Galilean boost symmetry[1]

D. E. Soper[2]
University of Oregon
4 April 2011

I offer here some background for Chapter 4 of J. J. Sakurai, *Modern Quantum Mechanics*. This chapter discusses symmetries other than translations and rotations, including parity, translations in a lattice, and time reversal. It omits boost invariance, so I discuss boost invariance here.

1 Introduction

Non-relativistic classical mechanics is invariant under a change of reference frame to one that is moving with a fixed velocity with respect to the reference frame originally considered. To be more precise, suppose that we describe a system in the "lab frame" using coordinates (t, x, y, z). Then we switch to a "moving frame" using coordinates (t', x', y', z'). The transformation is $t' = t$, $\vec{x}' = \vec{x} + \vec{v}\,t$. Note that if we were doing relativistic physics, we would have $t' \neq t$, but life is simpler with non-relativistic physics. For understanding quantum mechanics, it is simpler to think of leaving the coordinates alone but giving the system a boost by velocity $\vec{v}$: we simply add $\vec{v}$ to the velocity of every particle.

2 Boost operator

We are thus led to consider the unitary operator $U(\vec{v})$ that boosts the system. The operator $U(\vec{v})$ does nothing to spins. For this reason, we consider particles without spins. We consider a system made of particles 1 through N with masses m_1 through m_N. We begin with a momentum space description, in which the basis vectors are $\big|\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_N\big\rangle$.

The operator $U(\vec{v})$ adds $\vec{v}$ to the velocity of every particle. Thus for a particle i with mass m_i, $U(\vec{v})$ adds $m_i\vec{v}$ to the momentum of the particle:

$$U(\vec{v})\big|\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_N\big\rangle = \big|\vec{p}_1 + m_1\vec{v}, \vec{p}_2 + m_2\vec{v}, \ldots, \vec{p}_N + m_N\vec{v}\big\rangle \ . \qquad (1)$$

[2]soper@uoregon.edu

Now let's see what this does in a position space description. We have

$$\begin{aligned}
U(\vec{v})\big|\vec{x}_1,\vec{x}_2,\ldots,\vec{x}_N\big\rangle &= (2\pi)^{-3N/2}\int d\vec{p}_1\,\cdots d\vec{p}_N\;\exp\left(-\mathrm{i}\sum_i \vec{p}_i\cdot\vec{x}_i\right)\\
&\quad\times U(\vec{v})\big|\vec{p}_1,\vec{p}_2,\ldots,\vec{p}_N\big\rangle\\
&= (2\pi)^{-3N/2}\int d\vec{p}_1\,\cdots d\vec{p}_N\;\exp\left(-\mathrm{i}\sum_i \vec{p}_i\cdot\vec{x}_i\right)\\
&\quad\times\big|\vec{p}_1+m_1\vec{v},\vec{p}_2+m_2\vec{v},\ldots,\vec{p}_N+m_N\vec{v}\big\rangle\\
&= (2\pi)^{-3N/2}\int d\vec{k}_1\,\cdots d\vec{k}_N\\
&\quad\times\exp\left(-\mathrm{i}\sum_i(\vec{k}_i-m_i\vec{v})\cdot\vec{x}_i\right)\big|\vec{k}_1,\vec{k}_2,\ldots,\vec{k}_N\big\rangle\\
&= \exp\left(\mathrm{i}\sum_i m_i\vec{v}\cdot\vec{x}_i\right)(2\pi)^{-3N/2}\int d\vec{k}_1\,\cdots d\vec{k}_N\\
&\quad\times\exp\left(-\mathrm{i}\sum_i\vec{k}_i\cdot\vec{x}_i\right)\big|\vec{k}_1,\vec{k}_2,\ldots,\vec{k}_N\big\rangle\\
&= \exp\left(i\sum_i m_i\vec{v}\cdot\vec{x}_i\right)\big|\vec{x}_1,\vec{x}_2,\ldots,\vec{x}_N\big\rangle\ .
\end{aligned}\tag{2}$$

