RESEARCH STATEMENT

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1. Introduction

I am a number theorist whose research falls primarily under algebraic number theory. In particular, I study properties of number fields and groups with number theoretic connections, and I do this mainly by studying certain exponential sums. I have also recently begun an entirely new research project on explicit and computational aspects of automorphic forms on unitary groups.

One example of the sums that I study are Gaussian periods, which have been used in various areas of mathematics, including number theory, cryptography, and analysis. Motivated by the recent work of several mathematicians, I study Gaussian periods from a visual perspective by plotting them in the complex plane. These Gaussian period plots have many striking patterns, and I study the mathematical structures underlying these patterns in my research.

Additionally, one can view Gaussian periods as natural objects of study from the perspectives of character theory and class field theory. Using these perspectives as a guide, I have generalized Gaussian periods to other contexts. Specifically, I have used so-called supercharacter theory to study sums of characters on other groups, and I have used explicit class field theory for quadratic imaginary fields to study Galois sums acting on abelian extensions, which requires studying elliptic curves and complex multiplication. I study the visual aspects of both of these types of sums, where I attempt to glean number theoretic information by studying the emergent patterns and structures.

Overall, the driving philosophy of my research is that computation and experimentation have a mutually inspirational and informational relationship with new and interesting mathematics, and I wish to foster this relationship further in the continuance of my work.

2. Gaussian Periods and Analogues

In this section, we cover the definition of Gaussian periods and Gaussian period plots, some motivating examples, and discuss some of my own results. Following this, we discuss generalizations of Gaussian periods from both a supercharacter theory and class field theory perspective.

We note that throughout our discussion, we define $e(x) := e^{2\pi i x}$.

2.1. **Gaussian Periods.**

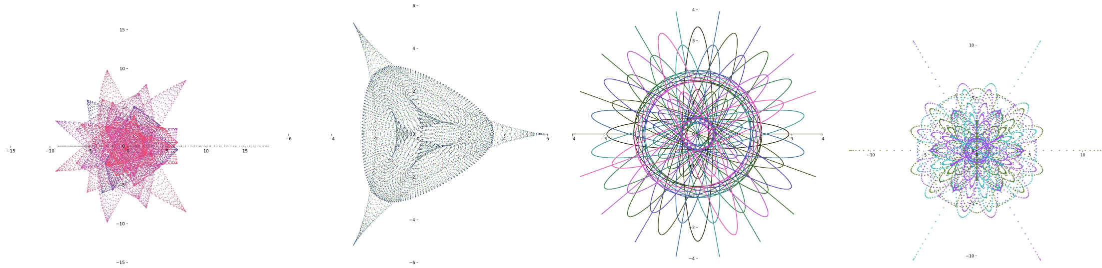
Definition 1. Let n be a positive integer and ω an integer coprime to n. Let d be the multiplicative order of ω modulo n. For an integer k, we define the following map:

$$\eta_{n,\omega} : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}, \qquad \eta_{n,\omega}(k) := \sum_{j=0}^{d-1} e\left(\frac{\omega^j k}{n}\right).$$

We call $\eta_{n,\omega}(k)$ a *Gaussian period* of modulus n and generator ω. Additionally, we call $\operatorname{img}(\eta_{n,\omega})$ the *Gaussian period plot* of modulus n and generator ω.

We provide examples of Gaussian period plots for various moduli n and generators ω in Figure 1. Each dot in the images represents a single Gaussian period for some $k \in \mathbb{Z}/n\mathbb{Z}$. For our discussion, we omit a description of the color scheme being used.

Gaussian periods have been studied for centuries, starting with Gauss while studying constructibility. They were also studied by Kummer and more recently by Lenstra and Pomerance. However,



(A) $n = 212979$, $\omega = 9247$ (B) $n = 52059$, $\omega = 766$ (C) $n = 478125$, $\omega = 3124$ (D) $n = 62160$, $\omega = 319$

FIGURE 1. Examples of Gaussian period plots for various choices of n and ω

it wasn't until the work of Brumbaugh et al., Duke, Garcia, Hyde, and Lutz [BBF$^+$14, BBGG$^+$13, DGL15, GHL15] in the last decade that mathematicians began to study Gaussian period *plots*.

