \cdots = 1 - i\theta'_k J_k + \cdots. \tag{28}$$

We know what θ' is, so let's use that:

$$1 - i\theta_k J_k - \phi_i\theta_j [J_i, J_j] + \cdots = 1 - i\theta_k J_k - i\epsilon_{ijk}\,\phi_i\theta_j\,J_k \cdots. \tag{29}$$

Matching terms, we have

$$[J_i, J_j] = i\epsilon_{ijk}\,J_k. \tag{30}$$

This is an important result. The generators of a representation of $O(3)$ have the same commutation relations as the generators of $O(3)$. (Of course, you already knew this, but maybe this derivation has been a little more careful than what you have seen.)

5 Irreducible representations

The problem of finding representations $U(R)$ of the rotation group can be simplified if we ask for *irreducible* representations.

We have been supposing that the operators $U(R)$ act on the entire space $\mathcal{H}$ that we are using for a quantum mechanics problem. But typically $\mathcal{H}$ can be decomposed into orthogonal subspaces $\mathcal{H}_i$ so that every $|\psi\rangle$ in$\mathcal{H}$ can be written as a sum

$$|\psi\rangle = \sum_i |\psi_i\rangle \quad \text{with} \quad |\psi_i\rangle \in \mathcal{H}_i \tag{31}$$

so that if $|\psi_i\rangle \in \mathcal{H}_i$ then $U(R)|\psi_i\rangle \in \mathcal{H}_i$. (Here "orthogonal" means that for every pair of subspaces $\{\mathcal{H}_i, \mathcal{H}_j\}$, every vector in $\mathcal{H}_i$ is orthogonal to every vector in $\mathcal{H}_j$.) If the representation is reducible, then we can build up the whole representation by working out what happens in each subspace.

An representation $U(R)$ acting on a space $\mathcal{H}_i$ is *irreducible* if there is no subspace of $\mathcal{H}_i$ that is left invariant by the operators $U(R)$. Thus if we want to know about the structure of representations of the rotation group, it suffices to study irreducible representations.

6 Representations of the Lie algebra

We have seen that finding a representation of the rotation group requires finding a representation of the *Lie algebra* for this group: the commutation relations for the infinitesimal generators. Thus we must find matrices J_i that satisfy

$$[J_i, J_j] = i\epsilon_{ijk}\, J_k. \tag{32}$$

Assuming that we want an irreducible representation of the rotation group, we should look for a representation of the Lie algebra that is irreducible: no subspace of the space $\mathcal{V}$ upon which the J_i act should be invariant under the J_i.

The way to find the irreducible representations of the rotation group is quite standard. First, we consider the operator

$$\vec{J}^2 = \sum_i J_i^2. \tag{33}$$

Since $\vec{J}^2$ is a hermitian operator, we can diagonalize it. That is, we can find eigenvalues λ and subspaces $\mathcal{V}_\lambda$ such that

$$\vec{J}^2|\psi_\lambda\rangle = \lambda|\psi_\lambda\rangle \quad \text{for} \quad |\psi_\lambda\rangle \in \mathcal{V}_\lambda. \tag{34}$$

Now in fact if there are different λ values, then the corresponding subspaces $\mathcal{V}_\lambda$ must be orthogonal. Then consider a vector $|\psi_\lambda\rangle$ in $\mathcal{V}_\lambda$ and the corresponding vectors

$$J_i|\psi_\lambda\rangle$$

that you get by acting with the generators J_i. From the commutator algebra we derive

$$[J_i, \vec{J}^2] = 0. \tag{35}$$

Thus

$$\vec{J}^2\, J_i|\psi_\lambda\rangle = J_i\vec{J}^2|\psi_\lambda\rangle = \lambda J_i|\psi_\lambda\rangle \tag{36}$$

so that $J_i|\psi_\lambda\rangle \in \mathcal{V}_\lambda$. That is, the subspaces $\mathcal{V}_\lambda$ corresponding to different eigenvalues are invariant under the operation of the J_i. We conclude that if the representation is irreducible, then there can be only *one* eigenvalue.

Exercise. Prove that $[J_i, \vec{J}^2] = 0$.

Now the familiar derivation proceeds by diagonalizing J_3. Call the eigenvalues m and the corresponding eigenvectors $|m\rangle$:

$$J_3|m\rangle = m|m\rangle. \tag{37}$$

One starts with the eigenvector with the smallest (most negative) eigenvalue m. Then one constructs the other eigenvectors by applying the "raising" operator $J_1 + iJ_2$ to $|m\rangle$, which gives an eigenvector with m increased by 1. (If there is more than one eigenvector with the smallest eigenvalue m then, with a little work, one finds that the representation is reducible.)

The net result is that λ, which labels the representation, must be of the form

$$\lambda = \ell(\ell+1) \quad \text{with} \quad \ell \in \{\,0, 1/2, 1, 3/2, \ldots\}. \tag{38}$$

Then the values of m are

$$m = -\ell, -\ell+1, -\ell+2, \ldots, +\ell. \tag{39}$$

We derive matrices $\mathcal{J}_i^{(\ell)}$ such that

$$J_i|m\rangle = \sum_{m'} (\mathcal{J}_i^{(\ell)})_{m'm}\,|m'\rangle \tag{40}$$

(There are some conventional phase conventions used.)