That is

$$U(\vec{v})\big|\vec{x}_1,\vec{x}_2,\ldots,\vec{x}_N\big\rangle = \exp\left(\mathrm{i}M\vec{v}\cdot\vec{R}\right)\big|\vec{x}_1,\vec{x}_2,\ldots,\vec{x}_N\big\rangle \tag{3}$$

where M is the total mass of the system,

$$M=\sum_i m_i \tag{4}$$

and $\vec{R}$ is the operator that gives the position of the center of mass of the system

$$\vec{R}=\frac{1}{M}\sum_i m_i\vec{x}_i\ . \tag{5}$$

We learn two things from this. First, the boost operator is very simple in the position representation. Second, the generator of boosts is $-M\vec{R}$. That is rather similar to the generator of space translations being $\vec{P}$.

3 Transformation of operators

These results tell us how the operators $\vec{x}_i$ and $\vec{p}_i$ transform under boosts. (We could call these $\vec{x}_{i,\mathrm{op}}$ and $\vec{p}_{i,\mathrm{op}}$, but we don't do that here because we are not in danger of confusing the operators with their eigenvalues.) We have

$$\begin{aligned} U(\vec{v})^{-1}\vec{x}_i U(\vec{v}) &= \vec{x}_i \ , \\ U(\vec{v})^{-1}\vec{p}_i U(\vec{v}) &= \vec{p}_i + m_i\vec{v} \ . \end{aligned} \tag{6}$$

One operator of some importance in quantum mechanics is the total momentum,

$$\vec{P} = \sum_i p_i \ . \tag{7}$$

It transforms according to

$$\begin{aligned} U(\vec{v})^{-1}\vec{P}U(\vec{v}) &= \sum_i U(\vec{v})^{-1}\vec{p}_i U(\vec{v}) \\ &= \sum_i (\vec{p}_i + m_i\vec{v}) \ . \end{aligned} \tag{8}$$

Thus

$$U(\vec{v})^{-1}\vec{P}U(\vec{v}) = \vec{P} + M\vec{v} \ . \tag{9}$$

Let us take the hamiltonian to be

$$H = \sum_i \frac{\vec{p}_i^{\,2}}{2m_i} + V(\vec{x}_1, \ldots, \vec{x}_N) \ . \tag{10}$$

With this form, we can see what the boosted hamiltonian is

$$\begin{aligned} U(\vec{v})^{-1}HU(\vec{v}) &= \sum_i \frac{(\vec{p}_i + m_i\vec{v})^{\,2}}{2m_i} + V(\vec{x}_1, \ldots, \vec{x}_N) \\ &= \sum_i \frac{\vec{p}_i^{\,2}}{2m_i} + V(\vec{x}_1, \ldots, \vec{x}_N) \\ &\quad + \sum_i \vec{p}_i \cdot \vec{v} + \sum_i \frac{1}{2} m_i \vec{v}^{\,2} \ . \end{aligned} \tag{11}$$

That is

$$U(\vec{v})^{-1}HU(\vec{v}) = H + \vec{P}\cdot\vec{v} + \frac{1}{2}\, M\vec{v}^{\,2} \ . \tag{12}$$

Thus the hamiltonian is *not* boost invariant. Rather, the hamiltonian and the total momentum operator transform in a simple way that is independent of the details of the potential.

4 The internal hamiltonian

Define

$$H_{\text{int.}} = H - \frac{1}{2M}\,\vec{P}^{\,2}\ . \tag{13}$$

Here "int." denotes internal, M is the total mass of the system and $\vec{P}$ is the total momentum of the system. The internal hamiltonian represents the energy in a reference frame in which the total momentum of the system vanishes. That is, if the system is in an eigenstate of $\vec{P}$, then we can boost it with velocity $\vec{v} = -\vec{P}/M$ and thus make the eigenvalue of $\vec{P}$ vanish. Then $H_{\text{int.}}$ is the hamiltonian for the suitably boosted system. Of course, the system in general will not be in an eigenstate of $\vec{P}$ and $\vec{v}$ is a numerical vector while $\vec{P}$ is an operator, so in general we cannot boost to make $\vec{P} = 0$. However, Eq. (13) is a relation among operators that can define what we mean by $H_{\text{int.}}$.

