Rotation group notes, exercise 22.1

For \( \vec{n} \) equal to the unit vector in the \( z \)-direction, the state we are looking for has the form

\[
|\psi\rangle = \alpha |\hat{z},+1\rangle \otimes |\hat{z},-1\rangle + \beta |\hat{z},0\rangle \otimes |\hat{z},0\rangle + \gamma |\hat{z},-1\rangle \otimes |\hat{z},+1\rangle .
\] (1)

If we apply \( J_+ \) to this, we should get zero:

\[
0 = J_+ |\psi\rangle = \alpha \sqrt{2} |\hat{z},+1\rangle \otimes |\hat{z},0\rangle + \beta \sqrt{2} |\hat{z},1\rangle \otimes |\hat{z},0\rangle + \gamma \sqrt{2} |\hat{z},0\rangle \otimes |\hat{z},+1\rangle .
\] (2)

The coefficients of the two linearly independent vectors must both vanish, so

\[
\alpha = -\beta , \quad \gamma = -\beta .
\] (3)

To make a normalized state, we need \( \beta = -1/\sqrt{3} \), or any phase factor times this. Thus

\[
|\psi\rangle = \frac{1}{\sqrt{3}} \left[ |\hat{z},+1\rangle \otimes |\hat{z},-1\rangle - |\hat{z},0\rangle \otimes |\hat{z},0\rangle + |\hat{z},-1\rangle \otimes |\hat{z},+1\rangle \right] .
\] (4)

The same coefficients can apply for any choice of quantization axis \( \vec{n} \). We just need to define \( |\vec{n},m\rangle \) by applying the rotation operator that rotates \( \hat{z} \) to \( \vec{n} \).

Rotation group notes, exercise 22.2

We have

\[
P(\vec{n}_A, +1; \vec{n}_B, +1) = |\langle \vec{n}_A, +1 | \otimes \langle \vec{n}_B, +1 | \rangle |\psi\rangle|^2 .
\] (5)
Let us take
\[ |\psi\rangle = \frac{1}{\sqrt{3}} \left[ |\vec{n}_B, +1\rangle \otimes |\vec{n}_B, -1\rangle - |\vec{n}_B, 0\rangle \otimes |\vec{n}_B, -1\rangle + |\vec{n}_B, -1\rangle \otimes |\vec{n}_B, +1\rangle \right]. \] (6)

Then
\[ P(\vec{n}_A, +1; \vec{n}_B, +1) = \frac{1}{3} |\langle \vec{n}_A, +1 | \vec{n}_B, -1 \rangle|^2. \] (7)

This can only depend on the angle \( \theta \) between \( \vec{n}_A \) and \( \vec{n}_B \). Therefore, let’s keep things simple by taking \( \vec{n}_A = \hat{z} \) and \( \vec{n}_B = \cos \theta \hat{z} + \sin \theta \hat{x} \).

A simple way to evaluate this is
\[ \langle \hat{z}, +1 | \vec{n}_B, -1 \rangle = \langle \hat{z}, +1 | U(R) | \hat{z}, -1 \rangle = D_{+1,-1}^{(1)}(R), \] (8)
where \( R\hat{z} = \vec{n}_B \). The SU(2) matrix corresponding to \( R \) is
\[ U = \cos(\theta/2) - i \sin(\theta/2) \sigma_y. \] (9)

That is
\[ U = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}. \] (10)

We can just use our simple method to construct the spin \( j \) rotation matrices from \( 2j \) copies of the spin 1/2 rotation matrices. This gives
\[ D_{+1,-1}^{(1)}(R) = (U^+)^2 = \sin^2(\theta/2). \] (11)

This gives
\[ P(\vec{n}_A, +1; \vec{n}_B, +1) = \frac{1}{3} \sin^4(\theta/2). \] (12)

Is this reasonable? For \( \theta = 0 \), Bob has to always get the opposite result from Alice, so we expect \( P(\vec{n}_A, +1; \vec{n}_B, +1) = 0 \) and that is what we get. For \( \theta = \pi \), Bob always gets the same result as Alice and Alice gets spin +1 one third of the time, so we expect \( P(\vec{n}_A, +1; \vec{n}_B, +1) = 1/3 \) and that is what we get.

