Lagrangian with electric and magnetic fields[1]
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1 The electric and magnetic fields

A particle of charge[3] $\mathcal{Q}$ in an electric field $\boldsymbol{E}(\boldsymbol{x}, t)$ and a magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$ feels a force

$$\boldsymbol{F} = \mathcal{Q}\boldsymbol{E} + \mathcal{Q}\,\dot{\boldsymbol{x}} \times \boldsymbol{B} \ . \tag{1}$$

What is the lagrangian that generates this?

What we need is the vector and scalar potential, which are related to $\boldsymbol{E}$ and $\boldsymbol{B}$ by

$$\boldsymbol{E}(\boldsymbol{x}, t) = -\boldsymbol{\nabla}\Phi(\boldsymbol{x}, t) - \frac{d}{dt}\boldsymbol{A}(\boldsymbol{x}, t) \ ,$$
$$\boldsymbol{B}(\boldsymbol{x}, t) = \boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{x}, t) \ . \tag{2}$$

2 The lagrangian

Let's start with our standard lagrangian for particles interacting with each other but with no external forces,

$$L = T - V \tag{3}$$

with

$$T = \sum_J \frac{1}{2} m_J \, \dot{\boldsymbol{x}}_J^2 \ ,$$
$$V = \frac{1}{2} \sum_{JK} V_{JK}(|\boldsymbol{x}_J - \boldsymbol{x}_K|) \ . \tag{4}$$

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[3]Instead of $\mathcal{Q}$ for the charge, one could use q. However that choice is rather unfortunate since we often have coordinates q_J. An alternative that one often sees is e. However, e is the charge of a proton, so that an electron has charge $-e$ and an up quark has charge $3e/3$. Thus e is not such a good choice.

Now we add an interaction with the external fields,

$$L = T - V + \sum_J L_J^{(e)} \ . \tag{5}$$

We just take a guess and see if it does what we want:

$$L_J^{(e)} = \mathcal{Q}_J \, \dot{\boldsymbol{x}}_J \cdot \boldsymbol{A}(\boldsymbol{x}_J, t) - \mathcal{Q}_J \, \Phi(\boldsymbol{x}_J, t) \ . \tag{6}$$

We need (using explicitly the components x_J^i of $\boldsymbol{x}_J$)

$$
\begin{aligned}
\frac{d}{dt} \frac{\partial L_J^{(e)}(\boldsymbol{x}_J, \dot{\boldsymbol{x}}_J, t)}{\partial \dot{x}_J^i} &= \frac{d}{dt} \, \mathcal{Q}_J A^i(\boldsymbol{x}_J, t) \\
&= \mathcal{Q}_J \frac{\partial A^i(\boldsymbol{x}_J, t)}{\partial t} + \mathcal{Q}_J \frac{\partial A^i(\boldsymbol{x}_J, t)}{\partial x_J^j} \, \dot{x}_J^j
\end{aligned}
\tag{7}
$$

and

$$\frac{\partial L_J^{(e)}(\boldsymbol{x}_J, \dot{\boldsymbol{x}}_J, t)}{\partial x_J^i} = \mathcal{Q}_J \dot{x}_J^j \frac{\partial A^j(\boldsymbol{x}_J, t)}{\partial x_J^i} - \mathcal{Q}_J \frac{\partial \Phi(\boldsymbol{x}_J, t)}{\partial x_J^i} \ . \tag{8}$$

Thus

$$
\begin{aligned}
-\frac{d}{dt} \frac{\partial L_J^{(e)}(\boldsymbol{x}_J, \dot{\boldsymbol{x}}_J, t)}{\partial \dot{x}_J^i} &+ \frac{\partial L_J^{(e)}(\boldsymbol{x}_J, \dot{\boldsymbol{x}}_J, t)}{\partial x_J^i} \\
&= -\mathcal{Q}_J \left[\frac{\partial \Phi(\boldsymbol{x}_J, t)}{\partial x_J^i} + \frac{\partial A^i(\boldsymbol{x}_J, t)}{\partial t} \right] \\
&\quad + \mathcal{Q}_J \left[\frac{\partial A^j(\boldsymbol{x}_J, t)}{\partial x_J^i} - \frac{\partial A^i(\boldsymbol{x}_J, t)}{\partial x_J^j} \right] \dot{x}_J^j \\
&= \mathcal{Q}_J E^i(\boldsymbol{x}_J, t) + \mathcal{Q}_J \epsilon_{ijk} B^k(\boldsymbol{x}_J, t) \, \dot{x}_J^j \\
&= \left[\mathcal{Q}_J \boldsymbol{E}(\boldsymbol{x}_J, t) + \mathcal{Q}_J \, \dot{\boldsymbol{x}}_J \times \boldsymbol{B}(\boldsymbol{x}_J, t) \right]^i \ .
\end{aligned}
\tag{9}
$$

