Midterm exam
PHYS 633, Quantum Mechanics
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D. E. Soper

There are three problems. Please answer them on separate sheets of paper provided. Please label each sheet with your name and the problem number. I am looking for not only an answer in the form of a number or a formula, but also a clear and concise explanation of your reasoning.

None of the problems requires a lot of computation, so if you are finding that your method of attack is requiring a lot of calculations, it might be best to go on to another problem and come back later to the one that was causing difficulties. Note that I set $\hbar = 1$.

Some of the questions relate to harmonic oscillators. Recall that a one dimensional harmonic oscillator with hamiltonian

$$H_0 = \frac{1}{2m}\, p^2 + \frac{m\omega^2}{2}\, x^2 \tag{1}$$

has energy eigenstates $|n\rangle$ with energies $(n+1/2)\omega$. The raising and lowering operators are

$$\begin{aligned} a^\dagger &= \sqrt{\frac{m\omega}{2}}\; x - i\sqrt{\frac{1}{2m\omega}}\; p \;\;, \\ a &= \sqrt{\frac{m\omega}{2}}\; x + i\sqrt{\frac{1}{2m\omega}}\; p \;\;. \end{aligned} \tag{2}$$

The raising and lowering operators obey $[a, a^\dagger] = 1$ and $[a, H_0] = \omega$. From this, one finds for the nth oscillator state

$$\langle n|x^2|n\rangle = \frac{n+1/2}{m\omega} \;\;. \tag{3}$$

1) Suppose that $v(t)$ is a matrix that depends on time t and that $v(t_1)$ at one time does not necessarily commute with $v(t_2)$ at a second time. Suppose further that $v(t)$ vanishes for large negative t. Let the matrix $U(t)$ be determined from $v(t)$ by the differential equation

$$\frac{d}{dt}U(t) = -i\lambda v(t)U(t) \tag{4}$$

with the boundary condition

$$U(t) \to 1 \qquad \text{as } t \to -\infty \quad . \tag{5}$$

If $\lambda v(t)$ is small, it is useful to expand $U(t)$ in a series containing powers of λ multiplying certain integrals of v. Find $U(t)$ correct to order λ^2.

2) Consider a charged particle that is in a three dimensional harmonic oscillator with hamiltonian

$$H_0 = \frac{1}{2m}\,\vec{p}^{\,2} + \frac{m\omega^2}{2}\,\vec{x}^{\,2} \quad . \tag{6}$$

Considering this as three copies of a one dimensional harmonic oscillator, we have energy eigenstates $\left|n_1, n_2, n_3\right\rangle$ with eigenvalues $(n_1 + n_2 + n_3 + 3/2)\,\omega$.

Suppose that this is subject to an additional potential

$$V = \frac{m\alpha^2}{2}(x_1 + x_2 + x_3)^2 \quad , \tag{7}$$

where $\alpha \ll \omega$, so that this constitutes a small perturbation.

1. Use first order perturbation theory to estimate the energy of the ground state of the perturbed system.
2. Is this perturbative estimate bigger or smaller than the true ground state energy? How do you know?

3) The three dimensional oscillator with hamiltonian (6) from problem 2 has a first excited energy level with energy $5\omega/2$. This energy level is triply degenerate. Suppose that the hamiltonian is perturbed with the potential (7). Use first order perturbation theory to estimate the change in the energy of the first excited energy level. Is the level still degenerate? (This problem is a little too difficult to do by brute force, so try to use symmetry to help.)