Position and momentum in quantum mechanics[1]

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1 Position

Let us consider a particle with no spin that can move in one dimension. Let us call the coordinate x. How can we describe this in quantum mechanics?

We postulate that if the particle is at x, its state can be represented by a vector $|x\rangle$. The particle could be anywhere, so we postulate that a general state should be represented as a linear combination of states $|x\rangle$ with different values of x.

If there were a discrete set of possible values for x, say x_i, we could just take over the structure that we had for spin, taking $\langle x_i | x_j \rangle = \delta_{x_i x_j}$. Since the possible values for x are continuous, we postulate instead that

$$\langle x' | x \rangle = \delta(x' - x) \quad . \tag{1}$$

Here $\delta(x' - x)$ is the Dirac delta function, defined by

$$\int dx \; f(x)\, \delta(x' - x) = f(x') \quad . \tag{2}$$

There is no actual function that does this, although one can think of $\delta(x' - x)$ as a sort of limit of ordinary functions that vanish when $x' - x$ is not very close to 0 but are very big when $x' - x$ is very close to 0, with the area under the graph of $\delta(x' - x)$ equal to 1. The precise way to think of it is that $\delta(x' - x)$ is a "distribution", where a distribution F maps nice well behaved functions f to (complex) numbers $F[f]$. For $F[f]$ we use the convenient notation $F[f] = \int dx f(x)\, F(x)$.

We postulate that the vectors $|x\rangle$ make a basis for the space of possible states, with the unit operator represented as

$$1 = \int dx \; |x\rangle\langle x| \quad . \tag{3}$$

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This is consistent with the inner product postulate:

$$\big|x'\big\rangle = 1\big|x'\big\rangle = \int dx\ \big|x\big\rangle\big\langle x\big|x'\big\rangle = \int dx\ \big|x\big\rangle\delta(x'-x) = \big|x'\big\rangle\ \ . \tag{4}$$

With the completeness relation, we can represent a general state $\big|\psi\big\rangle$ as a linear combination of our basis vectors $\big|x\big\rangle$:

$$\big|\psi\big\rangle = \int dx\ \big|x\big\rangle\big\langle x\big|\psi\big\rangle\ \ . \tag{5}$$

Then for any two states $\big|\psi\big\rangle$ and $\big|\phi\big\rangle$ we have

$$\big\langle\phi\big|\psi\big\rangle = \int dx\ \big\langle\phi\big|x\big\rangle\big\langle x\big|\psi\big\rangle = \int dx\ \big\langle x\big|\phi\big\rangle^*\big\langle x\big|\psi\big\rangle\ \ . \tag{6}$$

In particular, if $\big|\psi\big\rangle$ is normalized, we have

$$1 = \big\langle\psi\big|\psi\big\rangle = \int dx\ \big\langle\psi\big|x\big\rangle\big\langle x\big|\psi\big\rangle = \int dx\ \big|\big\langle x\big|\psi\big\rangle\big|^2\ \ . \tag{7}$$

With discrete states, we postulate in quantum mechanics that the probability that a system in state ψ will be found, if suitably measured, to be in state $\big|i\big\rangle$ is $|\big\langle i\ \big|\psi\big\rangle|^2$. We generalize this to continuous values x by postulating that the probability that a system in state ψ will be found, if suitably measured, to have position between x and $x + dx$ is is $|\big\langle x\ \big|\psi\big\rangle|^2 dx$. That is, $|\big\langle x\ \big|\psi\big\rangle|^2$ is the probability density, and the probability that the system will be found to be between position a and position b is

$$P(a,b) = \int_a^b dx\ \big|\big\langle x\big|\psi\big\rangle\big|^2 \tag{8}$$

This is consistent with the state normalization. The probability that the system is somewhere is

$$1 = P(-\infty,\infty) = \int_{-\infty}^{\infty} dx\ \big|\big\langle x\big|\psi\big\rangle\big|^2 \tag{9}$$

We can now introduce an operator $x_{\rm op}$ that measures x:

$$x_{\rm op}\big|x\big\rangle = x\big|x\big\rangle\ \ . \tag{10}$$

Then

$$x_{\rm op} = \int dx\ x\ |x\rangle\langle x|\ \ . \tag{11}$$

With this definition, $x_{\rm op}$ is a self-adjoint operator: $x_{\rm op}^\dagger = x_{\rm op}$. Its expansion in terms of eigenvectors and eigenvalues is Eq. (11).