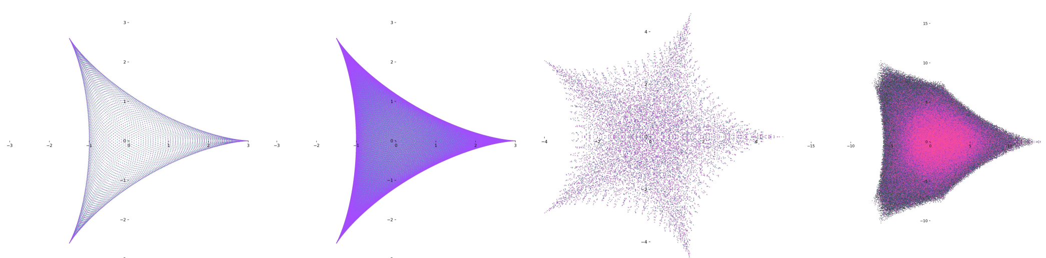
These plots have remarkable patterns and structures, inspiring curiosity into the mathematics underlying them. Many patterns have been explained, though many others remain a mystery. We provide one understood case for motivation, proved by Duke, Garcia, and Lutz [DGL15].

Theorem 2 (DGL15)**.** *Let $n = p^a$, where p is an odd prime. Choose ω with multiplicative order d dividing $p-1$. Then for $j \in \{0, \dots, d-1\}$ and $m \in \{0, \dots, \varphi(d)-1\}$, there exist explicitly defined constants c_{mj} such that img($\eta_{n,\omega}$) is contained in the image of the Laurent polynomial function*

$$g_d : \mathbb{T}^{\varphi(d)} \to \mathbb{C}, \qquad g_d(z_1, z_2, \dots, z_{\varphi(d)}) = \sum_{j=0}^{d-1} \prod_{m=0}^{\varphi(d)-1} z_{m+1}^{c_{mj}}.$$

Moreover, img(g_d) is "filled out" asymptotically by Gaussian periods as n goes to infinity.

In the case where d is itself also a prime, the image img(g_d) becomes a *d-sided hypocycloid.* We provide examples of this phenomenon (for d both prime and composite) in Figure 2.



(A) $n = 31^3$, $d = 3$ (B) $n = 13^5$, $d = 3$ (C) $n = 41^3$, $d = 5$ (D) $n = 31^5$, $d = 15$

FIGURE 2. Examples of Duke–Garcia–Lutz Theorem for various values of d

Thus far in our discussion, we have viewed Gaussian period plots as fixed final products. However, Benjamin Young and I discovered that these plots can also be viewed from a more dynamic perspective. Instead of plotting $\eta_{n,\omega}(k)$ for every $k \in \mathbb{Z}/n\mathbb{Z}$ all at once, we can plot them in batches. Stringing these batches together into a short animation gives us insight into the behavior of Gaussian periods. Some examples of these animations can be found on my website at pages.uoregon.edu/splatt3/research.html.

This perspective leads to the following proposition, which I prove in [Pla23, §3.1].

Proposition 3 (Pla23)**.** *Let $n = p^a$ be a power of an odd prime, d a prime dividing $p-1$, and $\omega \in (\mathbb{Z}/n\mathbb{Z})^\times$ an element of order d. Let $k \in \mathbb{Z}/n\mathbb{Z}$. Then the value of the k-th Gaussian period of modulus n and generator ω is contained in a $(d-1)$-sided hypocycloid centered at $e\left(\frac{k}{n}\right)$ and rotated* *by a factor of* $e\left(\frac{-k}{(d-1)n}\right)$. *That is,*

$$\eta_{n,\omega}(k) \in \left\{ e\left(\frac{k}{n}\right) + h \cdot e\left(\frac{-k}{(d-1)n}\right) \,:\, h \in H_{d-1} \right\},$$

where H_{d-1} *represents the filled-in* $(d-1)$*-sided hypocycloid centered at the origin.*

Since d is assumed to be prime in this proposition, then by the DGL Theorem, $\mathrm{img}(\eta_{n,\omega})$ is contained in the d-sided hypocycloid centered at the origin. This proposition then says that, as k increases from 0 to $n-1$, the behavior of Gaussian periods is that of a $(d-1)$-sided hypocycloid rolling smoothly counterclockwise along the interior of this d-sided hypocycloid. This fact relies on viewing Gaussian periods as trace maps of special unitary matrices $U \in \mathrm{SU}(d)$, demonstrating connections between Gaussian periods and more areas of mathematics.

