

Path integrals and the classical approximation¹

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I offer here some background for Sections 2.5 and 2.6 of J. J. Sakurai, *Modern Quantum Mechanics*.

1 Introduction

There is more than one way to understand quantum mechanics. In one way, we study a certain partial differential equation, the one particle Schrödinger equation. More generally, we have states $|\psi(t)\rangle$ in a vector space with evolution according to $i(d/dt)|\psi(t)\rangle = H|\psi(t)\rangle$. Here H is an operator on the space of states. Possible measurements and symmetry operations are represented by other operators. In the simplest case, this formulation is the same as the one particle Schrödinger equation. This is the Schrödinger picture for quantum mechanics. There is an alternative way of looking at the same physical content in which the states do not depend on time, but the operators do. This is the Heisenberg picture.

There is another way of understanding quantum mechanics, invented by Richard Feynman: the path integral formulation. In the simplest case, we can consider Feynman's formulation in the case of a single particle moving in three dimensions. We still have operators and states, but the emphasis is not on the operators and states and we do not use the Schrödinger equation directly. Instead, we express the amplitude for the particle to get from one place to another as an integral over all paths that it might have taken to do that. This formulation is not wonderfully useful from the point of view of calculation for non-relativistic quantum mechanics, but it does provide us with some insights. In relativistic quantum field theory, the path integral formulation is often the most useful way of expressing the theory. For a certain class of problems, it is also directly useful for calculations.

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2 Propagator for the Schrödinger equation

We consider the propagator

$$K(\vec{x}_F, t_F; \vec{x}_0, t_0) = \langle \vec{x}_F | e^{-iH(t_F - t_0)} | \vec{x}_0 \rangle . \quad (1)$$

We take the hamiltonian to be

$$H = \frac{1}{2m} \vec{p}_{\text{op}}^2 + V(\vec{x}_{\text{op}}) . \quad (2)$$

The subscripts “op” indicate that \vec{p}_{op} and \vec{x}_{op} are operators. (I often use hats for this, but hats and vector signs do not live happily together.)

The propagator $K(\vec{x}_F, t_F; \vec{x}_0, t_0)$ gives the amplitude for a particle at position \vec{x}_0 at an initial time t_0 to be found at position \vec{x}_F at a later time t_F . This definition is for $t_F > t_0$. For $t_F < t_0$, one usually defines $K = 0$. If the particle was originally in a general state $|\psi(t_0)\rangle$, with wave function $\langle \vec{x}_0 | \psi(t_0) \rangle$, then at time t_F its wave function is

$$\langle \vec{x}_F | \psi(t_F) \rangle = \int d\vec{x}_0 K(\vec{x}_F, t_F; \vec{x}_0, t_0) \langle \vec{x}_0 | \psi(t_0) \rangle . \quad (3)$$

Thus the propagator tells us quite generally how the particle propagates in time.

We can find this exactly for a free particle. For convenience we can set $t_0 = 0$ and $\vec{x}_0 = 0$. Then if $V = 0$ a simple calculation gives

$$K(\vec{x}, t; 0, 0) = \frac{1}{[2\pi i]^{3/2}} \left(\frac{m}{t} \right)^{3/2} \exp \left(i \frac{m}{2t} \vec{x}^2 \right) . \quad (4)$$

3 The path integral

We can derive the Feynman path integral in a simple and elegant fashion. Divide the time interval into small increments of size Δt . There are a very large number N of time increments, with $\Delta t = (t_F - t_0)/N$. We define intermediate times t_i for $i = 1, \dots, N-1$ and we denote $t_N = t_F$, so $t_i = t_{i-1} + \Delta t$ for $i = 1, \dots, N$. We then have

$$K(\vec{x}_F, t_F; \vec{x}_0, t_0) = \langle \vec{x}_F | e^{-iH\Delta t} \dots e^{-iH\Delta t} e^{-iH\Delta t} | \vec{x}_0 \rangle . \quad (5)$$

with N factors of $\exp(-iH\Delta t)$. Between each pair of factors $\exp(-iH\Delta t)$, we introduce a complete sum over position eigenstates,

$$K(\vec{x}_F, t_F; \vec{x}_0, t_0) = \int d\vec{x}_{N-1} \cdots \int d\vec{x}_2 \int d\vec{x}_1 \langle \vec{x}_F | e^{-iH\Delta t} | \vec{x}_{N-1} \rangle \cdots \times \langle \vec{x}_2 | e^{-iH\Delta t} | \vec{x}_1 \rangle \langle \vec{x}_1 | e^{-iH\Delta t} | \vec{x}_0 \rangle \quad (6)$$

Physically, \vec{x}_i is going to represent where the particle is at time t_i .

Now we approximate

$$\langle \vec{x}_{i+1} | e^{-iH\Delta t} | \vec{x}_i \rangle \approx \langle \vec{x}_{i+1} | \exp\left(-i\Delta t \frac{\vec{p}_{\text{op}}^2}{2m}\right) \exp(-i\Delta t V(\vec{x}_{\text{op}})) | \vec{x}_i \rangle \quad (7)$$

This is not exact because \vec{p}_{op} does not commute with \vec{x}_{op} . However, the error is of order $(\Delta t)^2$, which is small for $\Delta t \rightarrow 0$.³ After making this approximation, we can evaluate the matrix element without further approximation. We insert an integral over momentum eigenstates, then complete the square in the exponent. This gives

$$\begin{aligned} \langle \vec{x}_{i+1} | e^{-iH\Delta t} | \vec{x}_i \rangle &\approx \langle \vec{x}_{i+1} | \exp\left(-i\Delta t \frac{\vec{p}_{\text{op}}^2}{2m}\right) | \vec{x}_i \rangle \exp(-i\Delta t V(\vec{x}_i)) \\ &= \int d\vec{p} \langle \vec{x}_{i+1} | \exp\left(-i\Delta t \frac{\vec{p}_{\text{op}}^2}{2m}\right) | \vec{p} \rangle \langle \vec{p} | \vec{x}_i \rangle e^{-i\Delta t V(\vec{x}_i)} \\ &= \int d\vec{p} \langle \vec{x}_{i+1} | \vec{p} \rangle \langle \vec{p} | \vec{x}_i \rangle \exp\left(-i\Delta t \frac{\vec{p}^2}{2m}\right) e^{-i\Delta t V(\vec{x}_i)} \\ &= \int \frac{d\vec{p}}{(2\pi)^3} \exp(i\vec{p} \cdot (\vec{x}_{i+1} - \vec{x}_i)) \exp\left(-i\Delta t \frac{\vec{p}^2}{2m}\right) \\ &\quad \times e^{-i\Delta t V(\vec{x}_i)} \\ &= \left(\frac{m}{2\pi i\Delta t}\right)^{3/2} \\ &\quad \times \exp\left(i\Delta t \left[\frac{m}{2} \left(\frac{\vec{x}_{i+1} - \vec{x}_i}{\Delta t}\right)^2 - V(\vec{x}_i)\right]\right) . \end{aligned} \quad (8)$$