In a first course in quantum mechanics, one usually denotes $\langle x|\psi\rangle$ by $\psi(x)$ and calls it the "wave function." The wave function notation is helpful for many purposes and we will use it frequently. With the wave function notation,

$$\langle\phi|\psi\rangle = \int dx\ \phi(x)^*\,\psi(x)\ \ . \tag{12}$$

The state vector is expressed as a linear combination of basis kets $|x\rangle$ using Eq. (5),

$$|\psi\rangle = \int dx\ \psi(x)|x\rangle\ \ . \tag{13}$$

2 Translation in space

We can define an operator $U(x)$ that translates the system a distance a along the x-axis. The definition is simple:

$$U(a)|x\rangle = |x+a\rangle\ \ . \tag{14}$$

Evidently

$$U(b)U(a) = U(a+b)\ \ . \tag{15}$$

In particular

$$U(-a)U(a) = U(0) = 1\ \ , \tag{16}$$

so

$$U(-a) = U(a)^{-1}\ \ . \tag{17}$$

It will be helpful to express what $U(a)$ does to an arbitrary state $|\psi\rangle$ by using the wave function representation. If $U(a)\big|\psi\big\rangle = \big|\psi'\big\rangle$, we have

$$\begin{aligned}\int dx\ \psi'(x)\big|x\big\rangle &= \big|\psi'\big\rangle \\ &= U(a)\big|\psi\big\rangle \\ &= \int dy\ \psi(y)U(a)\big|y\big\rangle \\ &= \int dy\ \psi(y)\big|y+a\big\rangle \\ &= \int dx\ \psi(x-a)\big|x\big\rangle \ \ .\end{aligned} \tag{18}$$

From this, we identify

$$\psi'(x) = \psi(x-a) \ \ . \tag{19}$$

Note the minus sign.

If $U(a)\big|\psi\big\rangle = \big|\psi'\big\rangle$ and $U(a)\big|\phi\big\rangle = \big|\phi'\big\rangle$, then the inner product between $\big|\psi'\big\rangle$ and $\big|\phi'\big\rangle$ is

$$\begin{aligned}\big\langle\phi'\big|\psi'\big\rangle &= \int dx\ \phi'(x)^*\psi'(x) \\ &= \int dx\ \phi(x-a)^*\psi(x-a) \\ &= \int dy\ \phi(y)^*\psi(y) \\ &= \big\langle\phi\big|\psi\big\rangle \ \ .\end{aligned} \tag{20}$$

Thus $U(a)$ is unitary, $U(a)^\dagger = U(a)^{-1} = U(-a)$.

In particular, $\big\langle x\big|U(a)$ can be considered to be the conjugate of $U^\dagger(a)\big|x\big\rangle$, which is then $U(-a)\big|x\big\rangle = \big|x-a\big\rangle$. That is

$$\big\langle x\big|U(a) = \big\langle x-a\big| \ \ . \tag{21}$$

If $U(a)\big|\psi\big\rangle = \big|\psi'\big\rangle$, this gives

$$\psi'(x) = \big\langle x\big|U(a)\big|\psi\big\rangle = \big\langle x-a\big|\psi\big\rangle = \psi(x-a) \ \ . \tag{22}$$

This is our previous result, just looked at a different way.