**Sakurai, problem 3.30**

In this problem, I assume that \( U_i \) and \( V_j \) commute with each other. If they don’t, then we should use \( (U_i V_j + V_j U_i)/2 \) in place of \( U_i V_j \) everywhere below.
(a) Let’s use

\[ U_\pm = \mp \frac{1}{\sqrt{2}} (U_x \pm iU_y), \quad U_0 = U_z, \]
\[ \quad V_\pm = \mp \frac{1}{\sqrt{2}} (V_x \pm iV_y), \quad V_0 = V_z. \] (13)

Then the spin 1 combination is the antisymmetric combination such that the \( J_z \) values for \( U \) and \( V \) sum to the \( J_z \) values of \( T \):

\[ T_{+1}^{(1)} = U_+ V_0 - U_0 V_+ , \]
\[ T_0^{(1)} = U_+ V_- - U_- V_+ , \]
\[ T_{-1}^{(1)} = - U_- V_0 + U_0 V_- . \] (14)

The overall normalization is arbitrary, but the relative normalization of the three entries is fixed by the requirement that an angular momentum raising operator commuted with \( T_0^{(1)} \) gives \( T_{+1}^{(1)} \) and an angular momentum lowering operator commuted with \( T_0^{(1)} \) gives \( T_{-1}^{(1)} \). It is for that reason that the sign for \( T_{-1}^{(1)} \) is what is given above.

Written explicitly in terms of the cartesian components of \( U \) and \( V \), we find for \( T_{+1}^{(1)} \) the result

\[ T_{+1}^{(1)} = - \frac{1}{\sqrt{2}} [(U_x + iU_y)V_z - U_z(V_x + iV_y)] \]
\[ = - \frac{1}{\sqrt{2}} [(U_x V_z - U_z V_x) + i(U_y V_z - U_z V_y)] \]
\[ = - \frac{1}{\sqrt{2}} [-(\vec{U} \times \vec{V})_y + i(\vec{U} \times \vec{V})_x] \]
\[ = - \frac{i}{\sqrt{2}} [(\vec{U} \times \vec{V})_x + i(\vec{U} \times \vec{V})_y] . \] (15)

For \( T_{0}^{(1)} \) we find

\[ T_{0}^{(1)} = - \frac{1}{2} [(U_x + iU_y)(V_x - iV_y) - (U_x - iU_y)(V_x + iV_y)] \]
\[ = i[U_x V_y - U_y V_x] \]
\[ = i(\vec{U} \times \vec{V})_z . \] (16)
For $T^{(1)}_{-1}$ we find

$$T^{(1)}_{-1} = \frac{1}{\sqrt{2}}[-(U_x - iU_y)V_z + U_z(V_x - iV_y)]$$

$$= -\frac{1}{\sqrt{2}}[(U_xV_z - U_zV_x) - i(U_yV_z - U_zV_y)]$$

$$= -\frac{1}{\sqrt{2}}[-(\vec{U} \times \vec{V})_y - i(\vec{U} \times \vec{V})_x]$$

$$= i\frac{1}{\sqrt{2}}[(\vec{U} \times \vec{V})_x - i(\vec{U} \times \vec{V})_y].$$

(17)

Thus

$$T^{(1)}_{\pm 1} = \mp \frac{1}{\sqrt{2}} (W_x \pm iW_y),$$

$$T^{(1)}_0 = W_z,$$

(18)

where $W = i\vec{U} \times \vec{V}$.

(b) For spin 2, we take the completely symmetric combination

$$T^{(2)}_{+2} = U_+ V_+,$$

$$T^{(2)}_{+1} = \frac{1}{\sqrt{2}} (U_+ V_0 + U_0 V_+),$$

$$T^{(2)}_0 = \frac{1}{\sqrt{6}} (U_+ V_- + U_- V_+ + 2U_0 V_0),$$

$$T^{(2)}_{-1} = \frac{1}{\sqrt{2}} (U_- V_0 + U_0 V_-),$$

$$T^{(2)}_{-2} = U_- V_-.$$

(19)

The coefficients come from commuting with the angular lowering operator starting at the $m = +2$ operator and commuting with the angular raising operator starting at the $m = -2$ operator. This also gives the factor 2 multiplying $U_0 V_0$ in $T^{(2)}_0$. This is the same way that we constructed $Y^l_m$ for $l = 2$.

Sakurai, problem 3.31

(a) We define

$$T_\pm = \mp \frac{1}{\sqrt{2}} (x \pm iy),$$

$$T_0 = z.$$
Then
\[
\langle n', l', m' | T_q | n, l, m \rangle = \frac{\langle n', l' | T | n, l \rangle}{\sqrt{2l + 1}} \langle l', m' | 1, l, q, m \rangle
\] (21)

Where \( \langle j, m | j_1, j_2, m_1, m_2 \rangle \) are Clebsch-Gordan coefficients. The reduced matrix element \( \langle n', l' | T | n, l \rangle \) is independent of \( m', m, \) and \( q \).

We note from this that \( \langle n', l', m' | T_q | n, l, m \rangle = 0 \) unless \( l' = l, l' = l + 1 \) or \( l' = l - 1 \). The matrix element also vanishes unless \( m' = m + q \). We also note that the operator \( \vec{x} \) has odd parity, so \( \langle n', l', m' | T_q | n, l, m \rangle = 0 \) when \( l' = l \). That is, we have a non-zero matrix element only when \( l' = l + 1 \) or \( l' = l - 1 \). But considerations of parity are not covered until chapter 4.