How does $H_{\text{int.}}$ transform under boosts? We note that

$$U(\vec{v})^{-1}\frac{1}{2M}\,\vec{P}^{\,2}U(\vec{v}) = \frac{1}{2M}\,\vec{P}^{\,2} + \vec{P}\cdot\vec{v} + \frac{1}{2}\,M\vec{v}^{\,2}\ \ . \tag{14}$$

Thus $H_{\text{int.}}$ is boost invariant

$$U(\vec{v})^{-1}H_{\text{int.}}U(\vec{v}) = H_{\text{int.}}\ \ . \tag{15}$$

5 Translation invariant potential

For the remainder of these notes, we will take V will be a function only of the absolute values of differences of the coordinates, $|\vec{x}_i - \vec{x}_j|$. Thus it is invariant under space-translations and rotations. However, this invariance is not indicated explicitly in the notation. With a translation invariant potential, we have

$$[\vec{P}, V] = 0\ \ . \tag{16}$$

Then

$$[\vec{P}, H] = 0\ \ . \tag{17}$$

That means that we can choose $\vec{P}$ and H to both be diagonal.

From the definition (13), we have

$$H = H_{\text{int.}} + \frac{1}{2M}\,\vec{P}^{\,2}\ . \tag{18}$$

Choosing $\vec{P}$ to be diagonal makes the second term just the kinetic energy for a particle with momentum $\vec{P}$ and mass M. We can say that $H_{\text{int.}}$ gives the internal energy of the system and then the full energy equals the internal energy plus the kinetic energy from the motion of the center of mass. If we want the energy levels of a system, it is really $H_{\text{int.}}$ in which we are interested. The kinetic energy from the center of mass motion is trivial and uninteresting.

To construct $H_{\text{int.}}$, we simply need the momenta

$$\vec{k}_i = \vec{p}_i - \frac{m_i}{M}\,\vec{P} \ . \tag{19}$$

We note that the operators $\vec{k}_i$ are boost invariant,

$$U(\vec{v})^{-1}\vec{k}_i U(\vec{v}) = \vec{p}_i + m_i\,\vec{v} - \frac{m_i}{M}\,\vec{P} - \frac{m_i}{M}\,M\vec{v} = \vec{k}_i \ . \tag{20}$$

Furthermore, $\vec{k}_i = \vec{p}_i$ when $\vec{P} = 0$. Thus if the states happen to be eigenstates of $\vec{P}$, then $\vec{k}_i = \vec{p}_i$ if we boost the system to make the eigenvalue of $\vec{P}$ vanish. Note that, using $\sum \vec{p}_i = \vec{P}$ and $\sum m_i = M$, we have

$$\sum_i \vec{k}_i = 0 \ . \tag{21}$$

A simple calculation gives

$$\sum_i \frac{\vec{k}_i^{\,2}}{2m_i} = \sum_i \frac{\vec{p}_i^{\,2}}{2m_i} - \frac{\vec{P}^{\,2}}{2M} \ . \tag{22}$$

With this, we have

$$H_{\text{int.}} = \sum_i \frac{\vec{k}_i^{\,2}}{2m_i} + V(\vec{x}_1, \ldots, \vec{x}_N) \ . \tag{23}$$

6 Internal coordinates and momenta

We have already defined internal momenta $\vec{k}_i$,

$$\vec{k}_i = \vec{p}_i - \frac{m_i}{M}\,\vec{P} \ . \tag{24}$$

with

$$\sum_i \vec{k}_i = 0 \ . \tag{25}$$

Then the complete system can be described using the $\vec{k}_i$ and the total momentum $\vec{P}$.