3 Momentum

We now consider a translation through an infinitesimal distance δa. Since δa is infinitesimal, we expand in powers of δa and neglect terms of order $(\delta a)^2$ and higher. For $U(\delta a)$ we write

$$U(\delta a) = 1 - i\, p_{\rm op}\, \delta a + \cdots \quad . \tag{23}$$

This defines the operator $p_{\rm op}$, which we call the *momentum* operator. I take it as a postulate that $p_{\rm op}$, defined in this fashion as the infinitesimal generator of translations, represents the momentum in quantum mechanics.

We have

$$U(\delta a)^\dagger = 1 + i\, p_{\rm op}^\dagger\, \delta a + \cdots \quad . \tag{24}$$

and also

$$U(\delta a)^\dagger = U(-\delta a) = 1 + i\, p_{\rm op}\, \delta a + \cdots \quad . \tag{25}$$

Comparing these, we see that

$$p_{\rm op}^\dagger = p_{\rm op} \quad . \tag{26}$$

That is, $p_{\rm op}$ is self-adjoint.

Let us see what $p_{\rm op}$ does to an arbitrary state $\big|\psi\big\rangle$. We have

$$\big\langle x \big| U(\delta a) \big| \psi \big\rangle = \big\langle x - \delta a \big| \psi \big\rangle \tag{27}$$

but

$$\big\langle x \big| U(\delta a) \big| \psi \big\rangle = \big\langle x \big| 1 - i\, p_{\rm op}\, \delta a + \cdots \big| \psi \big\rangle \tag{28}$$

Thus

$$\big\langle x \big| \psi \big\rangle - i\delta a \big\langle x \big| p_{\rm op} \big| \psi \big\rangle + \cdots = \big\langle x - \delta a \big| \psi \big\rangle \quad . \tag{29}$$

If we expand the right hand side in a Taylor series, we have

$$\big\langle x \big| \psi \big\rangle - i\delta a \big\langle x \big| p_{\rm op} \big| \psi \big\rangle + \cdots = \big\langle x \big| \psi \big\rangle - \delta a \frac{\partial}{\partial x} \big\langle x \big| \psi \big\rangle + \cdots \quad . \tag{30}$$

Comparing terms gives

$$\big\langle x \big| p_{\rm op} \big| \psi \big\rangle = -i \frac{\partial}{\partial x} \big\langle x \big| \psi \big\rangle \quad . \tag{31}$$

With this result, we can easily compute the commutator of $x_{\rm op}$ and $p_{\rm op}$:

$$\begin{split}\big\langle x\big|[x_{\rm op},p_{\rm op}]\big|\psi\big\rangle &= \big\langle x\big|x_{\rm op}p_{\rm op}\big|\psi\big\rangle - \big\langle x\big|p_{\rm op}x_{\rm op}\big|\psi\big\rangle\\ &= x\big\langle x\big|p_{\rm op}\big|\psi\big\rangle + i\frac{\partial}{\partial x}\big\langle x\big|x_{\rm op}\big|\psi\big\rangle\\ &= -\,ix\,\frac{\partial}{\partial x}\big\langle x\big|\psi\big\rangle + i\frac{\partial}{\partial x}\,x\big\langle x\big|\psi\big\rangle\\ &= -\,ix\,\frac{\partial}{\partial x}\big\langle x\big|\psi\big\rangle + i\big\langle x\big|\psi\big\rangle + ix\,\frac{\partial}{\partial x}\big\langle x\big|\psi\big\rangle\\ &= +\,i\big\langle x\big|\psi\big\rangle\ \ .\end{split} \tag{32}$$

Since this works for any state $\big|\psi\big\rangle$, we have

$$[x_{\rm op},p_{\rm op}] = i\ \ . \tag{33}$$

Since every operator commutes with itself, we also have

$$\begin{split}[x_{\rm op},x_{\rm op}] &= 0\ \ ,\\ [p_{\rm op},p_{\rm op}] &= 0\ \ .\end{split} \tag{34}$$

These are known as the canonical commutation relations for $x_{\rm op}$ and $p_{\rm op}$.