We have already analyzed what the other terms in the lagrangian contribute to the equations of motion. Putting everything together, we have

$$m_J \ddot{\boldsymbol{x}}_J = \mathcal{Q}_J \boldsymbol{E}(\boldsymbol{x}_J, t) + \mathcal{Q}_J \, \dot{\boldsymbol{x}}_J \times \boldsymbol{B}(\boldsymbol{x}_J, t) - \sum_K \boldsymbol{\nabla} V_{JK}(|\boldsymbol{x}_J - \boldsymbol{x}_K|) \ . \tag{10}$$

We have the effect of internal forces, as previously, but now each particle feels an electromagnetic force of the form (1). We conclude that Eq. (6) was the right guess for the lagrangian.

For a single particle with charge $\mathcal{Q}$ and position $\boldsymbol{x}$, the lagrangian is then

$$L = \frac{1}{2} m \, \dot{\boldsymbol{x}}^2 + \mathcal{Q} \, \dot{\boldsymbol{x}} \cdot \boldsymbol{A}(\boldsymbol{x}, t) - \mathcal{Q} \, \Phi(\boldsymbol{x}, t) \;, \tag{11}$$

which gives the single particle equation of motion

$$m\ddot{\boldsymbol{x}} = \mathcal{Q}\boldsymbol{E} + \mathcal{Q}\,\dot{\boldsymbol{x}} \times \boldsymbol{B} \;\;. \tag{12}$$

Exercise 2.1: Consider the lagrangian Eq. (11) for just one particle. Suppose that Φ and $\boldsymbol{A}$ are independent of time (for fixed $\boldsymbol{x}$). Then there should be a conserved energy. What is it?

Exercise 2.2: Consider equation of motion (12) for just one particle. Suppose that $\boldsymbol{E} = 0$ and the magnetic field is uniform and time independent and in the z direction, $\boldsymbol{B} = B\hat{\boldsymbol{z}}$. Suppose that the particle starts at time 0 at $\boldsymbol{x} = 0$ with initial velocity $\boldsymbol{v}_0 = (v_0^1, v_0^2, v_0^3)$. Solve the equation of motion to find $\boldsymbol{x}(t)$ as a function of t. Verify that the energy that you found in Exercise 2.1 is conserved.

It would seem that this problem has symmetries under translations in the x^1, x^2, and x^3 directions and under rotations about the x^3 axis. The following exercises explore this issue. In the exercises, I suggest particular choices of $\boldsymbol{A}$ that simplify consideration of each symmetry. One can also use an arbitrary choice of $\boldsymbol{A}$ that gives the desired $\boldsymbol{B}$ for each of the symmetries. This is more elegant, but requires more cleverness.

Exercise 2.3: In the case of a particle in a uniform magnetic field as in Exercise 2.2, one could take

$$A^1 = -\frac{B}{2} x^2 \,, \qquad\qquad A^2 = \frac{B}{2} x^1, \qquad\qquad A^3 = 0 \,.$$

With this chooice of $\boldsymbol{A}$, the lagrangian is invariant under translations in the x^3 direction. Thus there ought to be a corresponding conserved quantity Q. What is Q? Verify directly from your solutions in Exercise 2.2 or from the equations of motion that this quantity is conserved.

Exercise 2.4: In the case of a particle in a uniform magnetic field as in Exercise 2.2, one could take

$$A^1 = 0 \, , \qquad A^2 = Bx^1 , \qquad A^3 = 0 \, .$$

With this chooice of $\boldsymbol{A}$, the lagrangian is invariant under translations in the x^2 direction. Thus there ought to be a corresponding conserved quantity Q. What is Q? Verify directly from your solutions in Exercise 2.2 or from the equations of motion that this quantity is conserved.

Exercise 2.5: In the case of a particle in a uniform magnetic field as in Exercise 2.2, one could take

$$A^1 = -\frac{B}{2} \, x^2 \, , \qquad A^2 = \frac{B}{2} \, x^1 , \qquad A^3 = 0 \, .$$

Show that with this chooice of $\boldsymbol{A}$, the lagrangian is invariant under rotations about the x^3 axis. Thus there ought to be a corresponding conserved quantity Q. What is Q? Verify directly from your solutions in Exercise 2.2 or from the equations of motion that this quantity is conserved.