Problem solutions, 18 April 2012[1]

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Problem 5.1 is pretty simple, so I do not write out the solutions.

Problem 5.2 The probability to find the unperturbed eigenstate $\big|k^{(0)}\big\rangle$ in the exact eigenstate $\big|k(\lambda)\big\rangle$ is

$$P = \frac{|\big\langle k^{(0)}\big|k(\lambda)\big\rangle|^2}{\big\langle k(\lambda)\big|k(\lambda)\big\rangle} \quad . \tag{1}$$

Note that we need the denominator because, with the conventions we are using, $\big|k(\lambda)\big\rangle$ is not normalized. In fact, our normalization convention for $\big|k(\lambda)\big\rangle$ is $\big\langle k^{(0)}\big|k(\lambda)\big\rangle = 1$, so

$$P = \frac{1}{\big\langle k(\lambda)\big|k(\lambda)\big\rangle} \quad . \tag{2}$$

Now with

$$\big|k(\lambda)\big\rangle = \big|k^{(0)}\big\rangle + \lambda\big|k^{(1)}\big\rangle + \lambda^2\big|k^{(2)}\big\rangle + \cdots \tag{3}$$

and $\big\langle k^{(0)}\big|k^{(0)}\big\rangle = 1$, $\big\langle k^{(n)}\big|k^{(0)}\big\rangle = 0$ for $n > 0$, we have

$$\big\langle k(\lambda)\big|k(\lambda)\big\rangle = 1 + \lambda^2\big\langle k^{(1)}\big|k^{(1)}\big\rangle + \cdots \quad . \tag{4}$$

Thus

$$P = 1 - \lambda^2\big\langle k^{(1)}\big|k^{(1)}\big\rangle + \cdots \quad . \tag{5}$$

Recall that

$$\big|k^{(1)}\big\rangle = \frac{Q_k}{E_k^{(0)} - H_0} V\big|n^{(0)}\big\rangle \quad . \tag{6}$$

Thus

$$\big\langle k^{(1)}\big|k^{(1)}\big\rangle = \big\langle k^{(0)}\big|V\frac{Q_k}{[E_k^{(0)} - H_0]^2}V\big|k^{(0)}\big\rangle \quad . \tag{7}$$

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Inserting a complete set of unperturbed eigenstates, this is

$$\left\langle k^{(1)}\middle|k^{(1)}\right\rangle = \sum_{l\neq k} \left\langle k^{(0)}\middle|V\middle|l^{(0)}\right\rangle \frac{1}{[E_k^{(0)} - E_l^{(0)}]^2} \left\langle l^{(0)}\middle|V\middle|k^{(0)}\right\rangle \quad . \tag{8}$$

Thus

$$P = 1 - \lambda^2 \sum_{l\neq k} \frac{|\left\langle l^{(0)}\middle|V\middle|k^{(0)}\right\rangle|^2}{[E_k^{(0)} - E_l^{(0)}]^2} + \cdots \quad . \tag{9}$$

Problem 5.3 Our particle is in a two dimensional box. The energy eigenfunction for the ground state is

$$\psi(x,y) = \frac{2}{L}\,\sin(\pi x/L)\sin(\pi y/L) \tag{10}$$

For the first excited energy level, there are two states

$$\begin{aligned} \psi(x,y) &= \frac{2}{L}\,\sin(2\pi x/L)\sin(\pi y/L) \quad , \\ \psi(x,y) &= \frac{2}{L}\,\sin(\pi x/L)\sin(2\pi y/L) \quad . \end{aligned} \tag{11}$$

Thus

The energy shift for the ground state is just

$$\begin{aligned} \Delta &= \int_0^L dx \int_0^L dy\ |\psi(x,y)|^2 \lambda x y \\ &= \frac{4\lambda}{L^2}\left(\int_0^L dx\ x \sin^2(\pi x/L)\right)^2 \\ &= \frac{4\lambda}{L^2}\left(\frac{1}{2}\int_0^L dx\ (x+L-x)\sin^2(\pi x/L)\right)^2 \\ &= \frac{4\lambda}{L^2}\left(\frac{L^2}{4}\right)^2 \\ &= \frac{\lambda L^2}{4} \quad . \end{aligned} \tag{12}$$

The zeroth order energy eigenfunction is just the unperturbed ground state wave function.

For the first excited energy level, there is a twofold degeneracy at zeroth order so we need degenerate perturbation theory. However, we note that the

perturbation is invariant under the operator P that interchanges x and y. Thus we can diagonalize P along with H. The eigenstates of P within the space spanned by the first excited states are

$$\begin{aligned}\psi_+(x,y) &= \frac{\sqrt{2}}{L}\left\{\sin(2\pi x/L)\sin(\pi y/L) + \sin(\pi x/L)\sin(2\pi y/L)\right\} \quad , \\ \psi_-(x,y) &= \frac{\sqrt{2}}{L}\left\{\sin(2\pi x/L)\sin(\pi y/L) - \sin(\pi x/L)\sin(2\pi y/L)\right\} \quad .\end{aligned} \tag{13}$$