4 Momentum eigenstates

Since $p_{\rm op}$ is self-adjoint, we can find a complete set of basis states $\big|p\big\rangle$ with

$$p_{\rm op}\big|p\big\rangle = p\big|p\big\rangle\ \ . \tag{35}$$

To find the wave functions $\big\langle x\big|p\big\rangle$, we just have to solve a very simple differential equation

$$-i\frac{\partial}{\partial x}\big\langle x\big|p\big\rangle = p\big\langle x\big|p\big\rangle\ \ . \tag{36}$$

The solution is

$$\big\langle x\big|p\big\rangle = \frac{1}{\sqrt{2\pi}}\ e^{ipx}\ \ . \tag{37}$$

Evidently, this solves the differential equation; the normalization factor $1/\sqrt{2\pi}$ is a convenient choice. We will see why presently.

Let us calculate the inner product

$$\langle p'|p\rangle = \int dx\ \langle p'|x\rangle\langle x|p\rangle = \frac{1}{2\pi}\int dx\ e^{i(p'-p)x} \quad . \tag{38}$$

You can look up the value of this integral in a book, but let's see if we can derive it.

The integral. We need

$$I(k) = \int dx\ e^{ikx} \tag{39}$$

This integral is not so well defined, but it is better defined if we treat it as a distribution. For that, we should integrate it against an arbitrary test function $h(k)$. However, we can cheat a little by just integrating against the function equal to 1 for $a < k < b$ and 0 otherwise. Thus we look at

$$\int_a^b dk\ I(k) = \int dx\ \int_a^b dk\ e^{ikx} = -i\int dx\ \frac{1}{x}\left[e^{ibx} - e^{iax}\right] \quad . \tag{40}$$

The integrand appears to have a pole at $x = 0$, but really it doesn't because $\exp(ibx)$ cancels $\exp(iax)$ at $x = 0$. For this reason, we can replace $1/x$ by $1/(x + i\epsilon)$ with $\epsilon > 0$ and take the limit $\epsilon \to 0$:

$$\int_a^b dk\ I(k) = -i\lim_{\epsilon\to 0}\int dx\ \frac{1}{x + i\epsilon}\left[e^{ibx} - e^{iax}\right] \quad . \tag{41}$$

Now, for finite ϵ the integral of each term exists separately and we can write

$$\int_a^b dk\ I(k) = -i\lim_{\epsilon\to 0}\left\{\int dx\ \frac{1}{x + i\epsilon}\ e^{ibx} - \int dx\ \frac{1}{x + i\epsilon}\ e^{iax}\right\} \quad . \tag{42}$$

Now we need the integral

$$f(b) = \int dx\ \frac{1}{x + i\epsilon}\ e^{ibx} \quad . \tag{43}$$

We can consider x to be a complex variable. We are integrating a function of x that is analytic except for a pole at $x = -i\epsilon$. Our integral runs along the real x-axis.

If $b > 0$, we can "close the contour in the upper half plane" by integrating over from x from $-R$ to R and adding an integration over a semicircle of radius R in the upper half x-plane. Then we take $R \to \infty$. The integral along the big semicircle has a $1/R$ from the $1/x$ and it is suppressed by

$$\exp(ibx) = \exp(ibR\cos\theta)\exp(-bR\sin\theta) \tag{44}$$

for $x = R\cos\theta + iR\sin\theta$. Thus the integral over the big semicircle gives zero in the limit $R \to \infty$ and we can add it for free. But now we have the integral over a closed contour of a function that is analytic (with no poles) inside the contour. The result is zero.