Similarly, we can define internal coordinates

$$\vec{r}_i = \vec{x}_i - \vec{R} \ . \tag{26}$$

with

$$\sum_i \frac{m_i}{M} \vec{r}_i = 0 \ . \tag{27}$$

Then the complete system can be described using the $\vec{r}_i$ and the position of the center of mass, $\vec{R}$.

A simple calculation shows that $\vec{R}$ and $\vec{P}$ are conjugate operators in the sense that their commutation relation is

$$[R^k, P^l] = \mathrm{i}\delta_{kl} \ . \tag{28}$$

A simple calculation also shows that $\vec{k}_i$ commutes with $\vec{R}$,

$$[R^k, k_i^l] = 0 \ . \tag{29}$$

(Indeed, $-M\vec{R}$ is the generator of boosts, so this says that $\vec{k}_i$ is boost invariant.) A simple calculation also shows that $\vec{r}_i$ commutes with $\vec{P}$,

$$[r_i^k, P^l] = 0 \ . \tag{30}$$

(Indeed, $\vec{P}$ is the generator of translations, so this says that $\vec{r}_i$ is translation invariant, which is evident because it is the difference of two positions.) Thus we have one set of operators, $\vec{R}$ and $\vec{P}$, that tell about the overall motion of the system and another set of operators, the $\vec{k}_i$ and $\vec{r}_i$, that tell about the internal motion. The internal motion operators commute with $\vec{R}$ and $\vec{P}$, so we can separate the dynamics of the quantum system into two independent parts, described by commuting sets of operators. Since we consider that the potential depends only on differences of the positions $\vec{x}_i$, it can be written as a function of the $\vec{r}_i$.

$$V(\vec{x}_1, \ldots, \vec{x}_N) = V(\vec{r}_1, \ldots, \vec{r}_N) \ . \tag{31}$$

Then the hamiltonian $H_{\text{int.}}$ depends only on the internal operators.

Unfortunately, the commutation relations between the $\vec{r}_i$ and the $\vec{k}_i$ are a little complicated since the operators $\vec{k}_i$ are not independent because of Eq. (25) and neither are the operators $\vec{r}_i$, because of Eq. (27). We have

$$[r_i^k, k_j^l] = \mathrm{i}\delta_{kl}(\delta_{ij} - \frac{m_j}{M}) \ . \tag{32}$$

Suppose that you want to solve for the internal motion of three particles. Then you need to reduce the three position vectors to just two, with two conjugate momentum vectors. There are more than one ways to do this, but we will not pursue them here. What we want to notice is that for any number of particles the internal motion factors from motion of the center of mass. Taking $\vec{P}$ to be diagonal, we can use a wave function

$$e^{\mathrm{i}\vec{P}'\cdot\vec{R}}\psi(\vec{r}_1, \cdots, \vec{r}_N) \ , \tag{33}$$

where here $\vec{P}'$ is the eigenvalue of the operator $\vec{P}$. To find $\psi(\vec{r}_1, \cdots, \vec{r}_N)$, we use $H_{\text{int.}}$.

7 The internal hamiltonian for two particles

With just two particles, this construction is pretty simple. With our assumption about V, the potential is a function of just $|\vec{x}_1 - \vec{x}_2|$, so that

$$H = \frac{\vec{p}_1^{\,2}}{2m_1} + \frac{\vec{p}_2^{\,2}}{2m_2} + V(|\vec{x}_1 - \vec{x}_2|) \ . \tag{34}$$

Then

$$H_{\text{int.}} = \frac{\vec{k}_1^{\,2}}{2m_1} + \frac{\vec{k}_2^{\,2}}{2m_2} + V(|\vec{x}_1 - \vec{x}_2|) \ . \tag{35}$$

Since $\vec{k}_1 + \vec{k}_2 = 0$, we can use just one momentum operator $\vec{k}$ with $\vec{k}_1 = \vec{k}$ and $\vec{k}_2 = -\vec{k}$. Then

$$H_{\text{int.}} = \left[\frac{1}{m_1} + \frac{1}{m_2}\right]\frac{1}{2}\vec{k}^{\,2} + V(|\vec{x}_1 - \vec{x}_2|) \ . \tag{36}$$