Thus these are the zeroth order energy eigenstates. The corresponding energy shifts are

$$\begin{aligned}\Delta_\pm &= \int_0^L dx \int_0^L dy \ |\psi_\pm(x,y)|^2 \lambda xy \\ &= \frac{2\lambda}{L^2}\left(\int_0^L dx \ x\sin^2(2\pi x/L)\right)\left(\int_0^L dy \ y\sin^2(\pi y/L)\right) \\ &\quad + \frac{2\lambda}{L^2}\left(\int_0^L dx \ x\sin^2(\pi x/L)\right)\left(\int_0^L dy \ y\sin^2(2\pi y/L)\right) \\ &\quad \pm \frac{4\lambda}{L^2}\left(\int_0^L dx \ x\sin(\pi x/L)\sin(2\pi x/L)\right) \\ &\quad\quad \times \left(\int_0^L dy \ y\sin(\pi y/L)\sin(2\pi y/L)\right) \\ &= \frac{2\lambda}{L^2}\left(\frac{L^2}{4}\right)\left(\frac{L^2}{4}\right) + \frac{2\lambda}{L^2}\left(\frac{L^2}{4}\right)\left(\frac{L^2}{4}\right) \pm \frac{4\lambda}{L^2}\left(\frac{8L^2}{9\pi^2}\right)\left(\frac{8L^2}{9\pi^2}\right) \\ &= \lambda L^2\left\{\frac{1}{4} \pm \frac{256}{81\,\pi^4}\right\} \quad .\end{aligned} \tag{14}$$

Problem 5.4 We have a harmonic oscillator in the x-direction and a harmonic oscillator in the y-direction. The energies are

$$E(n_x, n_y) = (n_x + n_y + 1)\omega \quad , \tag{15}$$

where n_x and n_y are non-negative integers. Thus the lowest energy levels are

$$\begin{aligned}E(0,0) &= \omega \quad , \\ E(1,0) &= 2\omega \quad , \\ E(0,1) &= 2\omega \quad .\end{aligned} \tag{16}$$

The second energy level has a double degeneraacy.

The first order energy shift for the ground state $\big|0,0\big\rangle$ is just

$$\Delta = \delta m\,\omega^2 \big\langle 0,0\big|xy\big|0,0\big\rangle = 0 \quad . \tag{17}$$

That is, the ground state energy remains equal to ω at first order.

For the first excited energy level, there is a twofold degeneracy at zeroth order so we need degenerate perturbation theory. However, we note that the perturbation is invariant under the operator P that interchanges x and y. Thus we can diagonalize P along with H. The eigenstates of P within the space spanned by the first excited states are

$$\big|\pm\big\rangle = \frac{1}{\sqrt{2}}\left\{\big|1,0\big\rangle \pm \big|0,1\big\rangle\right\} \quad . \tag{18}$$

Thus these are the zeroth order energy eigenstates. The corresponding first order energy shifts are

$$\begin{aligned}\Delta_\pm &= \frac{\delta m\,\omega^2}{2}\left\{\big\langle 1,0\big|xy\big|1,0\big\rangle + \big\langle 0,1\big|xy\big|0,1\big\rangle \pm \big\langle 1,0\big|xy\big|0,1\big\rangle \pm \big\langle 0,1\big|xy\big|1,0\big\rangle\right\} \\ &= \pm\,\delta m\,\omega^2\left\{\big\langle 1\big|x\big|0\big\rangle\big\langle 0\big|y\big|1\big\rangle + \big\langle 1\big|y\big|0\big\rangle\big\langle 0\big|x\big|1\big\rangle\right\}\end{aligned} \tag{19}$$

where in the second line the first matrix element refers to the x-oscillator and the second matrix refers to the y-oscillator. These matrix elements are

$$\big\langle 1\big|x\big|0\big\rangle = \big\langle 0\big|x\big|1\big\rangle = \big\langle 1\big|y\big|0\big\rangle = \big\langle 0\big|y\big|1\big\rangle = \frac{1}{\sqrt{2m\omega}} \tag{20}$$

Thus

$$\Delta_\pm = \pm\,\frac{\delta m\,\omega}{m} \quad . \tag{21}$$

The energy eigenvalues correct to first order in the perturbation are then

$$E_\pm = 2\omega \pm \frac{\delta m\,\omega}{m} \quad . \tag{22}$$

Now, let's solve this exactly. We have

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega^2}{2}(x^2 + y^2 + \frac{\delta m}{m}\,2xy) \tag{23}$$

This is

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{(m+\delta m)\omega^2}{4}(x+y)^2 + \frac{(m-\delta m)\omega^2}{4}(x-y)^2 \quad . \tag{24}$$

Let us define new coordinates

$$\begin{aligned} \bar{x} &= \frac{x+y}{\sqrt{2}} \quad , \\ \bar{y} &= \frac{x-y}{\sqrt{2}} \quad , \end{aligned} \tag{25}$$

Then

$$H = \frac{p_{\bar{x}}^2}{2m} + \frac{p_{\bar{y}}^2}{2m} + \frac{(m+\delta m)\omega^2}{2}\bar{x}^2 + \frac{(m-\delta m)\omega^2}{2}\bar{y}^2 \quad . \tag{26}$$

Now we have two independent oscillators. The total energy is

$$E(n_{\bar{x}}, n_{\bar{y}}) = \left(n_{\bar{x}} + \frac{1}{2}\right)\omega\sqrt{1+\frac{\delta m}{m}} + \left(n_{\bar{y}} + \frac{1}{2}\right)\omega\sqrt{1-\frac{\delta m}{m}} \quad . \tag{27}$$

Up to first order in δm, this is

$$E(n_{\bar{x}}, n_{\bar{y}}) \approx (n_{\bar{x}} + n_{\bar{y}} + 1)\,\omega + (n_{\bar{x}} - n_{\bar{y}})\,\omega\frac{\delta m}{m} \quad . \tag{28}$$

This agrees with what we found by applying first order perturbation theory directly.