If $b < 0$, we can "close the contour in the lower half plane" by integrating over from x from $-R$ to R and adding an integration over a semicircle of radius R in the lower half x-plane. Then we take $R \to \infty$. The integral along the big semicircle has a $1/R$ from the $1/x$ and it is suppressed by

$$\exp(ibx) = \exp(-i|b|R\cos\theta)\exp(-|b|R\sin\theta) \tag{45}$$

for $x = R\cos\theta - iR\sin\theta$. Thus the integral over the big semicircle gives zero in the limit $R \to \infty$ and we can add it for free. But now we have the integral over a closed contour of a function that is analytic inside the contour except for one pole, the one at $x = -i\epsilon$. The result is $-2\pi i$ times the residue of the pole:

$$\int dx\ \frac{1}{x + i\epsilon}\, e^{ibx} = -2\pi i e^{b\epsilon} \quad . \tag{46}$$

For $\epsilon \to 0$, this is just $-2\pi i$.

Putting this together for $b > 0$ and $b > 0$, we have

$$f(b) = -\theta(b < 0)\, 2\pi i\, e^{b\epsilon} \quad . \tag{47}$$

Applying this to both integrals in the integral of $I(k)$ and then taking the limit $\epsilon \to 0$, we have, assuming that $a < b$,

$$\int_a^b dk\ I(k) = 2\pi\left\{\theta(a < 0) - \theta(b < 0)\right\} = 2\pi\theta(a < 0\ \&\ b > 0) \quad . \tag{48}$$

That is to say, $I(k)$ vanishes on any interval that does not include $k = 0$, while if we integrate it over any interval that includes $k = 0$, its integral is 2π. We thus identify

$$I(k) = 2\pi\,\delta(k) \quad . \tag{49}$$

Well, perhaps you would have preferred to just pull the answer out of a math book. However, this style of derivation is useful in many circumstances. We will see derivations like this again. For that reason, it is worthwhile to learn how to do it right from the start of this course.

Our result is

$$\langle p'|p\rangle = \frac{1}{2\pi}\int dx\ e^{i(p'-p)x} = \frac{1}{2\pi}\ 2\pi\,\delta(p'-p)\ \ . \tag{50}$$

Since we included a factor $1/\sqrt{2\pi}$ in the normalization of $|p\rangle$, we have

$$\langle p'|p\rangle = \delta(p'-p)\ \ . \tag{51}$$

The vectors $|p\rangle$ are guaranteed to constitute a complete basis set. The completeness sum with the normalization that we have chosen is

$$1 = \int dp\ |p\rangle\langle p|\ \ . \tag{52}$$

5 Momentum space wave functions

If the system is in state $|\psi\rangle$, the amplitude for the particle to have momentum p is

$$\tilde{\psi}(p) = \langle p|\psi\rangle\ \ , \tag{53}$$

We call this the momentum-space wave function. If $|\psi\rangle$ and $|\phi\rangle$ represent two states, then by just inserting $1 = \int dp\ |p\rangle\langle p|$ between the two state vectors we can write the inner product $\langle\phi|\psi\rangle$ as

$$\langle\phi|\psi\rangle = \int dp\ \langle\phi|p\rangle\langle p|\psi\rangle = \int dp\ \tilde{\phi}(p)^*\psi(p)\ \ . \tag{54}$$

Assuming that $\langle\psi|\psi\rangle = 1$, we have

$$\int dp\ |\tilde{\psi}(p)|^2 = 1\ \ . \tag{55}$$

With our standard interpretation of probabilities, the probability that the system will be found to have momentum between a and b if we measure momentum is

$$P(a,b) = \int_a^b dp\ |\tilde{\psi}(p)|^2 \tag{56}$$

You are encouraged by this notation to think of the system as having one state $|\psi\rangle$, which can be represented by either $\langle x|\psi\rangle$ or $\langle p|\psi\rangle$, depending on what sort of analysis you want to do.

