Problem solutions, 2 May 2012[1]

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Here is a solution for problem 5.22, which seemed to cause the most difficulty.

Problem 5.22 The expectation value of x is

$$\langle x(t)\rangle = -i\big\langle 0\big|x_I(t)\int_0^t d\tau\ V_I(\tau)\big|0\big\rangle + i\big\langle 0\big|\int_0^t d\tau\ V_I(\tau)x_I(t)\big|0\big\rangle\ . \tag{1}$$

For the given $V(t)$, this is

$$\langle x(t)\rangle = -iF_0\int_0^t d\tau\ \cos(\omega\tau)\big\langle 0\big|x_I(t)x_I(\tau) - x_I(\tau)x_I(t)\big|0\big\rangle\ . \tag{2}$$

We can insert a complete set of states between the two x operators. The only state that contributes is the first excited oscillator state. Thus

$$\begin{aligned}\langle x(t)\rangle =\ & -iF_0\int_0^t d\tau\ \cos(\omega\tau)\\ & \Big\{\big\langle 0\big|x_I(t)\big|1\big\rangle\big\langle 1\big|x_I(\tau)\big|0\big\rangle - \big\langle 0\big|x_I(\tau)\big|1\big\rangle\big\langle 1\big|x_I(t)\big|0\big\rangle\Big\}\ .\end{aligned} \tag{3}$$

Including the interaction picture time dependence, this is

$$\begin{aligned}\langle x(t)\rangle =\ & -iF_0\int_0^t d\tau\ \cos(\omega\tau)\\ & \Big\{e^{-i\omega_0(t-\tau)}\big\langle 0\big|x\big|1\big\rangle\big\langle 1\big|x\big|0\big\rangle - e^{+i\omega_0(t-\tau)}\big\langle 0\big|x\big|1\big\rangle\big\langle 1\big|x\big|0\big\rangle\Big\}\ .\end{aligned} \tag{4}$$

Using the known value of $\big\langle 1\big|x\big|0\big\rangle$, this is

$$\langle x(t)\rangle = -iF_0\int_0^t d\tau\ \cos(\omega\tau)\frac{1}{2m\omega_0}\Big\{e^{-i\omega_0(t-\tau)} - e^{+i\omega_0(t-\tau)}\Big\}\ . \tag{5}$$

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That is

$$\langle x(t)\rangle = -\frac{F_0}{m\omega_0}\int_0^t d\tau\ \cos(\omega\tau)\sin(\omega_0(t-\tau)) \ . \tag{6}$$

We can simplify this to

$$\langle x(t)\rangle = -\frac{F_0}{2m\omega_0}\int_0^t d\tau\ \Big\{\sin(\omega_0 t+(\omega-\omega_0)\tau)-\sin(-\omega_0 t+(\omega+\omega_0)\tau)\Big\} \ . \tag{7}$$

Then the integral is

$$\begin{aligned}\langle x(t)\rangle = \frac{F_0}{2m\omega_0}\Big\{&\frac{1}{\omega-\omega_0}[\cos(\omega t)-\cos(\omega_0 t)] \\ &-\frac{1}{\omega+\omega_0}[\cos(\omega t)-\cos(\omega_0 t)]\Big\} \ .\end{aligned} \tag{8}$$

That is

$$\langle x(t)\rangle = \frac{F_0}{m(\omega^2-\omega_0^2)}[\cos(\omega t)-\cos(\omega_0 t)] \ . \tag{9}$$

Is this procedure OK for $\omega \approx \omega_0$? Generally, it is not. However in this case it gives the exact answer. See below.

Is this right? Let's check the equation of motion:

$$\begin{aligned} m\frac{d^2}{dt^2}\langle x(t)\rangle &= -\frac{F_0}{(\omega^2-\omega_0^2)}\left[\omega^2\cos(\omega t)-\omega_0^2\cos(\omega_0 t)\right] \\ &= -\frac{F_0}{(\omega^2-\omega_0^2)}\left[(\omega^2-\omega_0^2)\cos(\omega t)\right. \\ &\qquad \left.-\,\omega_0^2\cos(\omega_0 t)+\omega_0^2\cos(\omega t)\right] \\ &= -m\omega_0^2\langle x(t)\rangle - F_0\cos(\omega t) \ .\end{aligned} \tag{10}$$

This is the classical equation of motion with the applied force $-dV/dx$. The Heisenberg picture operator $x_H(t)$ should obey the classical equation of motion exactly. Thus $\big\langle 0\big|x_H(t)\big|0\big\rangle$ should obey the classical equation of motion exactly. Since the classical result is linear in F_0, first order perturbation theory should give the exact result. It does.