Problem solutions, 9 May 2012[1]

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16 May 2012

Here is a solution for problem 5.38, which seemed to cause the most difficulty. Please check the algebra: there could be errors.

Problem 5.38 With a potential $V_0 \cos(kz - \omega t)$, we write

$$V_0 \cos(kz - \omega t) = \frac{V_0}{2}\, e^{ikz} e^{-i\omega t} + \frac{V_0}{2}\, e^{-ikz} e^{+i\omega t} \quad . \tag{1}$$

The first term leads to transititions to states $\langle f|$ with $\omega_f = \omega_1 + \omega$, where ω_1 is the energy of the ground state atom that we start with. The second term leads to transititions to states $\langle f|$ with $\omega_f = \omega_1 - \omega$. Since the final state of interest has $\omega_{\rm f} > \omega_1$, this term does not contribute.

The transition rate is (using eq. (58) from our notes)

$$dR = (2\pi)\delta(\omega_{\rm f} - \omega_1 - \omega)\, d\vec{p}_{\rm f}\, \frac{V_0^2}{4}\, |\langle \vec{p}_{\rm f}| e^{ikz} |1\rangle|^2 \tag{2}$$

We can write this using $\omega_{\rm f} = p_{\rm f}^2/(2m)$ and

$$d\vec{p}_f = m p_{\rm f}\, d\omega_{\rm f}\, d\Omega_{\rm f} \quad . \tag{3}$$

This gives the transition rate per unit solid angle:

$$\frac{dR}{d\Omega_{\rm f}} = \frac{\pi m p_{\rm f} V_0^2}{2}\, |\langle \vec{p}_{\rm f}| e^{ikz} |1\rangle|^2 \tag{4}$$

Let's look at the matrix element $\langle \vec{p}_{\rm f}| e^{ikz} |1\rangle$. The final state wave function should really be an exact eigenstate for an electron in a Coulomb potential, but we approximate this as a plane wave. Then we have factors $\exp(-i\vec{p}_{\rm f}\cdot\vec{r})$ and $\exp(i\vec{k}\cdot\vec{r})$, where $\vec{k}$ points along the z-axis. The product of these is $\exp(-i\vec{q}\cdot\vec{r})$ where $\vec{q} = \vec{p}_{\rm f} - \vec{k}$.

$$\langle \vec{p}_{\rm f}| e^{ikz} |1\rangle = \frac{1}{(2\pi)^{3/2}} \int_0^\infty r^2 dr\, \psi_0(r) \int_{-1}^{1} d\cos\theta\, e^{-iqr\cos\theta} \int_0^{2\pi} d\phi \tag{5}$$

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Here we have chosen the coordinate system for the position $\vec{r}$ of the electron so that the z-axis is along $\vec{q}$. We have denoted the radial part of the wave function by $\psi_0(r)$:

$$\psi_0(r) = \frac{1}{\sqrt{\pi}} \left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0} \quad . \tag{6}$$

Now we can perform the ϕ and θ integrals:

$$\left\langle \vec{p}_{\rm f} \middle| e^{ikz} \middle| 1 \right\rangle = \frac{i}{(2\pi)^{1/2} q} \int_0^\infty r dr \, \psi_0(r) \left[e^{-iqr} - e^{iqr} \right] \quad . \tag{7}$$

We can perform the r integral using

$$\int_0^\infty r dr e^{-\alpha r} = \frac{1}{\alpha^2} \quad . \tag{8}$$

Thus

$$\left\langle \vec{p}_{\rm f} \middle| e^{ikz} \middle| 1 \right\rangle = \frac{i}{(2\pi)^{1/2} q} \frac{1}{\sqrt{\pi}} \left(\frac{Z}{a_0}\right)^{3/2} \left[\frac{1}{[(Z/a_0) - iq]^2} - \frac{1}{[(Z/a_0) + iq]^2} \right] \quad . \tag{9}$$

That is

$$\left\langle \vec{p}_{\rm f} \middle| e^{ikz} \middle| 1 \right\rangle = \frac{i}{(2\pi)^{1/2} q} \frac{1}{\sqrt{\pi}} \left(\frac{Z}{a_0}\right)^{3/2} \frac{4iZq/a_0}{[(Z^2/a_0^2) + q^2]^2} \quad . \tag{10}$$

This simplifies to

$$\left\langle \vec{p}_{\rm f} \middle| e^{ikz} \middle| 1 \right\rangle = -\frac{2^{3/2}}{\pi} \left(\frac{Z}{a_0}\right)^{5/2} \frac{1}{[(Z^2/a_0^2) + q^2]^2} \quad . \tag{11}$$

The dependence on angle is contained in

$$q^2 = p_{\rm f}^2 + k^2 - 2 p_{\rm f} k \cos\theta \quad , \tag{12}$$

where θ is the angle between $\vec{p}_{\rm f}$ and the original z-axis.

With an incoming electromagnetic wave, there is an operator $\vec{p}$, which becomes $\vec{p}_{\rm f}$ and there is a polarization vector $\vec{\epsilon}$. The amplitude is proportional to $\vec{p}_{\rm f} \cdot \vec{\epsilon}$, so there is a strong angular dependence favoring emission in the direction of the polarization, $\pm\vec{\epsilon}$. In the present problem, the angular dependence is weak if k is small compared to $p_{\rm f}$.