Decay width calculation[1]

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Abstract

I offer here a solution for exercise 12.1 in the notes on time dependent perturbation theory.

1 Problem setup

We need to calculate

$$\Gamma = \sum_\lambda \int d\vec{k}\ 2\pi\delta(E_2 + \omega - E_1)\ \left|\left\langle 1,0,0;\vec{k},\lambda\middle|V\middle|2,1,m;0\right\rangle\right|^2 \quad . \tag{1}$$

Here the atom states are denoted by $\left|n,l,m\right\rangle$ and there is an additional photon with polarization λ and momentum $\vec{k}$ that accompanies the final $\left|1,0,0\right\rangle$ state. The photon energy is $\omega = |\vec{k}|$. The atom states have energies

$$\begin{aligned} E_1 &= -E_0 \quad , \\ E_2 &= -\frac{E_0}{4} \quad , \end{aligned} \tag{2}$$

where I use the shorthand notation

$$E_0 = \frac{e^2}{2a_0} \approx 13.6\ \text{eV} \quad . \tag{3}$$

Thus

$$\omega = \frac{3}{4}\ E_0 \quad . \tag{4}$$

We have

$$\left\langle 1,0,0;\vec{k},\lambda\middle|V\middle|2,1,m;0\right\rangle = \frac{e}{2\pi m\sqrt{\omega}}\ \vec{\varepsilon}(\vec{k},\lambda)\cdot\left\langle 1,0,0\middle|e^{-i\vec{k}\cdot\vec{x}}\vec{p}\middle|2,1,m\right\rangle \quad . \tag{5}$$

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Furthermore, we can use the dipole approximation,

$$\big\langle 1,0,0\big|e^{-i\vec{k}\cdot\vec{x}}\vec{p}\big|2,1,m\big\rangle \approx -im\omega\,\big\langle 1,0,0\big|\vec{x}\big|2,1,m\big\rangle \ . \tag{6}$$

With this approximation,

$$\big\langle 1,0,0;\vec{k},\lambda\big|V\big|2,1,m;0\big\rangle = \frac{-ie\sqrt{\omega}}{2\pi}\ \vec{\varepsilon}(\vec{k},\lambda)\cdot\big\langle 1,0,0\big|\vec{x}\big|2,1,m\big\rangle \ . \tag{7}$$

Thus

$$\Gamma = \sum_{\lambda}\int d\vec{k}\ 2\pi\delta(E_2+\omega-E_1)\ \frac{e^2\omega}{4\pi^2}\big|\vec{\varepsilon}(\vec{k},\lambda)\cdot\vec{v}(m)\big|^2 \ , \tag{8}$$

where

$$\vec{v}(m) = \big\langle 1,0,0\big|\vec{x}\big|2,1,m\big\rangle \ . \tag{9}$$

2 Integrating over the emitted photon

Write this as (using $|\vec{k}| = \omega$)

$$\begin{aligned}\Gamma &= \sum_{\lambda}\int \omega^2\,d\omega\int d\Omega\ 2\pi\delta(E_2+\omega-E_1)\ \frac{e^2\omega}{4\pi^2}\ \big|\vec{\varepsilon}(\vec{k},\lambda)\cdot\vec{v}(m)\big|^2 \\ &= \sum_{\lambda}\int d\Omega\ \frac{e^2\omega^3}{2\pi}\ \big|\vec{\varepsilon}(\vec{k},\lambda)\cdot\vec{v}(m)\big|^2 \ ,\end{aligned} \tag{10}$$

Now

$$\sum_{\lambda}\big|\vec{\varepsilon}(\vec{k},\lambda)\cdot\vec{v}(m)\big|^2 = |\vec{v}(m)|^2 - |\vec{k}\cdot\vec{v}(m)|^2/\vec{k}^{\,2} \ . \tag{11}$$

Also we can use the fact that the tensor k^ik^j averaged over the angles of $\vec{k}$ is invariant under rotations, so must be proportional to δ^{ij}. Thus

$$\begin{aligned}\int d\Omega\ |\vec{k}\cdot\vec{v}(m)|^2 &= \int d\Omega\ [v(m)^*]^i v(m)^j k^i k^j \\ &= 4\pi\,[v(m)^*]^i v(m)^j\ \frac{1}{3}\delta^{ij}\vec{k}^{\,2} \\ &= \frac{4\pi}{3}|\vec{v}(m)|^2\vec{k}^{\,2} \ .\end{aligned} \tag{12}$$