Given the $\mathcal{J}_i^{(\ell)}$ we can form the representation for finite rotations

$$U(R)|m\rangle = \sum_{m'} \mathcal{D}^{(\ell)}(R)_{m'm}\,|m'\rangle. \tag{41}$$

Namely, for $R = \exp(-i\phi_i\mathcal{J}_i)$ we have

$$\mathcal{D}^{(\ell)}(R) = \exp(-i\phi_i\mathcal{J}_i^{(\ell)}). \tag{42}$$

7 The phase problem

Note that the representations for half integer ℓ of $O(3)$ derived in this way are not true representations but only representations up to a phase. That is because a rotation through 2π is represented by $U = \exp(i2\pi\ell)$, which is -1 if ℓ is a half integer. As we remarked earlier, this is not a problem for quantum mechanics. A nice way to formulate the representation theory is to regard the representations as (true) representations of $SU(2)$. The idea is that the group $SU(2)$ of 2×2 unitary matrices with determinant 1 has three generators, which we may take to be 1/2 times the familiar σ_i matrices. These are exactly the generators of the $\ell = 1/2$ representation of $O(3)$. Thus we can regard the map $R \to U(R)$ from $O(3)$ to $SU(2)$, which is a $1 \to 2$ map, as the backwards way to look at it. Instead we can use the corresponding $2 \to 1$ map from U to $R(U)$.

8 Notation for Lorentz covariant physics

We choose units in which $c = 1$. We denote the components of particle momenta by p^μ, q^μ, *etc.*. Here the indices μ run from 0 to 4; p^0 is the particle energy, p^1 is the x-component of the momentum, *etc.*.

The inner product

$$p\cdot q = p^\mu q^\nu g_{\mu\nu} \tag{43}$$

where

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{44}$$

has a reference frame independent meaning. For instance $p^2 = p^\mu p^\nu g_{\mu\nu}$ is the square of the mass of the particle. In a product with matching upper and lower indices, there is an implied sum; thus there are implied sums over μ and ν from 0 to 3 here.

Given a momentum, or any other vector, with an upper index, we define a corresponding vector with a lower index by using the metric tensor g:

$$p_\mu = g_{\mu\nu} p^\nu . \tag{45}$$

We denote the inverse matrix to g by using g with upper indices:

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu . \tag{46}$$

Evidently, this is the same matrix. (But in general relativity, $g_{\mu\nu}$ is more complicated and its inverse matrix $g^{\mu\nu}$ is a different matrix.) We can us the upper-index-g to get the upper index components of a vector from the lower index components:

$$p^\mu = g^{\mu\nu}\, p_\nu . \tag{47}$$

We typically use the raising and lowering of indices trick to hide the metric tensor. Thus

$$p \cdot q = p^\mu\, q_\mu . \tag{48}$$

9 The Lorentz group

We can specify a Lorentz transformation by giving a 4×4 real matrix Λ: if p^μ are the components of a particle momentum and p'^μ are the components of the transformed momentum then

$$p'^\mu = \Lambda^\mu{}_\nu\, p^\nu . \tag{49}$$

The key to Lorentz transformations is that they preserve the inner product $p \cdot q = p^\mu q^\nu g_{\mu\nu}$. Thus

$$p' \cdot q' = p \cdot q \tag{50}$$

so

$$\Lambda^{\mu}{}_{\alpha}\, p^{\alpha}\, \Lambda^{\nu}{}_{\beta}\, q^{\beta}\, g_{\mu\nu} = p^{\alpha}\, q^{\beta}\, g_{\alpha\beta}. \tag{51}$$

This should happen for every choice of p and q so

$$\Lambda^{\mu}{}_{\alpha}\, \Lambda^{\nu}{}_{\beta}\, g_{\mu\nu} = g_{\alpha\beta}. \tag{52}$$

A matrix that satisfies this equation, which is the analogue of $R^T R = 1$ for rotations, is called a Lorentz transformation matrix.

Note that the Lorentz transformations form a group. If Λ_1 and Λ_2 are Lorentz transformation matrices, then so is the matrix $\Lambda_3 = \Lambda_2\, \Lambda_1$,

$$(\Lambda_3)^{\mu}{}_{\nu} = (\Lambda_2)^{\mu}{}_{\alpha}\, (\Lambda_1)^{\alpha}{}_{\nu}. \tag{53}$$

Typically we also impose the conditions $\det \Lambda = +1$ and $\Lambda^0{}_0 > 0$ so as to eliminate the parity transformation,

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{54}$$

the time reversal transformation,

$$\Lambda = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{55}$$

and their product. Analysis of these transformations is then reserved for separate consideration. With these restrictions we have the *proper*, *orthochonous* Lorentz group.

10 Generators of the Lorentz group

Let

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \delta\Lambda^{\mu}{}_{\nu} + \cdots \tag{56}$$

where $\delta\Lambda$ is infinitesimal. Then the condition that Λ is a Lorentz transformation matrix gives

$$\delta\Lambda^{\mu}{}_{\alpha}\, g_{\mu\beta} + \delta\Lambda^{\nu}{}_{\beta}\, g_{\nu\alpha} = 0 \tag{57}$$

In terms of the matrix $\delta\Lambda$ with its first index lowered, this is

$$\delta\Lambda_{\beta\alpha} + \delta\Lambda_{\alpha\beta} = 0. \tag{58}$$

That is, $\delta\Lambda$ with its first index lowered is antisymmetric. Any such matrix can be written as a sum of coefficients times six linearly independent antisymmetric 4×4 matrices.