With this result, we have

$$K(\vec{x}_F, t_F; \vec{x}_0, t_0) = \int_{\substack{\vec{x}(t_N)=\vec{x}_F \\ \vec{x}(t_0)=\vec{x}_0}} \mathcal{D}[x] \exp(iS[x]) . \quad (9)$$

³We could make the error smaller by putting $\exp(-i\Delta t V(\vec{x}_{\text{op}})/2)$ on the left and $\exp(-i\Delta t V(\vec{x}_{\text{op}})/2)$ on the right, but that doesn't change the formalism very much.

Here

$$\int_{\substack{\vec{x}(t_N)=\vec{x}_F \\ \vec{x}(t_0)=\vec{x}_0}} \mathcal{D}[x] \cdots = \left(\frac{m}{2\pi i \Delta t} \right)^{3N/2} \prod_{i=1}^{N-1} \int d\vec{x}_i \cdots , \quad (10)$$

where we understand that in the integrand we should set $\vec{x}(t_N)$ and $\vec{x}(t_0)$ as indicated. Here also $S[x]$ represents the classical action associated with the path x :

$$\begin{aligned} S[x] &= \sum_{i=0}^{N-1} \Delta t \left\{ \frac{1}{2} m \left(\frac{x_{i+1} - x_i}{\Delta t} \right)^2 - V(\vec{x}_i) \right\} \\ &\approx \int_{t_0}^{t_F} dt \left\{ \frac{1}{2} m \left(\frac{d\vec{x}(t)}{dt} \right)^2 - V(\vec{x}(t)) \right\} . \end{aligned} \quad (11)$$

We use the discrete approximations shown to the integration over paths and to the action, then take the limit $\Delta t \rightarrow 0$. At least that is what we should do in principle. I do not address whether this process converges.

Thus with the path integral formulation, the propagator $K(\vec{x}_F, t_F; \vec{x}_0, t_0)$ is an integral over all paths that get from \vec{x}_0 to \vec{x}_F in time $t_F - t_0$. The integrand is $\exp(iS[x])$, where x here denotes the path and $S[x]$ is the classical action for that path.

In classical mechanics, the action $S[x]$ also appears. One considers all possibilities for paths from \vec{x}_0 to \vec{x}_F in time $t_F - t_0$. The path that nature chooses is the one for which $S[x]$ does not change if one makes a small variation δx away from the classical path x_{cl} .

We see that in quantum mechanics, all paths appear. The particle gets from \vec{x}_0 to \vec{x}_F in time $t_F - t_0$ every way it can. The amplitude is the sum of the amplitudes for each path x , weighted by $\exp(iS[x])$. There can be constructive interference among paths and there can be destructive interference among paths.

4 Method of stationary phase

Let us look at what seems to be a completely different subject, performing integrals by the method of stationary phase. Suppose that we want to approximately evaluate an integral of the form

$$I = \int d\phi \exp(i\nu S(\phi)) g(\phi) . \quad (12)$$

Here ϕ is a D -dimensional vector with components ϕ_i and the integral means

$$\int d\phi \cdots = \int_{-\infty}^{\infty} d\phi_1 \int_{-\infty}^{\infty} d\phi_2 \cdots \int_{-\infty}^{\infty} d\phi_D \cdots . \quad (13)$$

In the integrand, g is some well behaved function of ϕ . In the exponent of the integrand S is some well behaved function of ϕ , which I assume is real for real ϕ . The function $S(\phi)$ is multiplied by a parameter ν that we consider to be large, $\nu \rightarrow \infty$.

Suppose that at some point $\phi = \phi_{\text{cl}}$, the gradient of S vanishes,

$$\left[\frac{\partial}{\partial \phi_i} S(\phi) \right]_{\phi=\phi_{\text{cl}}} = 0 \quad (14)$$

This is a point of “stationary phase.” Suppose, for convenience, that there is only one point of stationary phase. Otherwise, we should apply the method to each such point and sum the results.

We recognize that contributions to the integral from regions of ϕ not near ϕ_{cl} will decrease as ν increases because of the rapidly oscillating phase. However, this cancellation is inhibited near the point of stationary phase. Therefore, we expand about $\phi = \phi_{\text{cl}}$. We have

$$S(\phi) = S(\phi_{\text{cl}}) + \frac{1}{2} \sum_{i,j} \Delta\phi_i \Delta\phi_j \mathcal{D}_{ij} + \cdots , \quad (15)$$

where

$$\Delta\phi_i = \phi_i - \phi_{\text{cl},i} \quad (16)$$

and

$$\mathcal{D}_{ij} = \left[\frac{\partial^2 S(\phi)}{\partial \phi_i \partial \phi_j} \right]_{\phi=\phi_{\text{cl}}} \quad (17)$$

This gives (after taking the factors that do not depend on $\Delta\phi$ out of the integral and changing the integration variables from ϕ to $\Delta\phi$),

$$I \sim \exp(i\nu S(\phi_{\text{cl}})) g(\phi_{\text{cl}}) \int d\Delta\phi \exp\left(i\nu \frac{1}{2} \sum_{i,j} \Delta\phi_i \mathcal{D}_{ij} \Delta\phi_j\right) . \quad (18)$$