Also

$$\int d\Omega\ = 4\pi \ . \tag{13}$$

Thus

$$\int d\Omega \sum_{\lambda} \left|\vec{\varepsilon}(\vec{k},\lambda)\cdot\vec{v}(m)\right|^2 = \frac{8\pi}{3}\,|\vec{v}(m)|^2 \quad . \tag{14}$$

These results give

$$\Gamma = \frac{4e^2\omega^3}{3}\,|\vec{v}(m)|^2 \quad , \tag{15}$$

3 The matrix element

Now we need

$$\vec{v}(m) = \left\langle 1,0,0\middle|\vec{x}\middle|2,1,m\right\rangle \quad . \tag{16}$$

This is easiest if we take the $m = 0$ case. We note that $\vec{v}(0)$ is invariant under rotations about the z-axis, so it must point in the z-direction. Thus, letting $\vec{u}(0)$ be a unit vector in the z-direction, we have

$$\vec{v}(0) = \vec{u}(0)\ \left\langle 1,0,0\middle|r\cos(\theta)\middle|2,1,0\right\rangle \quad . \tag{17}$$

The matrix element is a straightforward integral,

$$\begin{aligned}
\left\langle 1,0,0\middle|r\cos(\theta)\middle|2,1,0\right\rangle &= \int_0^\infty r^2 dr \int d\Omega\ r\cos(\theta) \\
&\quad \times R_{10}(r)Y_0^0(\theta,\phi)R_{21}(r)Y_1^0(\theta,\phi) \\
&= \int_0^\infty r^2 dr\ 2\pi \int_{-1}^{1} d\cos\theta\ r\cos(\theta) \\
&\quad \times \frac{1}{a_0^{3/2}}\,2\,e^{-r/a_0}\,\frac{1}{(4\pi)^{1/2}} \\
&\quad \times \frac{1}{(2a_0)^{3/2}}\,\frac{r}{3^{1/2}a_0}e^{-r/(2a_0)}\,\frac{3^{1/2}}{(4\pi)^{1/2}}\ \cos\theta \\
&= \frac{1}{2^{3/2}a_0^4}\int_0^\infty r^4 dr\ e^{-3r/(2a_0)} \\
&\quad \times \int_{-1}^{1} d\cos\theta\ \ \cos^2\theta \\
&= \frac{1}{2^{3/2}a_0^4}\ \frac{2^5 a_0^5}{3^5}\,2^3 3\ \frac{2}{3} \\
&= \frac{2^{15/2}}{3^5}\ a_0
\end{aligned} \tag{18}$$

Thus

$$\vec{v}(0) = \frac{2^{15/2}}{3^5}\ a_0\ \vec{u}(0) \quad . \tag{19}$$

For $m = \pm 1$, we get almost the same result, but with a unit vector

$$\vec{u}(\pm 1) = \pm \frac{1}{\sqrt{2}}(1, \pm i, 0) \quad . \tag{20}$$

Thus in each case, we get

$$\vec{v}(m) = \frac{2^{15/2}}{3^5}\ a_0\ \vec{u}(m) \quad . \tag{21}$$

where $|\vec{u}(m)|^2 = 1$.

4 Result

Putting this together, we have

$$\Gamma = \frac{4e^2\omega^3}{3}\ \frac{2^{15}}{3^{10}}\ a_0^2 = \frac{2^{17}}{3^{11}}\ e^2 a_0^2 \omega^3 \quad , \tag{22}$$

We recall that

$$\omega = \frac{3}{4}\ E_0 \tag{23}$$

with

$$E_0 = \frac{e^2}{2a_0} \quad . \tag{24}$$

Thus

$$\omega^3 = \frac{3^3 e^4}{2^8 a_0^2}\ E_0 \tag{25}$$

This gives

$$\Gamma = \frac{2^9}{3^8}\ e^6\, E_0 \quad , \tag{26}$$

With

$$\begin{aligned} E_0 &\approx 13.6 \text{ eV} \quad , \\ e^2 &= \frac{1}{137} \quad , \end{aligned} \tag{27}$$

this is

$$\Gamma \approx \frac{2^9}{3^8}\ \frac{1}{137^3}\, 13.6 \text{ eV} \approx 4.13 \times 10^{-7} \text{ eV} \quad . \tag{28}$$

See G. W. F. Drake, J. Kwela, and A. van Wijngaarden, Phys. Rev. A **46**, 113 (1992) for theory and experimental results.