Define the reduced mass μ by

$$\frac{1}{m_1} + \frac{1}{m_2} = \frac{1}{\mu} \ . \tag{37}$$

Then

$$H_{\text{int.}} = \frac{\vec{k}^{\,2}}{2\mu} + V(|\vec{x}_1 - \vec{x}_2|) \quad . \tag{38}$$

For space coordinates, we can use $\vec{R}$ and $\vec{r} = \vec{r}_1 - \vec{r}_2$:

$$\begin{aligned} \vec{R} &= \frac{1}{M}\left(m_1\vec{x}_1 + m_2\vec{x}_2\right) \quad . \\ \vec{r} &= \vec{x}_1 - \vec{x}_2 \quad . \end{aligned} \tag{39}$$

Then

$$H_{\text{int.}} = \frac{\vec{k}^{\,2}}{2\mu} + V(|\vec{r}|) \quad . \tag{40}$$

We know already that

$$\begin{aligned} [R^k, P^l] &= \mathrm{i}\delta_{kl} \quad , \\ [R^k, k^l] &= 0 \quad , \\ [r^k, P^l] &= 0 \quad . \end{aligned} \tag{41}$$

A simple computation gives

$$[r^k, k^l] = \mathrm{i}\delta_{kl} \quad . \tag{42}$$

Thus in a coordinate representation,

$$k^l = -\mathrm{i}\,\frac{\partial}{\partial r^l} \quad . \tag{43}$$

In a coordinate representation, we look for a wave function

$$e^{\mathrm{i}\vec{P}'\cdot\vec{R}}\psi(\vec{r}) \quad , \tag{44}$$

where here $\vec{P}'$ is the eigenvalue of the operator $\vec{P}$. We can diagonalize the operator $\vec{P}$ because it commutes with $H_{\text{int.}}$ and with $\vec{P}^{\,2}/(2M)$. Then $\psi(\vec{r})$ is the internal wave function. The Schrödinger equation for $\psi(\vec{r})$ is the equation that we are used to, except that the reduced mass μ appears in the kinetic energy.

8 Angular momentum

The angular momentum operator for our particles is

$$\vec{J} = \sum_i \vec{x}_i \times \vec{p}_i \ . \tag{45}$$

This transforms under translations and boosts according to

$$\begin{aligned} e^{\mathrm{i}\vec{P}\cdot\vec{a}}\vec{J}e^{-\mathrm{i}\vec{P}\cdot\vec{a}} &= \sum_i (\vec{x}_i + \vec{a}) \times \vec{p}_i \ , \\ U(\vec{v})^{-1}\vec{J}U(\vec{v}) &= \sum_i \vec{x}_i \times (\vec{p}_i + m_i\vec{v}) \ . \end{aligned} \tag{46}$$

That is,

$$\begin{aligned} e^{\mathrm{i}\vec{P}\cdot\vec{a}}\vec{J}e^{-\mathrm{i}\vec{P}\cdot\vec{a}} &= \vec{J} + \vec{a} \times \vec{P} \ , \\ U(\vec{v})^{-1}\vec{J}U(\vec{v}) &= \vec{J} + \vec{R} \times M\vec{v} \ . \end{aligned} \tag{47}$$

We can make up an internal angular momentum that is invariant under translations and boosts as

$$\vec{S} = \sum_i \vec{r}_i \times \vec{k}_i \ . \tag{48}$$

Since both $\vec{r}_i$ and $\vec{k}_i$ are invariant under translations and boosts, so is $\vec{S}$.