How can we perform this integral? We note that \mathcal{D} is a real, symmetric matrix. Therefore it has a complete set of real eigenvectors v_n ,

$$\sum_j \mathcal{D}_{ij} v_n^j = \lambda_n v_n^i . \quad (19)$$

The eigenvectors can be chosen to be normalized and orthogonal (with a real inner product),

$$\sum_i v_m^i v_n^i = \delta_{mn} . \quad (20)$$

We can write

$$\Delta\phi_i = \sum_n a_n v_n^i . \quad (21)$$

We change integration variables to the expansion coefficients a_n . For the exponent, we have

$$\begin{aligned} i\nu \frac{1}{2} \sum_{i,j} \Delta\phi_i \mathcal{D}_{ij} \Delta\phi_j &= i\nu \frac{1}{2} \sum_{n,m} a_n a_m \sum_{i,j} v_m^i \mathcal{D}_{ij} v_n^j \\ &= i\nu \frac{1}{2} \sum_{n,m} a_n a_m \lambda_n \sum_i v_m^i v_n^i \\ &= i\nu \frac{1}{2} \sum_{n,m} a_n a_m \lambda_n \delta_{mn} \\ &= i\nu \frac{1}{2} \sum_n \lambda_n a_n^2 . \end{aligned} \quad (22)$$

For the integration, we have

$$\prod_i d\Delta\phi_i = \det v \prod_n da_n . \quad (23)$$

Here we have the determinant of the matrix v with matrix elements v_n^i . This matrix is an orthogonal matrix, $\sum_i v_m^i v_n^i = \delta_{mn}$, so its determinant is 1. Thus our integral is

$$I \sim \exp(i\nu S(\phi_{\text{cl}})) g(\phi_{\text{cl}}) \int da \exp\left(i\nu \frac{1}{2} \sum_n \lambda_n a_n^2\right) . \quad (24)$$

The integral is now a product of many one dimensional integrals,

$$I \sim \exp(i\nu S(\phi_{\text{cl}})) g(\phi_{\text{cl}}) \prod_n \left[\int da_n \exp\left(i\nu \frac{1}{2} \lambda_n a_n^2\right) \right] . \quad (25)$$

We can perform each integral by changing variables to ξ_n ,

$$a_n = \sqrt{\frac{2i}{\lambda_n \nu}} \xi_n . \quad (26)$$

This gives

$$I \sim \exp(i\nu S(\phi_{\text{cl}})) g(\phi_{\text{cl}}) \prod_n \left[\sqrt{\frac{2i}{\lambda_n \nu}} \int d\xi_n \exp(-\xi_n^2) \right] . \quad (27)$$

That is

$$I \sim \exp(i\nu S(\phi_{\text{cl}})) g(\phi_{\text{cl}}) \prod_n \left[\sqrt{\frac{2\pi i}{\lambda_n \nu}} \right] . \quad (28)$$

We recognize that

$$\prod_n \lambda_n = \det \mathcal{D} . \quad (29)$$

Thus

$$I \sim \exp(i\nu S(\phi_{\text{cl}})) g(\phi_{\text{cl}}) \frac{1}{\sqrt{\det \mathcal{D}}} \left[\sqrt{\frac{2\pi i}{\nu}} \right]^D . \quad (30)$$

This is the leading result. You should verify for yourself that if you expand g and the remaining exponential factor in powers of $\Delta\phi$, you will generate corrections that modify the leading result by multiplying it by a power series $1 + c_1/\nu + c_2/\nu^2 + \dots$. The essential point is that, inside the integral, $\delta\phi$ is effectively of order $1/\sqrt{\nu}$.

Exercise 4.1 Consider the one dimensional integral

$$I = \int_{-\infty}^{\infty} d\phi \exp(i\nu S(\phi)) g(\phi) , \quad (31)$$

where

$$S(\phi) = 1 + \phi^2 + \phi^4 \quad (32)$$

and

$$g(\phi) = \frac{1}{1 + \phi^4} . \quad (33)$$

Use the method of stationary phase to evaluate I for large ν at the leading order. Taking $\nu = 5$, $\nu = 10$, and $\nu = 20$, compare to the exact integral evaluated by numerical integration. (If you use Mathematica for the numerical integration, I suggest `MaxRecursion` $\rightarrow 20$ to help it get a good answer with an oscillatory integrand.)

5 Classical approximation to the path integral

Consider our path integral for the propagator,

$$K(\vec{x}_F, t_F; \vec{x}_0, t_0) = \int_{\substack{\vec{x}(t_N)=\vec{x}_F \\ \vec{x}(t_0)=\vec{x}_0}} \mathcal{D}[x] \exp(iS[x]) . \quad (34)$$

Let $\vec{x}_{\text{cl}}(t)$ be the classical path with $\vec{x}_{\text{cl}}(t_0) = \vec{x}_0$ and $\vec{x}_{\text{cl}}(t_F) = \vec{x}_F$. Define

$$\delta\vec{x}(t) = \vec{x}(t) - \vec{x}_{\text{cl}}(t) . \quad (35)$$

Then

$$\delta\vec{x}(t_0) = \delta\vec{x}(t_F) = 0 . \quad (36)$$

We have

$$\begin{aligned} S[x] &= S[x_{\text{cl}}] \\ &\quad - \int_{t_0}^{t_F} dt \left\{ m \frac{d^2 \vec{x}_{\text{cl}}(t)}{dt^2} + \vec{\nabla} V(\vec{x}_{\text{cl}}(t)) \right\} \cdot \delta\vec{x}(t) \\ &\quad + \int_{t_0}^{t_F} dt \left\{ \frac{1}{2} m \left(\frac{d\delta\vec{x}(t)}{dt} \right)^2 - \frac{1}{2} \partial_i \partial_j V(\vec{x}_{\text{cl}}(t)) \delta x(t)^i \delta x(t)^j \right\} \\ &\quad + \mathcal{O}((\delta x)^3) . \end{aligned} \quad (37)$$

The first term is the classical action. In the second term, I have integrated by parts using $\delta\vec{x}(t_0) = \delta\vec{x}(t_F) = 0$. Then the second term vanishes because $\vec{x}_{\text{cl}}(t)$ obeys the classical equation of motion. This is the *principle of stationary action*. Since the action is the phase, we are now applying the method of stationary phase to the integral. In the third term, we can integrate by parts also, so that

$$\begin{aligned} S[x] &= S[x_{\text{cl}}] \\ &\quad + \int_{t_0}^{t_F} dt \delta x(t)^i \left\{ -\frac{1}{2} m \delta_{ij} \frac{d^2}{dt^2} - \frac{1}{2} \partial_i \partial_j V(\vec{x}_{\text{cl}}(t)) \right\} \delta x(t)^j \\ &\quad + \mathcal{O}((\delta x)^3) . \end{aligned} \quad (38)$$

Define the operator

$$\mathcal{D} = \left\{ -\frac{1}{2} m \delta_{ij} \frac{d^2}{dt^2} - \frac{1}{2} \partial_i \partial_j V(\vec{x}_{\text{cl}}(t)) \right\} . \quad (39)$$

Taking over our previous results, we have the classical approximation to our propagator

$$K(\vec{x}_F, t_F; \vec{x}_0, t_0) \sim \frac{\mathcal{N}}{\sqrt{\det \mathcal{D}}} \exp(iS[x_{\text{cl}}]) . \quad (40)$$

Here \mathcal{N} is a normalization factor coming from the definition of the integration over paths. We should let Δt be finite, so that we have a finite dimensional integral. Then \mathcal{D} is really a finite dimensional matrix. In principle, we are supposed to calculate with Δt finite and then take a limit $\Delta t \rightarrow 0$.