2.2. **Supercharacter Theory Perspective.** Gaussian periods can be viewed as certain sums of characters on the groups $\mathbb{Z}/n\mathbb{Z}$. In particular, note that every character χ_y of $\mathbb{Z}/n\mathbb{Z}$ corresponds to the map $\chi_y(k) = e\left(\frac{yk}{n}\right)$ for some $y \in \mathbb{Z}/n\mathbb{Z}$. Thus, if we sum over χ_1, χ_ω, $\ldots$, and $\chi_{\omega^{d-1}}$, then we exactly obtain Gaussian periods as described above.

The choice of using $1, \omega, \ldots, \omega^{d-1} \in \mathbb{Z}/n\mathbb{Z}$ relates to a generalization of character theory known as supercharacter theory, which was first defined by Diaconis and Isaacs [DI08]. We omit a technical definition of supercharacter theory, but we note that, as mentioned in [GHL15], it has been used in studying various objects, including random walks on upper triangular matrices and the Hopf algebra of symmetric functions of non-commuting variables.

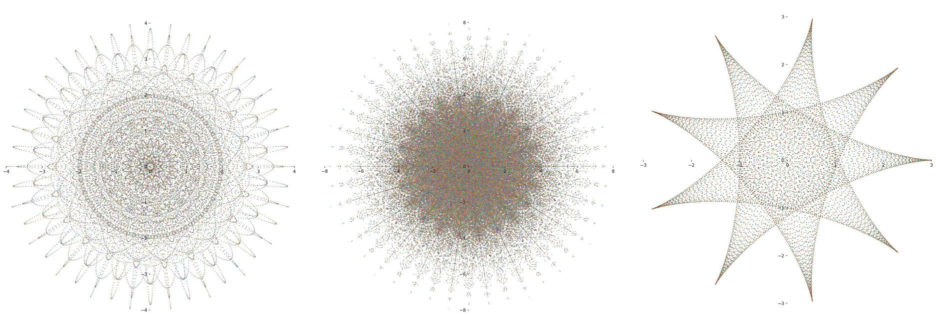
Given that Gaussian periods are a supercharacter theory on $\mathbb{Z}/n\mathbb{Z}$, I define in [Pla23, §3.2] an analogous construction of a supercharacter theory on $(\mathbb{Z}/n\mathbb{Z})^m$ for $m \in \mathbb{Z}_{\geq 1}$.

Definition 4 (Pla23)**.** Let n and m be positive integers, and let $A \in \mathrm{GL}_m(\mathbb{Z}/n\mathbb{Z})$ be a matrix of multiplicative order d. For $\mathbf{x} \in (\mathbb{Z}/n\mathbb{Z})^m$, we define the following supercharacter theory:

$$\theta_{n,m,A} : (\mathbb{Z}/n\mathbb{Z})^m \to \mathbb{C}, \qquad \theta_{n,m,A}(\mathbf{x}) := \sum_{j=0}^{d-1} e\left(\frac{(A^j \mathbf{1}) \cdot \mathbf{x}}{n}\right),$$

where $\mathbf{1} = (1, 1, \ldots, 1)^T \in (\mathbb{Z}/n\mathbb{Z})^m$ is viewed as a column vector.

Note that restricting to $m = 1$ gives the definition of Gaussian periods, where A is viewed as a 1×1 invertible matrix. Examples of plots of these supercharacter theories are shown in Figure 3.



(A) $A = \begin{pmatrix} 0 & 2 \\ 61 & 121 \end{pmatrix}$, $n = 715$ (B) $A = \begin{pmatrix} 310 & 229 \\ 458 & 295 \end{pmatrix}$, $n = 1155$ (C) $A = \begin{pmatrix} 51 & 224 & 140 \\ 168 & 268 & 14 \\ 203 & 203 & 170 \end{pmatrix}$, $n = 273$

Figure 3. Examples of supercharacter theory plots for a variety of m, n, and A.

My work then yields a natural generalization of the DGL Theorem.