Let us see how $\vec{S}$ is related to $\vec{J}$. We have

$$\begin{aligned} \vec{S} &= \sum_i (\vec{x}_i - \vec{R}) \times \left(\vec{p}_i - \frac{m_i}{M}\vec{P}\right) \\ &= \sum_i \vec{x}_i \times \vec{p}_i - \sum_i \frac{m_i}{M}\vec{x}_i \times \vec{P} - \sum_i \vec{R} \times \vec{p}_i + \sum_i \frac{m_i}{M}\vec{R} \times \vec{P} \\ &= \vec{J} - \vec{R} \times \vec{P} - \vec{R} \times \vec{P} + \vec{R} \times \vec{P} \\ &= \vec{J} - \vec{R} \times \vec{P} \ . \end{aligned} \tag{49}$$

That is

$$\vec{J} = \vec{L} + \vec{S} \ , \tag{50}$$

where

$$\vec{L} = \vec{R} \times \vec{P} \ . \tag{51}$$

Thus the total angular momentum $\vec{J}$ is composed of the orbital angular momentum $\vec{L}$ associated with the motion of the center of mass plus the internal angular momentum $\vec{S}$. The orbital angular momentum transforms under translations and boosts as an ordinary angular momentum,

$$\begin{aligned} e^{\mathrm{i}\vec{P}\cdot\vec{a}}\vec{L}e^{-\mathrm{i}\vec{P}\cdot\vec{a}} &= \vec{L} + \vec{a}\times\vec{P} \ , \\ U(\vec{v})^{-1}\vec{L}U(\vec{v}) &= \vec{L} + \vec{R}\times M\vec{v} \ . \end{aligned} \tag{52}$$

The internal angular momentum is invariant under translations and boosts,

$$\begin{aligned} e^{\mathrm{i}\vec{P}\cdot\vec{a}}\vec{S}e^{-\mathrm{i}\vec{P}\cdot\vec{a}} &= 0 \ , \\ U(\vec{v})^{-1}\vec{S}U(\vec{v}) &= 0 \ . \end{aligned} \tag{53}$$

It is easy to check that $\vec{L}$ has the usual angular momentum commutation relations with itself,

$$[L^k, L^l] = \mathrm{i}\epsilon_{kln}L^n \ . \tag{54}$$

Also, $\vec{L}$ commutes with $\vec{S}$,

$$[L^k, S^l] = 0 \ , \tag{55}$$

since both $\vec{R}$ and $\vec{P}$ commute with $\vec{S}$. Then

$$[J^k, J^l] = [L^k + S^k, L^l + S^l] = \mathrm{i}\epsilon_{kln}L^n + [S^k, S^l] \tag{56}$$

but

$$[J^k, J^l] = \mathrm{i}\epsilon_{kln}J^n \ , \tag{57}$$

so

$$[S^k, S^l] = \mathrm{i}\epsilon_{kln}(J^n - L^n) \ . \tag{58}$$

That is, the internal angular momentum has the usual angular momentum commutation relations with itself,

$$[S^k, S^l] = \mathrm{i}\epsilon_{kln}S^n \ . \tag{59}$$

To summarize, we have $\vec{J} = \vec{L} + \vec{S}$ where $\vec{L}$ and $\vec{S}$ have the usual angular momentum commutation relations with themselves and $\vec{L}$ commutes with $\vec{S}$. These are the same relations as we had for the orbital angular momentum and the spin angular momentum of a single particle.

We simplified our calculations by considering a bound system made of many spin 0 particles. We could have used spin 1/2 particles. Then the

operator we are calling $\vec{S}$ here would be made of the internal orbital angular momenta of the constituent particles, $\sum_i \vec{r}_i \times \vec{k}_i$, plus the sum of their spin angular momenta, $\sum_i \vec{s}_i$. For example, a simple model for a proton is that it is made of three spin 1/2 quarks combined, with some orbital angular momentum, to make a system with total angular momentum 1/2.

9 Motion of the system as a whole

We have considered a system of particles that interact with each other via a potential that depends only on absolute values of the separations between pairs of particles. The most important case is that there is a contribution

$$\frac{e_i e_j}{|\vec{x}_i - \vec{x}_j|}$$

for each pair of particles. The analysis that we have applied works for any state, but it is most useful for bound states. Thus, let's consider a bound state here.