We can note immediately that for the simple case of a free particle that starts at $\vec{x}_0 = 0$ at time $t_0 = 0$, the classical action is

$$S = \int_0^t d\tau \frac{m \vec{x}^2}{2 t^2} = \frac{m}{2t} \vec{x}^2 . \quad (41)$$

Thus the exponent in Eq. (40) matches the exponent in our exact solution for $K(\vec{x}, t; 0, 0)$ in Eq. (4).

Exercise 5.1 Another case for which it is pretty easy to evaluate the classical action is the simple harmonic oscillator in one dimension, $H = p^2/(2m) + (m\omega^2/2)x^2$. The classical approximation for the propagator $K(x_F, t_F; x_0, 0)$, should be exact: the action is a quadratic function of x so the order $(\delta x)^3$ contributions to the action vanish. Take $t_0 = 0$ and find $S[x_{\text{cl}}]$ for the path that starts at x_0 and reaches x_F at time t_F . Use this to calculate the exponential part of $K(x_F, t_F; x_0, 0)$. There is also a factor $\mathcal{N}/\sqrt{\det \mathcal{D}}$ that is independent of x_0 and x_F and that multiplies the exponential. Don't bother to calculate this factor. Compare your answer to the exact result given in Eq. (2.6.18) of Sakurai.

6 The action and the classical approximation

Let us try to derive this classical approximation another way. We seek an approximate solution to the Schrödinger equation

$$\left[i \frac{\partial}{\partial t} + \frac{1}{2m} \vec{\nabla}^2 - V(\vec{x}) \right] K(\vec{x}, t; \vec{x}_0, t_0) = 0 . \quad (42)$$

The boundary condition is that $K(\vec{x}, t; \vec{x}_0, t_0)$ approaches $\delta(\vec{x} - \vec{x}_0)$ at $t \rightarrow t_0$.

Given the result (40), we may suspect that the classical action for classical paths that go from \vec{x}_0 at time t_0 to \vec{x} at time t has something to do with this. Let us call this function $S(\vec{x}, t)$. That is

$$S(\vec{x}, t) = \int_{t_0}^t d\tau \left[\frac{m}{2} \left(\frac{d\vec{x}_{\text{cl}}(\tau)}{d\tau} \right)^2 - V(\vec{x}_{\text{cl}}(\tau)) \right] . \quad (43)$$

Here $\vec{x}_{\text{cl}}(\tau)$ is the classical path, with

$$\begin{aligned} \vec{x}_{\text{cl}}(t_0) &= \vec{x}_0 , \\ \vec{x}_{\text{cl}}(t) &= \vec{x} . \end{aligned} \quad (44)$$

We need some properties of this function. First, consider a new classical path $x_{\text{cl}}(\tau) + \delta\vec{x}_{\text{cl}}(\tau)$ that starts at \vec{x}_0 at time t_0 but gets to some new place $\vec{x} + \delta\vec{x}$ at time t . we have

$$\delta S(\vec{x}, t) = \int_{t_0}^t d\tau \left[m \frac{d\vec{x}_{\text{cl}}(\tau)}{d\tau} \cdot \frac{d\delta\vec{x}_{\text{cl}}(\tau)}{d\tau} - \delta\vec{x}_{\text{cl}}(\tau) \cdot \vec{\nabla} V(\vec{x}_{\text{cl}}(\tau)) \right] . \quad (45)$$

This is

$$\delta S(\vec{x}, t) = \int_{t_0}^t d\tau \delta\vec{x}_{\text{cl}}(\tau) \left[-m \frac{d^2\vec{x}_{\text{cl}}(\tau)}{d\tau^2} - \vec{\nabla} V(\vec{x}_{\text{cl}}(\tau)) \right] + m \delta\vec{x} \cdot \frac{d\vec{x}_{\text{cl}}(t)}{dt} . \quad (46)$$

The first term vanishes because $\vec{x}_{\text{cl}}(\tau)$ obeys the equation of motion. The second term remains because $\delta\vec{x}_{\text{cl}}(\tau)$ does not vanish at $\tau = t$. We conclude that

$$\delta S(\vec{x}, t) = \delta\vec{x} \cdot \vec{p}_{\text{cl}}(t) , \quad (47)$$

where

$$\vec{p}_{\text{cl}}(\tau) = m \frac{d\vec{x}_{\text{cl}}(\tau)}{d\tau} . \quad (48)$$

Since this holds for any $\delta\vec{x}$, we have

$$\vec{\nabla} S(\vec{x}, t) = \vec{p}_{\text{cl}}(t) . \quad (49)$$

We also need $\partial S(\vec{x}, t)/\partial t$. Consider keeping one path, $x_{\text{cl}}(\tau)$, but extending the time to $t + \delta t$, so that the particle gets to a new place, $\vec{x} + \delta\vec{x}$. Then

$$\delta\vec{x} = \frac{d\vec{x}_{\text{cl}}(t)}{dt} \delta t . \quad (50)$$

The action changes by

$$\delta S = \left[\frac{1}{2m} \vec{p}_{\text{cl}}(t)^2 - V(\vec{x}_{\text{cl}}(t)) \right] \delta t . \quad (51)$$

On the other hand,

$$\begin{aligned} \delta S &= \vec{\nabla} S(\vec{x}, t) \cdot \delta \vec{x} + \frac{\partial S(\vec{x}, t)}{\partial t} \delta t \\ &= \left[\vec{p}_{\text{cl}}(t) \cdot \frac{d\vec{x}_{\text{cl}}(t)}{dt} + \frac{\partial S(\vec{x}, t)}{\partial t} \right] \delta t \end{aligned} \quad (52)$$