Theorem 5 (Pla23)**.** *Let* $n \in \mathbb{Z}_{\geq 2}$ *and* $m \in \mathbb{Z}_{\geq 1}$. *Suppose* $d \mid (\#GL_m(\mathbb{Z}/n\mathbb{Z}))$, *and let* $\Phi_d(x)$ *be the* d*-th cyclotomic polynomial. Choose a matrix* $A \in GL_m(\mathbb{Z}/n\mathbb{Z})$ *of order* d *such that* $\Phi_d(A) = 0$

in $Mat_m(\mathbb{Z}/n\mathbb{Z})$. Then $img(\theta_{n,m,A})$ is contained in the image of the Laurent polynomial function $g_d : \mathbb{T}^{\varphi(d)} \to \mathbb{C}$ defined identically as in Theorem 2. Additionally, $img(g_d)$ is "filled out" asymptotically as n goes to infinity, assuming there exists a matrix $A \in GL_m(\mathbb{Z}/n\mathbb{Z})$ such that $\Phi_d(A) = 0$ mod n.

Remark 6. This theorem generalizes the original DGL Theorem in two ways. First, it looks at the group $(\mathbb{Z}/n\mathbb{Z})^m$ for any $m \geq 1$, rather than just the case $m = 1$. Second, it allows for moduli n which are not just powers of a prime, assuming there is some matrix $A \in \mathrm{GL}_m(\mathbb{Z}/n\mathbb{Z})$ such that $\Phi_d(A) = 0 \bmod n$. As an example of such an element A existing, if $n = 7 \cdot 13$, then $A = 16 \in (\mathbb{Z}/n\mathbb{Z})^\times$ has order 3 and satisfies $\Phi_3(A) = 0 \bmod n$. Thus, even in the case where $m = 1$, the theorem generalizes the DGL Theorem.

2.3. **Class Field Theory Perspective.** We can also view Gaussian periods through the lens of class field theory. In particular, the ray class field of $\mathbb{Q}$ of modulus n is $\mathbb{Q}(\zeta_n)$, where n is a primitive n-th root of unity. Additionally, the ray class group of $\mathbb{Q}$ of modulus n is naturally isomorphic to $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$. Since ω is an element of $(\mathbb{Z}/n\mathbb{Z})^\times$, then Gaussian periods are sums over the Galois action of $\langle\omega\rangle$ acting on the generators of the ray class field.

Thus, we would like to study Gaussian period analogues using base fields other than $\mathbb{Q}$. However, we must be able to describe explicitly the algebraic integers which generate abelian extensions, which is historically a difficult problem. In fact, it is Hilbert's 12th problem, and it has been solved in only two cases: when the base field is either quadratic imaginary or totally real, the latter being shown very recently by Dasgupta and Kakde in [DK21, DK23]. The case of quadratic imaginary fields is answered by the theory of elliptic curves and complex multiplication, and we restrict ourselves to this case.

Before continuing, we note that we assume knowledge of elliptic curves, complex multiplication (CM), and the class field theory of quadratic imaginary fields.

Let $K = \mathbb{Q}(\sqrt{-D})$, where $D \in \mathbb{Z}_{>0}$ is square-free. Let $\mathcal{O}_K$ be its ring of integers, E an elliptic curve with CM by $\mathcal{O}_K$, $\mathfrak{m} \subseteq \mathcal{O}_K$ an ideal, and $K[\mathfrak{m}]$ the ray class field modulus $\mathfrak{m}$. Then

$$K[\mathfrak{m}] = K(j(E), h(E[\mathfrak{m}])),$$

where $j(E)$ is the j-invariant, $E[\mathfrak{m}]$ is the set of $\mathfrak{m}$-torsion, and $h : E \to E/\mathrm{Aut}(E)$ is a Weber function. Additionally, if $K[1]$ is the Hilbert class field of K, then I show in [Pla23, §4.2] that

$$\mathrm{Gal}(K[\mathfrak{m}]/K[1]) \cong (\mathcal{O}_K/\mathfrak{m}\mathcal{O}_K)^\times/\mathcal{O}_K^\times.$$

I have defined in [Pla23, §4.3] the following analogue of Gaussian periods.