We can go from the x-representation to the p-representation by writing

$$\tilde{\psi}(p) = \langle p|\psi\rangle = \int dx\ \langle p|x\rangle\langle x|\psi\rangle = \frac{1}{\sqrt{2\pi}}\int dx\ e^{-ipx}\,\psi(x)\ . \tag{57}$$

The inverse transformation is

$$\psi(x) = \langle x|\psi\rangle = \int dp\ \langle x|p\rangle\langle p|\psi\rangle = \frac{1}{\sqrt{2\pi}}\int dp\ e^{ipx}\,\tilde{\psi}(p)\ . \tag{58}$$

This is known as the Fourier transformation and its inverse.

6 The translation operator again

Now that we know something, let's look at the translation operator $U(a)$ again. If a is a finite distance and δa is an additional infinitesimal distance, we have

$$U(a+\delta a) = U(\delta a)U(a)\ . \tag{59}$$

We have defined $U(\delta a)$ as

$$U(\delta a) = 1 - ip_{\rm op}\delta a + \cdots\ , \tag{60}$$

so

$$U(a+\delta a) = U(a) - ip_{\rm op}\delta a U(a) + \cdots\ . \tag{61}$$

That is

$$\frac{1}{\delta a}\left[U(a+\delta a) - U(a)\right] = -ip_{\rm op}\,U(a) + \cdots\ . \tag{62}$$

Taking the limit $\delta a \to 0$, this is

$$\frac{d}{da}U(a) = -ip_{\rm op}\,U(a)\ . \tag{63}$$

To see what this tells us, it is convenient to use the momentum representation. For an arbitrary state $|\psi\rangle$ we have

$$\frac{d}{da}\langle p|U(a)|\psi\rangle = -i\langle p|p_{\rm op}U(a)|\psi\rangle = -ip\,\langle p|U(a)|\psi\rangle \tag{64}$$

That's a differential equation that we know how to solve. Using the boundary condition $\langle p|U(0)|\psi\rangle = \langle p|\psi\rangle$, we have

$$\langle p|U(a)|\psi\rangle = e^{-ipa}\langle p|\psi\rangle \ . \tag{65}$$

This is the same thing as

$$\langle p|U(a)|\psi\rangle = \langle p|\exp(-ip_{\rm op}a)|\psi\rangle \ . \tag{66}$$

Here we define $\exp(-ip_{\rm op}a)$ by saying that, applied to a $p_{\rm op}$ eigenstate, it gives

$$\langle p|\exp(-ip_{\rm op}a) = \langle p|\exp(-ipa) \ . \tag{67}$$

Equally well, we can define

$$\exp(-ip_{\rm op}a) = \sum_{n=0}^{\infty}\frac{1}{n!}\,(-ia)^n p_{\rm op}^n \ . \tag{68}$$

Since Eq. (66) holds for any state $|\psi\rangle$ and any $\langle p|$, we have

$$U(a) = \exp(-ip_{\rm op}a) \ . \tag{69}$$

That is, $U(a)$ is an exponential of its infinitesimal generator $p_{\rm op}$. Note that this relation seems a little more mysterious if we think of $p_{\rm op}$ as represented by the differential operator $-i\partial/\partial x$. It is, however, perfectly sensible. To apply $\exp(-ip_{\rm op}a)$ to a wave function, you can Fourier transform the wave function, multiply by $\exp(-ipa)$ and then Fourier transform back. Try it!

7 The uncertainty relation

There is a general relation between the commutator of two self-adjoint operators A and B and how precisely the values of A and B can be known in a single state $|\psi\rangle$. Let us pick a (normalized) state $|\psi\rangle$ of interest and define

$$\begin{aligned}\langle A\rangle &= \langle\psi|A|\psi\rangle \ , \\ \langle B\rangle &= \langle\psi|B|\psi\rangle \ .\end{aligned} \tag{70}$$

Then consider the quantity

$$\langle\psi|\big(A-\langle A\rangle\big)^2|\psi\rangle \ .$$

We can call this the variance of A in the state $\big|\psi\big\rangle$; its square root can be called the uncertainty of A in the state. To understand this, expand in eigenvectors $\big|i\big\rangle$ of A with eigenvalues a_i. We have

$$\big\langle A\big\rangle = \sum_i a_i \big|\big\langle i\big|\psi\big\rangle\big|^2 \tag{71}$$