Comparing these, we have

$$\begin{aligned} \frac{\partial S(\vec{x}, t)}{\partial t} &= \frac{1}{2m} \vec{p}_{\text{cl}}(t)^2 - V(\vec{x}_{\text{cl}}(t)) - \vec{p}_{\text{cl}}(t) \cdot \frac{d\vec{x}_{\text{cl}}(t)}{dt} \\ &= - \frac{1}{2m} \vec{p}_{\text{cl}}(t)^2 - V(\vec{x}_{\text{cl}}(t)) . \end{aligned} \quad (53)$$

The right hand side is the negative of the energy E on the path. Thus

$$\frac{\partial S(\vec{x}, t)}{\partial t} = -E . \quad (54)$$

Now we can return to the Schrödinger equation. Let

$$K(\vec{x}, t; \vec{x}_0, t_0) = \exp[iA(\vec{x}, t)] . \quad (55)$$

Then

$$-\frac{\partial A}{\partial t} - \frac{1}{2m} \left(\vec{\nabla} A \right)^2 + \frac{i}{2m} \vec{\nabla}^2 A - V(\vec{x}) = 0 . \quad (56)$$

I propose that under certain circumstances (the “semiclassical approximation”), we can neglect $\vec{\nabla}^2 A$. Let’s try it and then come back to see under what conditions this is a good approximation. With the semiclassical approximation, we have

$$-\frac{\partial A}{\partial t} - \frac{1}{2m} \left(\vec{\nabla} A \right)^2 - V(\vec{x}) \approx 0 . \quad (57)$$

Let’s try $A(\vec{x}, t) = S(\vec{x}, t)$. I claim that

$$-\frac{\partial S}{\partial t} - \frac{1}{2m} \left(\vec{\nabla} S \right)^2 - V(\vec{x}) = 0 . \quad (58)$$

To see this, just use our previous results, giving

$$E - \frac{1}{2m} \vec{p}_{\text{cl}}(t)^2 - V(\vec{x}_{\text{cl}}(t)) = 0 \quad . \quad (59)$$

Since E is just the kinetic energy $\vec{p}^2/(2m)$ plus the potential energy V , Eq. (58) is indeed true.

In Eq. (59), we consider the path $\vec{x}_{\text{cl}}(\tau)$ that gets from \vec{x}_0 at time t_0 to \vec{x} at time t . We then evaluate $\vec{x}_{\text{cl}}(\tau)$ and $\vec{p}_{\text{cl}}(\tau)$ at $\tau = t$. The results depend on \vec{x} and t , so we could write this equation as

$$E(\vec{x}, t) - \frac{1}{2m} \vec{p}(\vec{x}, t)^2 - V(\vec{x}) = 0 \quad . \quad (60)$$

Now, let's look at the semiclassical approximation. Suppose that E , V , and $\vec{p}^2/(2m)$ are large. Then our wave function has closely spaced wiggles in space and time. In particular, $|\vec{p}|$ is big, or the quantum wavelength $\lambda = 1/|\vec{p}|$ is small. However, let's suppose that $V(\vec{x})$ is slowly varying and that $|\vec{x} - \vec{x}_0| \gg \lambda$, so that \vec{x} is far from the singular point where all the paths start. Let's denote the distance characteristic of the variation of V or $|\vec{x} - \vec{x}_0|$ by R . Then we consider the situation in which $\lambda \ll R$. The distance characteristic of variation of $p(\vec{x}, t)$ is controlled by classical mechanics, so it should be R :

$$\vec{\nabla} \cdot \vec{p}(\vec{x}, t) \sim \frac{1}{R} |\vec{p}(\vec{x}, t)| \quad . \quad (61)$$

That is

$$\vec{\nabla} \cdot (\vec{\nabla} S) \sim \frac{1}{R} |\vec{\nabla} S| \quad , \quad (62)$$

while

$$|\vec{\nabla} S| \sim \frac{1}{\lambda} \quad . \quad (63)$$

Then since $R \gg \lambda$ we have $\vec{\nabla} \cdot (\vec{\nabla} S) \sim 1/(R\lambda)$ while $(\vec{\nabla} S)^2 \sim 1/\lambda^2$, so

$$\vec{\nabla} \cdot (\vec{\nabla} S) \ll (\vec{\nabla} S)^2 \quad , \quad (64)$$

That is our approximation.

We see that the solution to the Schrödinger equation in the classical limit is

$$K(\vec{x}, t; \vec{x}_0, t_0) = \sqrt{\rho(\vec{x}, t)} \exp[iS(\vec{x}, t)] \quad . \quad (65)$$

The classical action appears in the exponent. There is a normalization factor $\sqrt{\rho}$ in front. If ρ is real and if we call $K(\vec{x}, t; \vec{x}_0, t_0) = \psi(\vec{x}, t)$, then $\rho = |\psi|^2$. To find ρ , we need the next order approximation. We will return to that question after we have learned a little more.

7 More about the classical action

It is easy to interpret the classical action as an approximate quantum mechanical phase. We write, using $\vec{p}_{\text{cl}} = m d\vec{x}/dt$,

$$\begin{aligned}
 S(\vec{x}, t) &= \int_{t_0}^t dt \left\{ \frac{1}{2} m \left(\frac{d\vec{x}_{\text{cl}}(t)}{dt} \right)^2 - V(\vec{x}_{\text{cl}}(t)) \right\} \\
 &= \int_{t_0}^t dt \left\{ m \left(\frac{d\vec{x}_{\text{cl}}(t)}{dt} \right)^2 - \frac{1}{2} m \left(\frac{d\vec{x}_{\text{cl}}(t)}{dt} \right)^2 - V(\vec{x}_{\text{cl}}(t)) \right\} \\
 &= \int_{t_0}^t dt \frac{d\vec{x}_{\text{cl}}(t)}{dt} \cdot \vec{p}_{\text{cl}}(t) - \int_{t_0}^t dt E \\
 &= \int_{\vec{x}_0}^{\vec{x}} d\vec{x} \cdot \vec{p}_{\text{cl}}(\vec{x}) - (t - t_0)E .
 \end{aligned} \tag{66}$$

In the first term we integrate along the path in space that the particle takes, with \vec{p} being a vector pointing along the path whose magnitude is

$$|\vec{p}_{\text{cl}}(\vec{x})| = \sqrt{2m(E - V(\vec{x}))} , \tag{67}$$

where E is the classical energy for the path. We accumulate phase from a factor $\exp(i\vec{p} \cdot d\vec{x})$ for each step along the path. In the second term, we have the classical energy E times the total time needed to traverse the path, corresponding to a factor $\exp(-iEdt)$ for each step of the path.