Definition 7 (Pla23)**.** Let K be a quadratic imaginary field, $\alpha \in K$ in the upper half-plane so that $\mathcal{O}_K = \mathbb{Z}[\alpha]$, and E the elliptic curve isomorphic to $\mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z} + \mathbb{Z}\alpha$ (so E has CM by $\mathcal{O}_K$). Choose $A \in (\mathcal{O}_K/\mathfrak{m}\mathcal{O}_K)^\times/\mathcal{O}_K^\times$, let d be the order of A, and let $\wp(z) := \wp(z; \Lambda)$ be the Weierstrass $\wp$-function. Let $z \in \mathcal{O}_K/\mathfrak{m}\mathcal{O}_K$, seen as an element of $\mathbb{C}/\Lambda$. Then we define the following map:

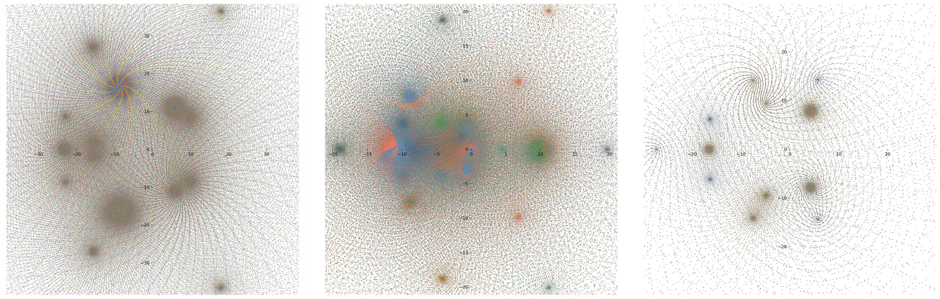
$$\eta_{K,\mathfrak{m},A} : \mathcal{O}_K/\mathfrak{m}\mathcal{O}_K \to \mathbb{C}, \qquad \eta_{K,\mathfrak{m},A}(z) = \sum_{j=0}^{d-1} \wp(A^j z).$$

We call $\eta_{K,\mathfrak{m},A}(z)$ a *ray class field period* (RCFP) of modulus $\mathfrak{m}$ and generator A. Additionally, we call $\mathrm{img}(\eta_{K,\mathfrak{m},A})$ the *ray class field period plot* of modulus $\mathfrak{m}$ and generator A.

Examples of RCFP plots are shown in Figure 4. For simplicity, we use $K = \mathbb{Q}(\sqrt{-7})$ and moduli $\mathfrak{m} = (m)$ for $m \in \mathbb{Z}$.

Several distinctive patterns emerge in these images. In particular, every example has areas where points accumulate more densely. Additionally, when $m = p^a$ and $d = p$, these accumulation areas have well-defined spirals. I provide more plots and discussions in [Pla23, §4].

While these RCFP plots have interesting patterns, we run into problems when trying to study them mathematically. In particular, note that there exists some rational function $f_A(x) \in K[x]$ such that $\wp(Az) = f_A(\wp(z))$. The functions f_A are determined by *division polynomials*, which are defined recursively on A and are dependent on the specific elliptic curve under consideration.



(A) $\mathfrak{m} = (2401)$, $d = 7$ (B) $m = 1265$, $d = 3$ (C) $m = 625$, $d = 5$

Figure 4. RCFP plots for $K = \mathbb{Q}(\sqrt{-7})$, modulus (m), and A of order d

The study of division polynomials for general A (and not just integers) is a rather underdeveloped area of mathematics, though algorithms do exist [Sat04, Küç15]. However, in order to understand RCFPs, we need to study the arithmetic dynamics of the rational functions f_A, which is a difficult and under-researched problem.

3. Future Work

I have several avenues of exploration in mind for my future research. First, there remain several unanswered questions about Gaussian periods, general cyclic supercharacter theories, and ray class field periods. I state many of these open questions in [Pla23, §5.1], but I highlight a few of them here.

- What are the unexplained mechanisms behind Gaussian periods for n and ω outside of the conditions described in Theorem 5?
- Can the geometry of $\operatorname{img}(g_d)$ be described succinctly when d is not a prime power? On this note, are these images the traces of some subgroup of $\mathrm{SU}(d)$?
- Can the rational functions f_A feasibly be studied in a general way? If so, can the arithmetic dynamics of f_A be described?

Finally, I would also like to mention that I have begun a new research project. This project is still in its beginning stages, but it focuses on explicit and computational aspects of automorphic forms on unitary groups. While this has connections to my previous research, it also has a completely different focus. However, I believe this project will broaden my research contributions in mathematics.

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