There are two notations that we can use to express this. First, we can write

$$\delta\Lambda^{\mu}{}_{\nu} = -i\,\tfrac{1}{2}\,\Theta^{\alpha\beta}(\mathcal{M}_{\alpha\beta})^{\mu}{}_{\nu} \tag{59}$$

with

$$(\mathcal{M}_{\alpha\beta})^{\mu}{}_{\nu} = i(\delta^{\mu}_{\alpha}\,g_{\beta\nu} - \delta^{\mu}_{\beta}\,g_{\alpha\nu}). \tag{60}$$

In this covariant notation, the $\mathcal{M}_{\alpha\beta}$ are the infinitesimal generators. Since $\mathcal{M}_{\alpha\beta} = -\mathcal{M}_{\beta\alpha}$, there are six independent generators. The angle tensor $\Theta^{\alpha\beta}$ is also antisymmetric, so there are six independent angles. In the sum over α and β, each of the independent contributions gets counted twice, so we include a compensating factor 1/2. Evidently the matrix $M_{\alpha\beta}$ generates a Lorentz transformation that mixes p^{α} and p^{β} while leaving the other two components of p^{μ} unchanged.

In the second notation, we define generators $\mathcal{J}_i$ and $\mathcal{K}_i$ by

$$\begin{aligned} \mathcal{M}_{ij} &= \epsilon_{ijk}\,\mathcal{J}_k \\ \mathcal{M}_{i0} &= \mathcal{K}_i. \end{aligned} \tag{61}$$

Then

$$\delta\Lambda = -i(\theta_i\mathcal{J}_i + \omega_i\mathcal{K}_i) \tag{62}$$

The three matrices $\mathcal{J}_i$ generate rotations. For instance, $\mathcal{J}_3$ generates rotations about the 3-axis. The three matrices $\mathcal{K}_i$ generate boosts. For instance, $\mathcal{K}_3$ generates boosts in the 3-direction.

Just as with the rotation group, we see that finite Lorentz transformations can be represented as

$$\Lambda = \exp\{-i(\theta_i\mathcal{J}_i + \omega_i\mathcal{K}_i)\} \tag{63}$$

The following exercises show just that the finite rotations and boosts have the forms that should be familiar to you.

Exercise. Prove that

$$\mathcal{J}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{64}$$

Exercise. Prove that

$$\mathcal{K}_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \tag{65}$$

Exercise. Prove that

$$\exp\{-i\theta\mathcal{J}_3\} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{66}$$

Exercise. Prove that

$$\exp\{-i\omega\mathcal{K}_3\} = \begin{pmatrix} \cosh\omega & 0 & 0 & \sinh\omega \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\omega & 0 & 0 & \cosh\omega \end{pmatrix} \tag{67}$$

11 The Lie algebra for the Lorentz group

By explicit calculation we find

$$\begin{aligned}
[\mathcal{J}_i, \mathcal{J}_j] &= i\,\epsilon_{ijk}\,\mathcal{J}_k \\
[\mathcal{J}_i, \mathcal{K}_j] &= i\,\epsilon_{ijk}\,\mathcal{K}_k \\
[\mathcal{K}_i, \mathcal{K}_j] &= -i\,\epsilon_{ijk}\,\mathcal{J}_k
\end{aligned} \tag{68}$$

The first equation gives the Lie algebra for the rotation group, which is a subgroup of the Lorentz group. The second equation says that the components of $\mathcal{K}$ form a vector under rotations. Then the third equation says that if you perform two boosts in different directions, the boost operators don't commute and you get a rotation mixed in.

Exercise. Starting from the generator matrices, derive the commutation relations for the Lorentz group.

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12 Finite dimensional representations of the Lorentz group

Later, we will look for unitary representations of the Lorentz group – in fact, of the Lorentz group combined with translations. But now we pause to characterize the finite dimensional irreducible representations of the Lorentz group. As we will see, none of these representations are unitary.

We have found the commutation relations of the generators of the Lorentz group. Let us rewrite these using linear combinations of the generators,

$$\begin{aligned}
\mathcal{J}_i^A &= \tfrac{1}{2}(\mathcal{J}_i - i\mathcal{K}_i) \\
\mathcal{J}_i^B &= \tfrac{1}{2}(\mathcal{J}_i + i\mathcal{K}_i).
\end{aligned} \tag{69}$$

Then

$$\begin{aligned}
[\mathcal{J}_i^A, \mathcal{J}_j^A] &= i\,\epsilon_{ijk}\,\mathcal{J}_k^A \\
[\mathcal{J}_i^B, \mathcal{J}_j^B] &= i\,\epsilon_{ijk}\,\mathcal{J}_k^B \\
[\mathcal{J}_i^A, \mathcal{J}_j^B] &= 0.
\end{aligned} \tag{70}$$

We see that we have two copies of the Lie algebra of $O(3)$, and the two sets of generators commute with each other.

We can use this. We know all of the finite representations of $O(3)$, so we just have to put them together. Let's take the ℓ_A representation of $O(3)_A$ and the ℓ_B representation of $O(3)_B$ and make a representation $\{\ell_A, \ell_B\}$ of the Lorentz group from that.

To make this concrete, consider vectors ψ in a space $\mathcal{V}_{\{\ell_A,\ell_B\}}$ upon which the the generators J_k^A and J_k^B act. We want to define what the generators J_k^A and J_k^B do when acting on these vectors. Let the vectors ψ have components ψ_{ij}, with $i = -\ell_A, \ldots, +\ell_A$ and $j = -\ell_B, \ldots, +\ell_B$. For example, for $\ell_A = 1$ and $\ell_B = 1/2$, we are using one index that takes 3 values and another that takes 2 values instead of using one index that takes 6 values. Let

$$J_k^A \psi = \psi' \tag{71}$$

with

$$\psi'_{ij} = (\mathcal{J}_k^{(\ell_A)})_{ii'}\ \delta_{jj'}\ \psi_{i'j'}. \tag{72}$$

Here the matrices $\mathcal{J}_k^{(\ell_A)}$ are the standard generator matrices for the ℓ_A representation of $O(3)$. As indicated these matrices act on the first index.