That is, $\langle A\rangle$ is the expectation value of A, the average of the eigenvalues a_i weighted by the probability that the system will be found in the state with that eigenvalue. Then

$$\big\langle\psi\big|\big(A - \big\langle A\big\rangle\big)^2\big|\psi\big\rangle = \sum_i \big(a_i - \big\langle A\big\rangle\big)^2 \big|\big\langle i\big|\psi\big\rangle\big|^2 \ .$$

This is the average value of the square of the difference between the eigenvalue a_i and the average value $\big\langle A\big\rangle$. That is what one calls the variance of a distribution of values in statistics. If the variance is small, then we know the value of A very well; if the variance is large, then the value of A is very uncertain.

The general relation concerns the product of the variance of A and the variance of B in the state $\big|\psi\big\rangle$:

$$\big\langle\psi\big|\big(A - \big\langle A\big\rangle\big)^2\big|\psi\big\rangle\ \big\langle\psi\big|\big(B - \big\langle B\big\rangle\big)^2\big|\psi\big\rangle \geq \frac{1}{4}\big|\big\langle\psi\big|[A,B]\big|\psi\big\rangle\big|^2 \ . \tag{72}$$

The proof is given in Sakurai.

We have seen that

$$[x_{\rm op}, p_{\rm op}] = i \ . \tag{73}$$

Thus

$$\big\langle\psi\big|\big(x_{\rm op} - \big\langle x_{\rm op}\big\rangle\big)^2\big|\psi\big\rangle\ \big\langle\psi\big|\big(p_{\rm op} - \big\langle p_{\rm op}\big\rangle\big)^2\big|\psi\big\rangle \geq \frac{1}{4} \ . \tag{74}$$

Thus if we know the position of a particle very well, then we cannot know its momentum well; if we know the momentum of a particle very well, then we cannot know its position well.

There is a class of functions for which the "$\geq$" becomes "$=$": gaussian wave packets. Lets see how this works. Define

$$\psi(x) = \frac{1}{(2\pi)^{1/4}}\ \frac{1}{\sqrt{a}}\ e^{ik_0(x-x_0)} \exp\left(-\frac{(x-x_0)^2}{4a^2}\right) \ . \tag{75}$$

This function is normalized to 1:

$$
\begin{aligned}
\int dx\ |\psi(x)|^2 &= \frac{1}{(2\pi)^{1/2}}\ \frac{1}{a}\ \int dx\ \ \exp\left(-\frac{(x-x_0)^2}{2a^2}\right) \\
&= \frac{1}{(2\pi)^{1/2}}\ \frac{1}{a}\ \int dy\ \ \exp\left(-\frac{y^2}{2a^2}\right) \\
&= \frac{1}{(2\pi)^{1/2}}\ \sqrt{2}\int dz\ \ \exp\left(-z^2\right) \\
&= 1\ \ .
\end{aligned}
\tag{76}
$$

Here we have used the integral

$$
\int dz\ \ \exp\left(-z^2\right) = \sqrt{\pi}\ \ , \tag{77}
$$

which one needs often.

It is pretty much self evident that $\langle x_{\rm op}\rangle = x_0$.