We can write this as

$$S(\vec{x}, t) = W(\vec{x}, E) - (t - t_0)E . \tag{68}$$

where

$$W(\vec{x}, E) = \int_{\vec{x}_0}^{\vec{x}} d\vec{x} \cdot \vec{p}_{\text{cl}}(\vec{x}) . \tag{69}$$

We consider this to be a function of the final position \vec{x} and the energy E .

Let us find how $W(\vec{x}, E)$ varies if we vary \vec{x} and E :

$$\begin{aligned}
 \delta W(\vec{x}, E) &= \delta S + E\delta t + (t - t_0)\delta E \\
 &= [\vec{p} \cdot \delta\vec{x} - E\delta t] + E\delta t + (t - t_0)\delta E \\
 &= \vec{p} \cdot \delta\vec{x} + (t - t_0)\delta E .
 \end{aligned} \tag{70}$$

Thus

$$\begin{aligned}\vec{\nabla}W(\vec{x}, E) &= \vec{p}_{\text{cl}}(\vec{x}) \ , \\ \frac{\partial W(\vec{x}, E)}{\partial E} &= (t - t_0) \ .\end{aligned}\tag{71}$$

We use $W(\vec{x}, E)$ if we want to find an energy eigenfunction in the classical limit. For instance, we can think of a wave function that represents waves that start from a source at location \vec{x}_0 . If we move \vec{x}_0 to somewhere far away near the $-z$ axis, then we would have waves that start as plane waves for large negative z and then get bent by a potential $V(\vec{x})$ that is zero for large $|\vec{x}|$ but can bend the waves when they get to the region where the potential acts. This corresponds to unbound quantum states.

We can also think of bound states, so that we want to find bound state energy eigenvalues. Then our approximation is part of the WKB approximation described in Sakurai. However, for a bound state in one dimension, there is a point where the particle runs out of kinetic energy and turns around. At the turning point, $p_{\text{cl}} = 0$. At that point, the semiclassical approximation is not working. Thus we need to use real quantum mechanics to match wave functions in the classically allowed region and the classically forbidden region.

To find an energy eigenstate, we want to solve

$$\left[-\frac{1}{2m} \vec{\nabla}^2 + V(\vec{x}) - E \right] \psi(\vec{x}) = 0 \ .\tag{72}$$

We put

$$\psi(\vec{x}) = \exp[iA(\vec{x})] \ .\tag{73}$$

Then

$$\frac{1}{2m} \left(\vec{\nabla}A \right)^2 - \frac{i}{2m} \vec{\nabla}^2 A + V(\vec{x}) - E = 0 \ .\tag{74}$$

We approximate this in the small wavelength limit as

$$\frac{1}{2m} \left(\vec{\nabla}A \right)^2 + V(\vec{x}) - E \approx 0 \ .\tag{75}$$

We try the solution $A(\vec{x}) = W(\vec{x}, E)$. Since $\vec{\nabla}W = \vec{p}_{\text{cl}}$, we get

$$\frac{1}{2m} \vec{p}_{\text{cl}}(\vec{x})^2 + V(\vec{x}) - E = 0 \ .\tag{76}$$

Thus this ansatz solves our equation. We see that the solution to the Schrödinger equation for an energy eigenfunction in the classical limit is

$$\psi(\vec{x}) = \sqrt{\rho(\vec{x}, E)} \exp[iW(\vec{x}, E)] . \quad (77)$$

The function W from Eq. (69) appears in the exponent. There is a normalization factor $\sqrt{\rho}$ in front. If ρ is real, then $\rho = |\psi|^2$. To find ρ , we need the next order approximation.

8 The next order approximation

We have two semiclassical treatments. One considers quantum particles that start at \vec{x}_0 at time t_0 and reach position \vec{x} at time t . These particles can have any energy. The other treatment considers particles that have energy E . They travel from \vec{x}_0 to \vec{x} , but we do not ask how long it takes. In each case, there is a normalization ρ such that $|\psi|^2 = \rho$. Our treatment has not been at the level of approximation that enables us to find ρ , but we can analyze that now.

The constant energy case is easiest. Let

$$\psi(\vec{x}) = \exp \left(iW(\vec{x}, E) + \frac{1}{2} \log(\rho(\vec{x}, E)) \right) . \quad (78)$$

The Schrödinger equation (72) is equivalent to

$$-\frac{1}{2m} \left(i\vec{\nabla}W + \frac{1}{2\rho} \vec{\nabla}\rho \right)^2 - \frac{1}{2m} \left(i\vec{\nabla}^2W + \frac{1}{2} \vec{\nabla}^2 \log \rho \right) + V(\vec{x}) - E = 0 . \quad (79)$$

Counting ρ and $\vec{\nabla}W$ as being slowly varying, we can approximate this by

$$\frac{1}{2m} (\vec{\nabla}W)^2 - \frac{i}{2m} \frac{1}{\rho} \vec{\nabla}\rho \cdot \vec{\nabla}W - \frac{i}{2m} \vec{\nabla}^2W + V(\vec{x}) - E = 0 . \quad (80)$$

With $\vec{\nabla}W = \vec{p}_{\text{cl}}(\vec{x})$, this is

$$\frac{1}{2m} \vec{p}_{\text{cl}}(\vec{x})^2 - \frac{i}{2m} \frac{1}{\rho} \vec{\nabla}\rho \cdot \vec{p}_{\text{cl}}(\vec{x}) - \frac{i}{2m} \vec{\nabla} \cdot \vec{p}_{\text{cl}}(\vec{x}) + V(\vec{x}) - E = 0 . \quad (81)$$

We have kept the “big” terms here and the first not-so-big terms. As we have already arranged, the big terms cancel ($E = p^2/(2m) + V$) and we are left with the not so big terms,

$$-\frac{i}{2m} \frac{1}{\rho(\vec{x})} (\vec{\nabla}\rho(\vec{x})) \cdot \vec{p}_{\text{cl}}(\vec{x}) - \frac{i}{2m} \vec{\nabla} \cdot \vec{p}_{\text{cl}}(\vec{x}) = 0 . \quad (82)$$