For $O(3)_B$, let

$$J_k^B \psi = \psi' \tag{73}$$

with

$$\psi'_{ij} = \delta_{ii'}\ (\mathcal{J}_k^{(\ell_B)})_{jj'}\ \psi_{i'j'}. \tag{74}$$

Here the matrices $\mathcal{J}_k^{(\ell_B)}$ are the standard generator matrices for the ℓ_B representation of $O(3)$. As indicated these matrices act on the second index.

You should verify for yourself that these definitions give operators J_i^A and J_i^B that indeed have the desired commutation relations.

13 Representations of the Lorentz group including parity

We have found quite explicitly the finite dimensional irreducible representations or the proper orthochronous Lorentz group. These representations are labelled by two numbers ℓ_A and ℓ_B chosen from $0, 1/2, 1, 3/2, \ldots$. Thus we have representations $\{0,0\}$, $\{1/2,0\}$, $\{0,1/2\}$, $\{1/2,1/2\}$, ... Now consider the matrix $\mathcal{P}$ that reverses the space components of 4-vectors. We have

$$\begin{aligned} \mathcal{P}\mathcal{J}_j\mathcal{P} &= \mathcal{J}_j \\ \mathcal{P}\mathcal{K}_j\mathcal{P} &= -\mathcal{K}_j \end{aligned} \tag{75}$$

We say that $\vec{\mathcal{K}}$ is a vector while $\vec{\mathcal{J}}$ is a pseudovector.

Exercise. Prove that the rotation generators are unchanged under a parity transformation while the boost generators change sign under a parity transformation in the sense stated above.

From these relations we derive

$$\begin{aligned} \mathcal{P}\mathcal{J}_j^A\mathcal{P} &= \mathcal{J}_j^B \\ \mathcal{P}\mathcal{J}_j^B\mathcal{P} &= \mathcal{J}_j^A \end{aligned} \tag{76}$$

If we want a representation that includes a parity transformation the $\{\ell_1,\ell_2\}$ representation by itself won't work unless $\ell_1 = \ell_2$. For $\ell_1 \neq \ell_2$ we will have to combine the $\{\ell_1,\ell_2\}$ representation with the $\{\ell_2,\ell_1\}$ to make a representation

$$\{\ell_1,\ell_2\} \oplus \{\ell_2,\ell_1\}. \tag{77}$$

This is easy, even though the notation is a little cumbersome. A vector Ψ in the combined space $\mathcal{V}_{\{\ell_1,\ell_2\}} \oplus \mathcal{V}_{\{\ell_2,\ell_1\}}$ is made up of a pair $\{\psi,\phi\}$ of vectors, with $\psi \in \mathcal{V}_{\{\ell_1,\ell_2\}}$ and $\phi \in \mathcal{V}_{\{\ell_2,\ell_1\}}$. When a generator J_k^A acts on Ψ we get a new vector $\Psi' = \{\psi',\phi'\}$ with

$$\begin{aligned} \psi'_{ij} &= (\mathcal{J}_k^{(\ell_1)})_{ii'}\ \delta_{jj'}\ \psi_{i'j'}. \\ \phi'_{ij} &= (\mathcal{J}_k^{(\ell_2)})_{ii'}\ \delta_{jj'}\ \phi_{i'j'}. \end{aligned} \tag{78}$$

When a generator J_k^B acts on Ψ we get a new vector $\Psi' = \{\psi',\phi'\}$ with

$$\begin{aligned} \psi'_{ij} &= \delta_{ii'}\ (\mathcal{J}_k^{(\ell_2)})_{jj'}\ \psi_{i'j'} \\ \phi'_{ij} &= \delta_{ii'}\ (\mathcal{J}_k^{(\ell_1)})_{jj'}\ \phi_{i'j'}. \end{aligned} \tag{79}$$

When the parity operator acts on Ψ we get a new vector $\Psi' = \{\psi',\phi'\}$ with the roles of ψ and ϕ reversed

$$\begin{aligned} \psi'_{ij} &= \phi_{ji} \\ \phi'_{ij} &= \psi_{ji}. \end{aligned} \tag{80}$$

Exercise. Prove that the commutation relations of rotations and boost generators with the parity operator are correct as given by the construction above in the case of the $\{1/2,0\} \oplus \{0,1/2\}$ representation (for which the notation is simpler since an index that takes zero values can simply be eliminated.

In the case of the $\{\ell,\ell\}$ representation, we can include parity without enlarging the space by defining $P\psi = \psi'$ with

$$\psi'_{ij} = \psi_{ji}. \tag{81}$$

Exercise. Prove that the commutation relations of rotations and boost generators with the parity operator are correct as given by the construction above in the case of the $\{1/2,1/2\}$ representation.

14 Comments on some of the representations of the Lorentz group

The $\{0,0\}$ representation is one dimensional. An object ϕ transforming under this representation doesn't transform and is called a scalar.

The $\{1/2,0\} \oplus \{0,1/2\}$ representation is 4 dimensional $(4 = 2+2)$. An object ψ transforming under this representation is called a Dirac spinor.

