Final exam
PHYS 633, Quantum Mechanics
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There are five problems. Please answer them on the sheets of paper provided. Please label each sheet with your name clearly label the problem numbers. I am looking for not only an answer in the form of a number or a formula, but also a clear and concise explanation of your reasoning.

None of the problems requires a lot of computation, so if you are finding that your method of attack is requiring a lot of calculations, it might be best to go on to another problem and come back later to the one that was causing difficulties. Note that I set $\hbar = 1$.

Here is some possibly useful information:

$$
\begin{aligned}
P_0(\cos\theta) &= 1 \ , \\
P_1(\cos\theta) &= \cos\theta \ , \\
P_2(\cos\theta) &= \frac{1}{2}\,[3\cos^2\theta - 1] \ , \\
P_l(1) &= 1 \ , \\
\int_{-1}^{1} d\cos\theta \ P_l(\cos\theta)P_{l'}(\cos\theta) &= \frac{2}{2l+1}\ \delta_{l,l'} \ .
\end{aligned}
\tag{1}
$$

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1) Two electrons are confined in a box that has the shape of a cube with sides of length L. The hamiltonian is

$$
H = -\frac{1}{2m}\,\vec{\nabla}_1^2 - \frac{1}{2m}\,\vec{\nabla}_2^2 + A\ \vec{s}_1\cdot\vec{s}_2 \ , \tag{2}
$$

where the terms $-\vec{\nabla}_i^2/(2m)$ represent the kinetic energy of the two electrons and, in addition there is an interaction betweens the spins $\vec{s}_1$ and $\vec{s}_2$ of the two electrons. Here A is a positive constant with the dimensions of energy. In a more realistic model, A would depend on the separation between the electrons, but here A is just a constant in order to have a simple problem.

Find the ground state energy of the two electron system as a function of m, L, and A.

2) Suppose that $v(t)$ is a matrix that depends on time t and that $v(t_1)$ at one time does not necessarily commute with $v(t_2)$ at a second time. Suppose further that $v(t)$ vanishes for large negative t. Let the matrix $U(t)$ be determined from $v(t)$ by the differential equation

$$\frac{d}{dt}U(t) = -i\lambda v(t)U(t) \tag{3}$$

with the boundary condition

$$U(t) \to 1 \qquad \text{as } t \to -\infty \quad . \tag{4}$$

If $\lambda v(t)$ is small, it is useful to expand $U(t)$ in a series containing powers of λ multiplying certain integrals of v. Find $U(t)$ correct to order λ^2.

3) Recall that we calculated the partial width for the decay of a state $\big|n, l, m\big\rangle$ of a hydrogen to decay to the ground state $\big|1, 0, 0\big\rangle$ by emitting a photon with momentum $\vec{k}$ and polarization λ. We found that the partial width, calculated in first order perturbation theory, is proportional to the square of the matrix element that we expressed as

$$\big\langle 1, 0, 0; \vec{k}, \lambda\big|V\big|n, l, m\big\rangle = \frac{e}{2\pi m_e\sqrt{\omega}}\ \vec{\epsilon}(\vec{k}, \lambda) \cdot \big\langle 1, 0, 0\big|e^{-i\vec{k}\cdot\vec{x}}\vec{p}\big|n, l, m\big\rangle \tag{5}$$

Here m_e is the electron mass, the electron charge is $-e$, $\omega = |\vec{k}|$ is the photon energy, $\epsilon(\vec{k}, \lambda)$ is the photon polarization vector, $\vec{x}$ is the electron position operator and $\vec{p}$ is the electron momentum operator. In this problem, the electron spin is not relevant and we leave it out of the calculation.

(a) For this sort of a problem, one has $ka_0 \ll 1$, where a_0 is the Bohr radius. Show how to use this approximation to relate the matrix element in Eq. (5) to the matrix element (which is easier to calculate)

$$\vec{v}(n, l, m) = \big\langle 1, 0, 0\big|\vec{x}\big|n, l, m\big\rangle \quad . \tag{6}$$

(b) We were interested in $\big|n, l, m\big\rangle = \big|2, 1, 0\big\rangle$. In what direction does $\vec{v}(2, 1, 0)$ point?

(c) Suppose that we had been interested in $\big|n, l, m\big\rangle = \big|2, 0, 0\big\rangle$. What can you say about the size of $\vec{v}(2, 0, 0)$ compared to $\vec{v}(2, 1, 0)$?

4) Consider a quantum system governed by a hamiltonian $H_0 + V$ where H_0 is represented by the matrix

$$H_0 = \mathcal{E}_0 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \tag{7}$$

and V is represented by the matrix

$$V = \mathcal{E}_1 \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad . \tag{8}$$

Here $\mathcal{E}_0$ and $\mathcal{E}_1$ are constants with the dimension of energy and $\mathcal{E}_1 \ll \mathcal{E}_0$. What is the ground state energy of this system calculated to lowest order in $\mathcal{E}_1$?

5) Recall that we based much the theory for scattering a spinless particle from a potential on the scattering amplitude $f(\vec{k}_{\rm F}, \vec{k}_{\rm I})$, defined by

$$(2\pi)^{3/2}\psi_+(R\vec{n}) \sim e^{i\vec{k}_{\rm I}\cdot\vec{n}\,R} + \frac{1}{R}\, e^{ik_{\rm F}R} f(k_{\rm F}\vec{n}, \vec{k}_{\rm I}) \quad . \tag{9}$$

for $R \to \infty$. Here $\vec{k}_{\rm I}$ is along the z-axis and $\vec{k}_{\rm F} = k_{\rm F}\vec{n}$ makes an angle θ with the z-axis. Energy conservation gives $|\vec{k}_{\rm I}| = |\vec{k}_{\rm F}|$, so it is convenient to denote $k = |\vec{k}_{\rm I}| = |\vec{k}_{\rm F}|$.

(a) What is the relation between $f(\vec{k}_{\rm F}, \vec{k}_{\rm I})$ and the differential cross section $d\sigma/d\Omega$?

(b) What is the relation between $f(\vec{k}_{\rm F}, \vec{k}_{\rm I})$ and the total cross section σ_T according to the optical theorem? (There is a proportionality factor involved here, but if you don't remember it, that's OK because you will derive it in the next part of the question.)

(c) Recall that we can represent $f(\vec{k}_{\rm F}, \vec{k}_{\rm I})$ using a partial wave expansion as

$$f(\vec{k}_{\rm F}, \vec{k}_{\rm I}) = \frac{1}{2ik} \sum_l (2l+1)\, P_l(\cos\theta)\, \left[e^{2i\delta_l(k)} - 1\right] \quad . \tag{10}$$

The quantity $\delta_l(k)$ is the phase shift for the lth partial wave. Use this information to prove the optical theorem.