This is more understandable if we write

$$\vec{p}_{\text{cl}}(\vec{x}) = m\vec{v}_{\text{cl}}(\vec{x}) . \quad (83)$$

Then

$$\vec{v}_{\text{cl}}(\vec{x}) \cdot \vec{\nabla} \rho(\vec{x}) + \rho(\vec{x}) \vec{\nabla} \cdot \vec{v}_{\text{cl}}(\vec{x}) = 0 . \quad (84)$$

Given $\vec{v}_{\text{cl}}(\vec{x})$, one could pretty easily solve this for ρ , at least numerically. To see what the equation means, write it as

$$\vec{\nabla} \cdot [\rho(\vec{x}) \vec{v}_{\text{cl}}(\vec{x})] = 0 . \quad (85)$$

Imagine lots of particles. The particles at \vec{x} have density $\rho(\vec{x})$ and move with velocity $\vec{v}_{\text{cl}}(\vec{x})$. This equation says that the number of particles is conserved. For example, if $\vec{v}_{\text{cl}}(\vec{x})$ always points in the same direction, then ρ is smallest where v_{cl} is biggest. If $|\vec{v}_{\text{cl}}|$ is constant, but all of the particles are moving radially outward from the origin, then $\rho \propto 1/|\vec{x}|^2$.

Now let's try the time dependent case. We write

$$\psi(\vec{x}) = \exp \left(iS(\vec{x}, t) + \frac{1}{2} \log(\rho(\vec{x}, t)) \right) . \quad (86)$$

The the Schrödinger equation (72) is equivalent to

$$\begin{aligned} 0 = & -\frac{\partial S}{\partial t} + \frac{i}{2\rho} \frac{\partial \rho}{\partial t} + \frac{1}{2m} \left(i\vec{\nabla} S + \frac{1}{2\rho} \vec{\nabla} \rho \right)^2 \\ & + \frac{1}{2m} \left(i\vec{\nabla}^2 S + \frac{1}{2} \vec{\nabla}^2 \log \rho \right) - V(\vec{x}) . \end{aligned} \quad (87)$$

Counting ρ and $\vec{\nabla} W$ as being slowly varying, we can approximate this by

$$\begin{aligned} 0 = & -\frac{\partial S}{\partial t} + \frac{i}{2\rho} \frac{\partial \rho}{\partial t} - \frac{1}{2m} \left(\vec{\nabla} S \right)^2 + \frac{i}{2m\rho} \vec{\nabla} S \cdot \vec{\nabla} \rho \\ & + \frac{i}{2m} \vec{\nabla}^2 S - V(\vec{x}) . \end{aligned} \quad (88)$$

With $\partial S / \partial t = -E$ and $\vec{\nabla} S = \vec{p}_{\text{cl}}(\vec{x})$, this is

$$0 = E + \frac{i}{2\rho} \frac{\partial \rho}{\partial t} - \frac{1}{2m} \vec{p}_{\text{cl}}^2 + \frac{i}{2m\rho} \vec{p}_{\text{cl}} \cdot \vec{\nabla} \rho + \frac{i}{2m} \vec{\nabla} \cdot \vec{p}_{\text{cl}} - V(\vec{x}) . \quad (89)$$

As we have already arranged, the big terms cancel ($E = p^2/(2m) + V$) and we are left with the not-so-big terms,

$$0 = \frac{i}{2\rho} \frac{\partial \rho}{\partial t} + \frac{i}{2m\rho} \vec{p}_{\text{cl}} \cdot \vec{\nabla} \rho + \frac{i}{2m} \vec{\nabla} \cdot \vec{p}_{\text{cl}} . \quad (90)$$

This is more understandable if we write

$$\vec{p}_{\text{cl}}(\vec{x}, t) = m\vec{v}_{\text{cl}}(\vec{x}, t) . \quad (91)$$

Then

$$\frac{\partial \rho}{\partial t} + \vec{v}_{\text{cl}} \cdot \vec{\nabla} \rho + \rho \vec{\nabla} \cdot \vec{v}_{\text{cl}} = 0 . \quad (92)$$

To see what the equation means, write it as

$$\frac{\partial \rho(\vec{x}, t)}{\partial t} + \vec{\nabla} \cdot [\rho(\vec{x}, t) \vec{v}_{\text{cl}}(\vec{x}, t)] = 0 . \quad (93)$$

This equation says that the number of particles is conserved. We need some boundary conditions to actually solve for ρ . For example, for free particles starting from $\vec{x}_0 = 0$ at time $t_0 = 0$, we have $\vec{v} = \vec{x}/t$ and, using Eq. (4), $\rho \propto 1/t^3$. This does satisfy the conservation equation.

9 The path integral with a magnetic field

Our derivation of the path integral was for a hamiltonian $-\vec{\nabla}^2/(2m) + V$. For a particle in a time independent electric and magnetic field, the hamiltonian is

$$H = \frac{1}{2m} \left(-i\vec{\nabla} - q\vec{A}(\vec{x}) \right)^2 + q\phi(\vec{x}) . \quad (94)$$

Here q is the charge of the particle and ϕ and \vec{A} are the scalar and vector potentials

$$\begin{aligned} -\vec{\nabla}\phi &= \vec{E} , \\ \vec{\nabla} \times \vec{A} &= \vec{B} . \end{aligned} \quad (95)$$

We can derive the Feynman path integral in this case also. As before, we divide the time interval into a large number N of small increments of size Δt . We then have

$$K(\vec{x}_{\text{F}}, t_{\text{F}}; \vec{x}_0, t_0) = \langle \vec{x}_{\text{F}} | e^{-iH\Delta t} \dots e^{-iH\Delta t} e^{-iH\Delta t} | \vec{x}_0 \rangle . \quad (96)$$

with N factors of $\exp(-iH\Delta t)$. Between each pair of factors $\exp(-iH\Delta t)$, we introduce a complete sum over position eigenstates,

$$K(\vec{x}_F, t_F; \vec{x}_0, t_0) = \int d\vec{x}_{N-1} \cdots \int d\vec{x}_2 \int d\vec{x}_1 \langle \vec{x}_F | e^{-iH\Delta t} | \vec{x}_{N-1} \rangle \cdots \times \langle \vec{x}_2 | e^{-iH\Delta t} | \vec{x}_1 \rangle \langle \vec{x}_1 | e^{-iH\Delta t} | \vec{x}_0 \rangle \quad (97)$$

Physically, \vec{x}_i is going to represent where the particle is at time t_i .