Exercise. The generators of the $\ell = 1/2$ representation of the rotation group are $\mathcal{J}_k^{(1/2)} = \frac{1}{2}\sigma_k$, where the σ_k are the usual Pauli matrices. Choosing a simple basis, what are the 4×4 generator matrices J_k and K_k for the $\{1/2,0\} \oplus \{0,1/2\}$ representation of the Lorentz group.

The $\{1/2,1/2\}$ representation is 4 dimensional $(4 = 2\times 2)$, and it is also a 4 dimensional irreducible representation of the proper orthochronous Lorentz group. A four-vector A^μ (like the electromagnetic potential) evidently transforms under a 4 dimensional irreducible representation of the proper orthochronous Lorentz group. The $\{1/2,1/2\}$ representation is the only such representation. Thus, "four-vector" representation and "$\{1/2,1/2\}$" representation are the same thing.

The $\{1,0\} \oplus \{0,1\}$ representation is 6 dimensional ($6 = 3+3$). An antisymmetric rank two tensor $F^{\mu\nu}$ (like the electromagnetic field) transforms under a 6 dimensional irreducible representation of the Lorentz group. The $\{1,0\} \oplus \{0,1\}$ is the only such representation, so we conclude that antisymmetric rank two tensors transform under the $\{1,0\} \oplus \{0,1\}$ representation of the Lorentz group.

The $\{1,1\}$ representation is 9 dimensional ($9 = 3 \times 3$). A symmetric rank two tensor $T^{\mu\nu}$ with zero trace, $T^\mu_\mu = 0$, transforms under a 9 dimensional irreducible representation of the Lorentz group. The $\{1,1\}$ is the only such representation, so we conclude that traceless symmetric rank two tensors transform under the $\{1,1\}$ representation of the Lorentz group.

15 The Poincaré group

The invariance group of physics is bigger than the Lorentz group. We can also translate the system in space-time by an amount a^μ, so that a event at position x gets moved to x', with $(x')^\mu = x^\mu + a^\mu$. The group consisting of Lorentz transformations and translations together is usually called the Poincaré group.

In applications to quantum mechanics, the generators of translations are called the components of the four-momentum, P^μ. The operator that translates the system through displacement a^μ is

$$\exp(iP_\mu a^\mu). \tag{82}$$

Translations in different directions commute, so

$$[P^\mu, P^\nu] = 0. \tag{83}$$

On the other hand

$$\exp(i\Theta^{\alpha\beta}M_{\alpha\beta})\, P^\mu \exp(-i\Theta^{\alpha\beta}M_{\alpha\beta}) = \Big(\exp(-i\Theta^{\alpha\beta}\mathcal{M}_{\alpha\beta})\Big)^\mu{}_\nu \; P^\nu. \tag{84}$$

where, we recall,

$$(\mathcal{M}_{\alpha\beta})^\mu{}_\nu = i\left(\delta^\mu_\alpha\, g_{\beta\nu} - \delta^\mu_\beta\, g_{\alpha\nu}\right) \tag{85}$$

so

$$[M_{\alpha\beta}, P^\mu] = -\,(\mathcal{M}_{\alpha\beta})^\mu{}_\nu \; P^\nu. \tag{86}$$

Note that these commutation relations simply say that P^μ is a four-vector. Thus we would have analogous commutation relations for any four-vector.

Exercise. Show that these commutation relations amount to

$$\begin{aligned}
[J_i, P^0] &= 0 \\
[J_i, P^j] &= i\epsilon_{ijk} P^k \\
[K_i, P^0] &= -iP^i \\
[K_i, P^j] &= -i\delta_{ij} P^0.
\end{aligned} \tag{87}$$

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16 Particle states for spinless particles

If we want to describe the quantum mechanics of a spinless particle of mass m, how do we do it? There should be some space $\mathcal{H}$ of states and we should have a unitary representation of the Poincaré group on $\mathcal{H}$. That's easy. Just diagonalize the momentum operators P^μ. Call the eigenstates $|k\rangle$.

$$P^\mu |k\rangle = k^\mu |k\rangle. \tag{88}$$

Here $k^\mu k_\mu = m^2$. Thus one could consider each state to be labelled by three real numbers k^1, k^2, and k^3, with k^0 determined by the mass-shell condition. However, in order to have a Lorentz covariant notation, we label the states with all four components of the momentum.

Define the action of a Lorentz transformation by

$$U(\Lambda)|k\rangle = |\Lambda k\rangle. \tag{89}$$

Then we will have

$$\begin{aligned}
U(\Lambda_2)\, U(\Lambda_1)|k\rangle &= U(\Lambda_2)\, |\Lambda_1 k\rangle \\
&= |\Lambda_2 \Lambda_1 k\rangle \\
&= U(\Lambda_2 \Lambda_1)|k\rangle
\end{aligned} \tag{90}$$

so we have a genuine representation of the Lorentz group. Also

$$\begin{aligned}
U(\Lambda^{-1})\, P^\mu\, U(\Lambda)|k\rangle &= U(\Lambda^{-1})\, P^\mu\, |\Lambda k\rangle \\
&= \Lambda^\mu{}_\nu\, k^\nu U(\Lambda^{-1})|\Lambda k\rangle \\
&= \Lambda^\mu{}_\nu\, k^\nu |k\rangle \\
&= \Lambda^\mu{}_\nu\, P^\nu |k\rangle
\end{aligned} \tag{91}$$

so

$$U(\Lambda^{-1})\, P^\mu\, U(\Lambda) = \Lambda^\mu{}_\nu\, P^\nu \tag{92}$$

and we get the right commutation relations between the momentum components and the generators of Lorentz transformations.