Let us evaluate the variance. We have

$$
\begin{aligned}
\langle\psi\big|\big(x_{\rm op}-\langle x_{\rm op}\rangle\big)^2\big|\psi\rangle &= \frac{1}{(2\pi)^{1/2}}\ \frac{1}{a}\ \int dx\ (x-x_0)^2\exp\left(-\frac{(x-x_0)^2}{2a^2}\right) \\
&= \frac{1}{(2\pi)^{1/2}}\ \frac{1}{a}\ \int dy\ y^2\exp\left(-\frac{y^2}{2a^2}\right) \\
&= \frac{2a^2}{\pi^{1/2}}\ \int dz\ z^2\exp\left(-z^2\right) \\
&= -\ \frac{2a^2}{\pi^{1/2}}\ \left[\frac{d}{d\lambda}\int dz\ \ \exp\left(-\lambda z^2\right)\right]_{\lambda=1} \\
&= -\ \frac{2a^2}{\pi^{1/2}}\ \left[\frac{d}{d\lambda}\sqrt{\frac{\pi}{\lambda}}\right]_{\lambda=1} \\
&= \frac{2a^2}{\pi^{1/2}}\ \frac{\sqrt{\pi}}{2} \\
&= a^2\ \ .
\end{aligned}
\tag{78}
$$

Thus the variance is just a^2.

Now, let's look at what the same state looks like in momentum space. To find $\tilde{\psi}(p)$, we need to "complete the square" in the exponent. This is often a useful manipulation. We find

$$
\begin{aligned}
\tilde{\psi}(p) &= \frac{1}{\sqrt{2\pi}} \int dx\ e^{-ipx} \psi(x) \\
&= \frac{1}{(2\pi)^{3/4}}\ \frac{1}{\sqrt{a}}\ \int dx\ e^{-ipx} e^{ik_0(x-x_0)} \exp\left(-\frac{(x-x_0)^2}{4a^2}\right) \\
&= \frac{1}{(2\pi)^{3/4}}\ \frac{1}{\sqrt{a}}\ e^{-ipx_0} \int dx\ e^{-ipy} e^{ik_0 y} \exp\left(-\frac{y^2}{4a^2}\right) \\
&= \frac{1}{(2\pi)^{3/4}}\ \frac{1}{\sqrt{a}}\ e^{-ipx_0} \int dy\ \exp\left(-i(p-k_0)y - \frac{y^2}{4a^2}\right) \\
&= \frac{1}{(2\pi)^{3/4}}\ 2\sqrt{a}\ e^{-ipx_0} \int dz\ \exp\left(-i2a(p-k_0)z - z^2\right) \\
&= \frac{1}{(2\pi)^{3/4}}\ 2\sqrt{a}\ e^{-ipx_0} e^{-a^2(p-k_0)^2} \int dz\ \exp\left(-[z + ia(p-k_0)]^2\right) \\
&= \frac{1}{(2\pi)^{3/4}}\ 2\sqrt{a}\ e^{-ipx_0} e^{-a^2(p-k_0)^2} \int dw\ \exp\left(-w^2\right) \\
&= \frac{1}{(2\pi)^{3/4}}\ 2\sqrt{a}\ e^{-ipx_0} e^{-a^2(p-k_0)^2} \sqrt{\pi} \\
&= \frac{1}{(2\pi)^{1/4}} \sqrt{2a}\ e^{-ipx_0} \exp\left(-a^2(p-k_0)^2\right) \quad .
\end{aligned}
\tag{79}
$$

By comparing to the definition of $\psi(x)$, we see that $\tilde{\psi}(p)$ is a properly normalized wave function with

$$
\langle p_{\rm op} \rangle = k_0 \quad . \tag{80}
$$

This same comparison shows that

$$
\langle \psi | \left(p_{\rm op} - \langle p_{\rm op} \rangle \right)^2 | \psi \rangle = \frac{1}{4a^2} \quad . \tag{81}
$$

Thus the uncertainty product is

$$
\langle \psi | \left(x_{\rm op} - \langle x_{\rm op} \rangle \right)^2 | \psi \rangle\ \langle \psi | \left(p_{\rm op} - \langle p_{\rm op} \rangle \right)^2 | \psi \rangle = a^2\ \frac{1}{4a^2} = \frac{1}{4} \quad . \tag{82}
$$

That is, the uncertainty product is as small as it is allowed to be. This example shows us rather directly how making the state more concentrated in x makes it more spread out in p.