We can make the system more complicated by supposing that there is an additional force that acts on our system. The simplest case is that we have a bound state, an atom, say, and we place it in an external electric field. Then we have an additional potential

$$\Delta V = \sum_i e_i \Phi(\vec{x}_i) \ . \tag{60}$$

Assuming that our potential is slowly varying, we can approximate this by

$$\Delta V \approx Q\, \Phi(\vec{R}) \ , \tag{61}$$

where $Q = \sum e^i$ is the total charge of the atom. Then we have

$$H = H_{\text{int.}} + \frac{1}{2M}\, \vec{P}^{\,2} + Q\, \Phi(\vec{R}) \ . \tag{62}$$

Here $H_{\text{int.}}$ is the same hamiltonian for the internal motion that we had before. We can imagine solving for the internal motion in the form

$$H_{\text{int.}} \big| \Psi \big\rangle = E_{\text{int.}} \big| \Psi \big\rangle \ . \tag{63}$$

Then the full (time independent) Schrödinger equation is

$$\left[\frac{1}{2M}\,\vec{P}^{\,2} + Q\,\Phi(\vec{R})\right]\big|\Psi\big\rangle = (E - E_{\text{int.}})\big|\Psi\big\rangle \quad . \tag{64}$$

Our atom moves in the potential $\Phi(\vec{R})$.

A more exact approximation to the interaction of the atom with the external field is

$$\Delta V \approx Q\,\Phi(\vec{R}) - \sum_i e_i \vec{r}_i \cdot \vec{E}(\vec{R}) \quad , \tag{65}$$

where

$$\vec{E} = -\vec{\nabla}\Phi \quad . \tag{66}$$

This next approximation is especially important if the lowest order approximation is zero because the atom is electrically neutral, $Q = 0$. Now we have an interaction between the external field and the electric dipole moment of the atom. This couples the internal and external motions. Perhaps we can come back to this when we deal with approximation methods in the next quarter.

10 Group multiplication law

There is a peculiarity with symmetry operations involving the boost operator. Suppose that we multiply together

$$\mathcal{U} = e^{+\mathrm{i}\vec{P}\cdot\vec{a}} U(\vec{v})^{-1} e^{-\mathrm{i}\vec{P}\cdot\vec{a}} U(\vec{v}) \quad . \tag{67}$$

This corresponds to the following sequence of coordinate transformations:

$$\begin{aligned}
\vec{x} &\to \vec{x} + \vec{v}\,t && \text{a boost} \\
&\to \vec{x} + \vec{v}\,t + \vec{a} && \text{a translation} \\
&\to \vec{x} + \vec{a} && \text{an inverse boost} \\
&\to \vec{x} && \text{an inverse translation} \quad .
\end{aligned} \tag{68}$$

That is, the product of these transformations leaves the coordinates alone. If our quantum operators obeyed the group multiplication laws, we would have $\mathcal{U} = 1$. However, we know how $\vec{P}$ transforms under boosts, so

$$U(\vec{v})^{-1} e^{-\mathrm{i}\vec{P}\cdot\vec{a}} U(\vec{v}) = e^{-\mathrm{i}(\vec{P} + M\vec{v})\cdot\vec{a}} \quad . \tag{69}$$

Thus

$$\mathcal{U} = e^{-\mathrm{i}M\vec{v}\cdot\vec{a}} \quad . \tag{70}$$

We do *not* get the unit operator. But what we get is a number with absolute value 1, a phase factor. This phase factor is applied to every vector in the space of quantum mechanical states. It then cancels whenever we calculate a probability $|\langle\phi|\psi\rangle|^2$.

We see that the quantum theory provides a representation of the Galilei group up to a phase. This phase doesn't matter. When we dealt with rotations and translations by themselves, we could have had something similar, but there we got a "true" representation with no phase. At least that's what we got if you didn't have spin 1/2 particles. With a spin 1/2 particle, a rotation through 2π, which is the same as not rotating at all, gives a phase factor -1.