We would like our representation to be unitary. This amounts to saying that the inner product on $\mathcal{H}$ should be Lorentz invariant. We define the inner product between basis vectors to be

$$\langle k|p\rangle = (2\pi)^3 2p^0\ \delta(\vec{p}-\vec{k}). \tag{93}$$

The 2π here is purely conventional. This does not look very covariant, but it is. We have

$$\begin{aligned}
2p^0\,\delta(\vec{p}-\vec{k})\delta(p^2-k^2) &= 2p^0\,\delta(\vec{p}-\vec{k})\delta((p^0)^2-\vec{p}^2-(k^0)^2-\vec{k}^2) \\
&= 2p^0\,\delta(\vec{p}-\vec{k})\delta((p^0)^2-(k^0)^2) \\
&= 2p^0\,\delta(\vec{p}-\vec{k})\frac{1}{2p^0}\delta(p^0-k^0) \\
&= \delta^{(4)}(p-k)
\end{aligned} \tag{94}$$

The corresponding covariant integration is

$$\int \frac{d^4p}{(2\pi)^4} 2\pi\delta(p^2-m^2)\ f(p) = \int \frac{d^3\vec{p}}{(2\pi)^3 2p^0}\ f(p). \tag{95}$$

This all works without change if $m = 0$.

17 Particle states for massive particles with spin

Now let's try to describe the quantum mechanics of a particle of mass m that has spin. First, diagonalize the momentum operators P^μ. Call the eigenstates $|k,s\rangle$.

$$P^\mu|k,s\rangle = k^\mu|k,s\rangle. \tag{96}$$

Here $k^\mu k_\mu = m^2 > 0$. We include an index s in anticipation that there may be more than one eigenstate of P^μ with a certain eigenvalue k^μ. We would like our representation to be unitary, so we generalize the previous inner product between basis vectors to be

$$\langle k, s' | p, s \rangle = \delta_{ss'} (2\pi)^3 2p^0 \; \delta(\vec{p} - \vec{k}). \tag{97}$$

What should $U(\Lambda)$ do to our states? Clearly we should have

$$U(\Lambda)|k, s\rangle = A_{s's} |\Lambda k, s'\rangle \tag{98}$$

But it is not so obvious what the matrix A here should be in order to get all of the required commutation relations right. Eugene Wigner figured out what to do in his paper Ann. Math. **40**, 149 (1939) and we follow his treatment. First, define a standard momentum k_0^μ. It is convenient to take

$$k_0 = (m, 0, 0, 0). \tag{99}$$

There are certain Lorentz transformations that leave k_0 invariant. These are the rotations. If we apply a rotation $U(R)$ applied to one of the standard states, we must get back a linear combination of the standard states

$$U(R)|k_0, s\rangle = A(R)_{s's} |k_0, s'\rangle. \tag{100}$$

Applying two rotations in a row, we have

$$\begin{aligned} A(R_2 R_1)_{s''s} |k_0, s''\rangle &= U(R_2 R_1)|k_0, s\rangle \\ &= U(R_2)\, U(R_1)|k_0, s\rangle \\ &= A(R_1)_{s's}\, U(R_2)|k_0, s'\rangle \\ &= A(R_2)_{s''s'} A(R_1)_{s's} |k_0, s''\rangle \end{aligned} \tag{101}$$

Thus

$$A(R_2 R_1)_{s''s} = A(R_2)_{s''s'} A(R_1)_{s's}. \tag{102}$$

That is, the matrices $A(R)$ form a representation of the rotation group. The simplest possibility is that it is one of the irreducible representations of the rotation group. One could build more complicated choices by combining irreducible representations if one wanted. Thus let us take one of the irreducible representations, with a label ℓ:

$$U(R)|k_0, s\rangle = \mathcal{D}^{(\ell)}(R)_{s's} |k_0, s'\rangle. \tag{103}$$

Now, for any momentum k^μ other than k_0^μ, define a standard Lorentz transformation $\Lambda_0(k)$ such that

$$\Lambda_0(k)^\mu{}_\nu\, k_0^\nu = k^\mu. \tag{104}$$

A convenient choice is

$$\Lambda_0(k) = \exp\left(-i\,\omega_i\,\mathcal{K}_i)\right) \tag{105}$$

with

$$\omega_i = (k^i/|\vec{k}|)\sinh^{-1}(|\vec{k}|/m). \tag{106}$$

That is, $\Lambda_0(k)$ is a boost in the direction of $\vec{k}$. Now we *define* the state $|k,s\rangle$ in terms of the state $|k_0,s\rangle$ using

$$|k,s\rangle = U(\Lambda_0(k))|k_0,s\rangle. \tag{107}$$

By definition, this boost operation on this state does not mix states with different spin indices.

What happens if we apply a general Lorentz transformation Λ to a general state $|k,s\rangle$? We have no choice:

$$\begin{aligned} U(\Lambda)|k,s\rangle &= U(\Lambda)U(\Lambda_0(k))|k_0,s\rangle \\ &= U(\Lambda_0(\Lambda k))U(\Lambda_0(\Lambda k)^{-1})U(\Lambda)U(\Lambda_0(k))|k_0,s\rangle \\ &= U(\Lambda_0(\Lambda k))\,U(\Lambda_0(\Lambda k)^{-1}\,\Lambda\,\Lambda_0(k))|k_0,s\rangle \end{aligned} \tag{108}$$

Now if we apply the Lorentz transformation

$$\Lambda_0(\Lambda k)^{-1}\,\Lambda\,\Lambda_0(k) \tag{109}$$

to the vector k_0, we get k_0 back again:

$$\Lambda_0(\Lambda k)^{-1}\,\Lambda\,\Lambda_0(k)\,k_0 = \Lambda_0(\Lambda k)^{-1}\,\Lambda\,k = k_0. \tag{110}$$