We can choose the matrix element with \( m' = q = m = 0 \) as a standard matrix element to determine the reduced matrix element. Then
\[
\langle n', l', m' | T_q | n, l, m \rangle = \frac{\langle n', l', 0 | T_0 | n, l, 0 \rangle}{\langle l', 0 | 1, l, 0, 0 \rangle} \langle l', m' | 1, l, q, m \rangle
\] (22)

This gives all of the matrix elements for varying \( m', m, \) and \( q \) (with \( m' = m + q \)) in terms of one of them.

(b) The matrix elements can be written as integrals. Then one can evaluate \( \langle n', l', 0 | T_0 | n, l, 0 \rangle \). The angular momentum structure follows from the known integrals of three \( Y_l^m \) functions. I think that we can be excused from performing the radial integrals.

**Sakurai, problem 3.32**

(a) Using our results from problem 3.30, we define the components of an angular momentum 2 irreducible tensor operator as

\[
\begin{align*}
T_{+2}^{(2)} &= r_+ r_+ , \\
T_{+1}^{(2)} &= \frac{1}{\sqrt{2}} (r_+ r_0 + r_0 r_+) , \\
T_0^{(2)} &= \frac{1}{\sqrt{6}} (r_+ r_- + r_- r_+ + 2r_0 r_0) , \\
T_{-1}^{(2)} &= \frac{1}{\sqrt{2}} (r_- r_0 + r_0 r_-) , \\
T_{-2}^{(2)} &= r_- r_- .
\end{align*}
\] (23)
That is
\[ T_{+2}^{(2)} = \frac{1}{2} (x^2 - y^2 + 2ixy) , \]
\[ T_{+1}^{(2)} = -(x + iy)z , \]
\[ T_{0}^{(2)} = \frac{1}{\sqrt{6}} (3z^2 - r^2) , \]
\[ T_{-1}^{(2)} = (x - iy)z , \]
\[ T_{-2}^{(2)} = \frac{1}{2} (x^2 - y^2 - 2ixy) . \]

Thus
\[ xy = -\frac{i}{2} (T_{+2}^{(2)} - T_{-2}^{(2)}) , \]
\[ xz = \frac{1}{2} (T_{+1}^{(2)} - T_{+1}^{(2)}) , \]
\[ x^2 - y^2 = T_{+2}^{(2)} + T_{-2}^{(2)} , \]
\[ 3z^2 - r^2 = \sqrt{6} T_{0}^{(2)} . \]

(b) We define
\[ Q = e\langle \alpha, j, m|3z^2 - r^2|\alpha, j, j \rangle \] (26)
as the dipole moment for an atomic state. That is,
\[ \langle \alpha, j, j|T_{0}^{(2)}|\alpha, j, j \rangle = \frac{Q}{\sqrt{6} e} . \] (27)

With this notation
\[ e\langle \alpha, j, m'|x^2 - y^2|\alpha, j, j \rangle = e\langle \alpha, j, m'|T_{+2}^{(2)} + T_{-2}^{(2)}|\alpha, j, j \rangle \] (28)

The Wigner-Eckart theorem gives
\[ e\langle \alpha, j, m'|x^2 - y^2|\alpha, j, j \rangle = e\langle \alpha, j, m'|T_{0}^{(2)}|\alpha, j, j \rangle \times \left[ \frac{\langle j, m'|2, j, 2, j \rangle}{\langle j, j|2, j, 0, j \rangle} + \frac{\langle j, m'|2, j, -2, j \rangle}{\langle j, j|2, j, 0, j \rangle} \right] . \] (29)
That is
\[
e \langle \alpha, j, m'| x^2 - y^2 | \alpha, j, j \rangle = \frac{Q}{\sqrt{6}} \left[ \frac{\langle j, m'| 2, j, 2, j \rangle}{\langle j, j | 2, j, 0, j \rangle} + \frac{\langle j, m'| 2, j, -2, j \rangle}{\langle j, j | 2, j, 0, j \rangle} \right].
\]

(30)

The only non-zero matrix elements are for \( m' = j \pm 2 \) and there is no state with \( m' = j + 2 \). Thus the only non-zero matrix element is for \( m' = j - 2 \).

It is
\[
e \langle \alpha, j, j - 2 | x^2 - y^2 | \alpha, j, j \rangle = \frac{Q}{\sqrt{6}} \frac{\langle j, j - 2| 2, j, -2, j \rangle}{\langle j, j | 2, j, 0, j \rangle}.
\]

(31)