Now we examine

$$\begin{aligned} & \langle \vec{x}_{i+1} | e^{-iH\Delta t} | \vec{x}_i \rangle \\ &= \langle \vec{x}_{i+1} | \exp \left(-i\Delta t \left[\frac{1}{2m} \left(\vec{p}_{\text{op}} - q\vec{A}(\vec{x}_{\text{op}}) \right)^2 + q\phi(\vec{x}_{\text{op}}) \right] \right) | \vec{x}_i \rangle . \end{aligned} \quad (98)$$

We need to approximate this. We set each instance of \vec{x}_{op} equal to a number $\vec{x} = (\vec{x}_{i+1} + \vec{x}_i)/2$. This is not exact, but it is OK to first order in the small quantity Δt . Then

$$\begin{aligned} & \langle \vec{x}_{i+1} | e^{-iH\Delta t} | \vec{x}_i \rangle \\ &= \langle \vec{x}_{i+1} | \exp \left(-i\Delta t \left[\frac{1}{2m} \left(\vec{p}_{\text{op}} - q\vec{A}(\vec{x}) \right)^2 + q\phi(\vec{x}) \right] \right) | \vec{x}_i \rangle . \end{aligned} \quad (99)$$

After making this approximation, we can evaluate the matrix element without further approximation. We insert an integral over momentum eigenstates, giving

$$\begin{aligned} & \langle \vec{x}_{i+1} | e^{-iH\Delta t} | \vec{x}_i \rangle \\ &\approx \int d\vec{p} \langle \vec{x}_{i+1} | \vec{p} \rangle \exp \left(-i\Delta t \left[\frac{1}{2m} \left(\vec{p} - q\vec{A}(\vec{x}) \right)^2 + q\phi(\vec{x}) \right] \right) \langle \vec{p} | \vec{x}_i \rangle \\ &= \frac{1}{(2\pi)^3} \int d\vec{p} e^{i\vec{p} \cdot (\vec{x}_{i+1} - \vec{x}_i)} \exp \left(-i\Delta t \left[\frac{1}{2m} \left(\vec{p} - q\vec{A}(\vec{x}) \right)^2 + q\phi(\vec{x}) \right] \right) \\ &= \frac{1}{(2\pi)^3} \int d\vec{p} \exp \left(-i\Delta t \left[\frac{1}{2m} \left(\vec{p} - q\vec{A}(\vec{x}) \right)^2 - \vec{v} \cdot \vec{p} + q\phi(\vec{x}) \right] \right) . \end{aligned} \quad (100)$$

Here we have defined

$$\vec{v} = \frac{\vec{x}_{i+1} - \vec{x}_i}{\Delta t} . \quad (101)$$

We can complete the square in the exponent:

$$\begin{aligned} \frac{1}{2m} \left(\vec{p} - q\vec{A}(\vec{x}) \right)^2 - \vec{v} \cdot \vec{p} &= \frac{1}{2m} \left(\vec{p} - m\vec{v} - q\vec{A}(\vec{x}) \right)^2 \\ &\quad - \frac{1}{2} m\vec{v}^2 - q\vec{v} \cdot \vec{A}(\vec{x}) . \end{aligned} \quad (102)$$

Then

$$\begin{aligned} \langle \vec{x}_{i+1} | e^{-iH\Delta t} | \vec{x}_i \rangle &\approx \exp \left(i\Delta t \left[\frac{1}{2} m\vec{v}^2 + q\vec{v} \cdot \vec{A}(\vec{x}) - q\phi(\vec{x}) \right] \right) \\ &\quad \frac{1}{(2\pi)^3} \int d\vec{k} \exp \left(-i\Delta t \frac{1}{2m} \vec{k}^2 \right) \\ &= \left(\frac{m}{2\pi i\Delta t} \right)^{3/2} \exp \left(i\Delta t \left[\frac{1}{2} m\vec{v}^2 + q\vec{v} \cdot \vec{A}(\vec{x}) - q\phi(\vec{x}) \right] \right) . \end{aligned} \quad (103)$$

With this result, we have

$$K(\vec{x}_F, t_F; \vec{x}_0, t_0) = \int_{\substack{\vec{x}(t_N)=\vec{x}_F \\ \vec{x}(t_0)=\vec{x}_0}} \mathcal{D}[x] \exp(iS[x]) . \quad (104)$$

Here

$$\int_{\substack{\vec{x}(t_N)=\vec{x}_F \\ \vec{x}(t_0)=\vec{x}_0}} \mathcal{D}[x] \cdots = \left(\frac{m}{2\pi i\Delta t} \right)^{3N/2} \prod_{i=1}^{N-1} \int d\vec{x}_i \cdots , \quad (105)$$

where we understand that in the integrand we should set $\vec{x}(t_N)$ and $\vec{x}(t_0)$ as indicated. Here $S[x]$ represents the classical action associated with the path x :

$$S[x] = \int_{t_0}^{t_F} dt \left\{ \frac{1}{2} m \left(\frac{d\vec{x}(t)}{dt} \right)^2 + q \frac{d\vec{x}(t)}{dt} \cdot \vec{A}(\vec{x}(t)) - q\phi(\vec{x}(t)) \right\} . \quad (106)$$

This is the action for a smooth path. What we get from our derivation is a discrete approximation to this, in which the integral over t is really as sum.

If you study classical mechanics with a charged particle moving in an electric and magnetic field, then this S is the action that you use.

We can note something very interesting about this. In the term involving \vec{A} we can write

$$\int dt \frac{d\vec{x}(t)}{dt} \cdots = \int d\vec{x} \cdots . \quad (107)$$

Then

$$S[x] = \int_{t_0}^{t_F} dt \left\{ \frac{1}{2} m \left(\frac{d\vec{x}(t)}{dt} \right)^2 - q \phi(\vec{x}(t)) \right\} + q \int d\vec{x} \cdot \vec{A}(\vec{x}) . \quad (108)$$

The first term depends on how the motion of the particle along the path depends on the time t , but the second term is more geometrical: it is the line integral of \vec{A} along the path that the particle traces out, but the time dependence doesn't matter. This is especially interesting when the particle is moving in a region in which there is no magnetic field. Then in a classical approximation, the classical path is determined by the electric field $-\vec{\nabla}\phi$, while the vector potential \vec{A} does not affect the classical path. Nevertheless, the quantum wave function accumulates a phase $q \int d\vec{x} \cdot \vec{A}(\vec{x})$. This is the basis of the Aharonov-Bohm effect, which is described in Sakurai.