Thus this matrix is a rotation, called the "Wigner rotation." We already know what happens when a rotation to one of the standard states. Thus

$$\begin{aligned} U(\Lambda)|k,s\rangle &= U(\Lambda_0(\Lambda k))\,\mathcal{D}^{(\ell)}(\Lambda_0(\Lambda k)^{-1}\,\Lambda\,\Lambda_0(k))_{s's}|k_0,s'\rangle \\ &= \mathcal{D}^{(\ell)}(\Lambda_0(\Lambda k)^{-1}\,\Lambda\,\Lambda_0(k))_{s's}|\Lambda k,s'\rangle \end{aligned} \tag{111}$$

In general, the Wigner rotation is a rather complicated function of k and Λ, but it is simple if Λ is itself a rotation.

Exercise. Show that

$$\Lambda_0(Rk)^{-1}\, R\,\Lambda_0(k) = R. \tag{112}$$

Conclusion: relativity allows massive particles to have spin 0,1/2,1,3/2,.... A particle with spin ℓ that is at rest transforms under rotations just as in non-relativistic physics. Even if the particle is not at rest, the transformation law under rotations is the same as in non-relativistic physics, as long as one follows the conventions used above. However, the transformation law under general Lorentz transformations is more complicated: the spin indices get transformed according to the Wigner rotation.

18 Particle states for massless particles with spin

Now let's try to describe the quantum mechanics of a particle of mass 0 that has spin First, diagonalize the momentum operators P^μ. Call the eigenstates $|k,s\rangle$.

$$P^\mu|k,s\rangle = k^\mu|k,s\rangle. \tag{113}$$

Here $k^\mu k_\mu = 0$. We include an index s in anticipation that there may be more than one eigenstate of P^μ with a certain eigenvalue k^μ. We would like our representation to be unitary, so take inner product between basis vectors to be

$$\langle k,s'|p,s\rangle = \delta_{ss'}(2\pi)^3 2p^0\;\delta(\vec{p}-\vec{k}). \tag{114}$$

What should $U(\Lambda)$ do to our states? Eugene Wigner also figured out what to do in this case and we follow his treatment. First, define a standard momentum k_0^μ. We cannot take a particle at rest! It is convenient to take instead

$$k_0 = (E_0, 0, 0, E_0). \tag{115}$$

There are certain Lorentz transformations that leave k_0 invariant. These transformations evidently form a group, $G(k_0)$, called the little group. For *these* transformations,

$$U(\Lambda)|k_0,s\rangle = A(\Lambda)_{s's}|k_0,s'\rangle. \tag{116}$$

As in the case of the Lorentz transformations that leave the vector $(m, 0, 0, 0)$ unchanged, we argue that the matrices $A(\Lambda)$ should form a representation of the little group:

$$A(\Lambda_2 \lambda_1)_{s''s} = A(\Lambda_2)_{s''s'} A(\Lambda_1)_{s's}. \tag{117}$$

We do not yet know the irreducible representations of the little group of k_0, but for the moment, let us simply suppose that we have one with a label m:

$$U(\Lambda)|k_0, s\rangle = \mathcal{D}^{(m)}(\Lambda)_{s's}|k_0, s'\rangle \tag{118}$$

Now, for any momentum k^μ other than k_0^μ, define a standard Lorentz transformation $\Lambda_0(k)$ such that

$$\Lambda_0(k)^\mu{}_\nu\, k_0^\nu = k^\mu. \tag{119}$$

A convenient choice for $\Lambda_0(k)$ is a rotation in the plane of $\vec{k}$ and the z-axis that rotates $\vec{k}$ to a vector $\vec{k}'$ along the z-axis, followed by a boost in the z direction to adjust $|\vec{k}'|$ to E_0.

Now we *define* the state $|k, s\rangle$ in terms of the state $|k_0, s\rangle$ using

$$|k, s\rangle = U(\Lambda_0(k))|k_0, s\rangle. \tag{120}$$

By definition, this boost operation does not mix states with different spin indices.

What happens if we apply a general Lorentz transformation Λ to a general state $|k, s\rangle$? We have no choice:

$$\begin{aligned} U(\Lambda)|k, s\rangle &= U(\Lambda)U(\Lambda_0(k))|k_0, s\rangle \\ &= U(\Lambda_0(\Lambda k))U(\Lambda_0(\Lambda k)^{-1})U(\Lambda)U(\Lambda_0(k))|k_0, s\rangle \\ &= U(\Lambda_0(\Lambda k))\, U(\Lambda_0(\Lambda k)^{-1}\, \Lambda\, \Lambda_0(k))|k_0, s\rangle. \end{aligned} \tag{121}$$

Now if we apply the Lorentz transformation

$$\Lambda_0(\Lambda k)^{-1}\, \Lambda\, \Lambda_0(k) \tag{122}$$

to the vector k_0, we get k_0 back again:

$$\Lambda_0(\Lambda k)^{-1}\, \Lambda\, \Lambda_0(k)\, k_0 = \Lambda_0(\Lambda k)^{-1}\, \Lambda\, k = k_0. \tag{123}$$

Thus this matrix is an element of the little group of k_0. We already know what happens when we apply such a Lorentz transformation to one of the standard states. Thus

$$\begin{aligned} U(\Lambda)|k,s\rangle &= U(\Lambda_0(\Lambda k))\,\mathcal{D}^{(m)}(\Lambda_0(\Lambda k)^{-1}\,\Lambda\,\Lambda_0(k))_{s's}|k_0,s'\rangle \\ &= \mathcal{D}^{(m)}(\Lambda_0(\Lambda k)^{-1}\,\Lambda\,\Lambda_0(k))_{s's}|\Lambda k,s'\rangle \end{aligned} \tag{124}$$

It remains to figure out what are the (unitary) representations of the little group of a lightlike vector.

19 The little group for a lightlike vector and its representations

What Lorentz transformations leave the vector $k_0 = (E_0, 0, 0, E_0)$ unchanged? Put another way, what (linear combinations of the) generators of the Lorentz group give 0 when acting on k_0?

First, it should be clear that if you rotate k_0 about the z-axis, it stays the same. Thus

$$\mathcal{J}_3 = \mathcal{M}_{12} \tag{125}$$

should give

$$(\mathcal{J}_3)^\mu{}_\nu\, k_0^\nu = 0. \tag{126}$$

Let us define

$$\begin{aligned} \mathcal{B}_1 &= \frac{1}{\sqrt{2}}\,(\mathcal{M}_{01} + \mathcal{M}_{31}) \\ \mathcal{B}_2 &= \frac{1}{\sqrt{2}}\,(\mathcal{M}_{02} + \mathcal{M}_{32}) \end{aligned} \tag{127}$$

Then

$$(\mathcal{B}_i)^\mu{}_\nu\, k_0^\nu = 0. \tag{128}$$

Thus $\mathcal{J}_3$, $\mathcal{B}_1$, and $\mathcal{B}_2$ are the generators that we were looking for. The commutation relations among these generators are

$$\begin{aligned} [\mathcal{B}_1, \mathcal{B}_2] &= 0 \\ [\mathcal{J}_3, \mathcal{B}_i] &= i\epsilon_{3ij}\,\mathcal{B}_j. \end{aligned} \tag{129}$$

Exercise. Using our explicit expression for $(\mathcal{M}_{\alpha\beta})^{\mu}{}_{\nu}$, show that $(\mathcal{J}_3)^{\mu}{}_{\nu}\, k_0^{\nu} = 0$, $(\mathcal{B}_1)^{\mu}{}_{\nu}\, k_0^{\nu} = 0$, and $(\mathcal{B}_2)^{\mu}{}_{\nu}\, k_0^{\nu} = 0$.

Exercise. Using our explicit expression for $(\mathcal{M}_{\alpha\beta})^{\mu}{}_{\nu}$ or using the commutation relations among the $\mathcal{M}_{\alpha\beta}$, prove that these commutation relations for $\{\mathcal{J}_3, \mathcal{B}_1, \mathcal{B}_2\}$ hold.

Now we need some irreducible representation matrices $\mathcal{D}^{(m)}$ for the little group.

One possibility is that we diagonalize B_1 and B_2. Then we would have states $|k_0, \theta\rangle$ with

$$\begin{aligned} B_1|k_0,\theta\rangle &= b\cos\theta|k_0,\theta\rangle \\ B_2|k_0,\theta\rangle &= b\sin\theta|k_0,\theta\rangle. \end{aligned} \tag{130}$$

Here b labels the irreducible representation and is defined by

$$(B_1^2 + B_1^2)|k_0,\theta\rangle = b^2|k_0,\theta\rangle. \tag{131}$$

For J_3 acting on the states we would have

$$e^{-i\phi J_3}|k_0,\theta\rangle = |k_0,\theta+\phi\rangle. \tag{132}$$

Thus we would have a sort of continuous spin index, θ that ranges between $-\pi$ and π. Nature does not seem to have made use of this possibility.

The other possibility is that the states are $|k_0\rangle$ and

$$\begin{aligned} B_1|k_0\rangle &= 0 \\ B_2|k_0\rangle &= 0. \end{aligned} \tag{133}$$

Then we can diagonalize J_3:

$$J_3|k_0\rangle = m|k_0\rangle. \tag{134}$$

In order for the finite transformations to work correctly, we need m to be an integer or half-integer, but I will not go into the required reasoning here.

Note that the representations are one dimensional: there is a single state $|k_0\rangle$. The eigenvalue m labels the representation. The matrices $\mathcal{D}^{(m)}$ are just complex numbers of the form $e^{-im\phi}$.

If we want to include a representation of the parity operator, then we need a two dimensional representation. In addition to the state $|k_0, m\rangle \equiv |k_0\rangle$ we need a state $|k_0, -m\rangle$ with the opposite eigenvalue of J_z. The easiest way to see that is to note that m is the eigenvalue of the helicity operator

$$\vec{J} \cdot \vec{P}/|\vec{P}|. \tag{135}$$

Under a parity reflection, $\vec{J}$ stays the same while $\vec{P}$ is reversed, so the eigenvalue m must be reversed.

We conclude that for massless particles can be labelled by their helicities, which can take the values $m = 0, 1/2, -1/2, 1, -1, \ldots$. Familiar examples are the following.

1) Neutrinos in the Standard Model. $m = -1/2$ for neutrinos, $m = +1/2$ for antineutrinos. *But beware, recent experiments have indicated that the Standard Model is wrong in that neutrinos have mass.*

2) Photons. $m = +1$ and $m = -1$. (Note that there is no $m = 0$ state, as there would be for a massive spin 1 particle.)

2) Gravitons. $m = +2$ and $m = -2$. (Note that there are no $m = -1, 0, +1$ states, as there would be for a massive spin 2 particle.) *But then, no one has seen a graviton.*