

Coherent states, beam splitters and photons

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1. Each mode of the electromagnetic (radiation) field with frequency ω is described mathematically by a 1D harmonic oscillator with frequency ω . We know one nice basis to describe this system, $\{|n\rangle\}$, the set of number states, eigenstates of the Hamiltonian for that one mode

$$\hat{H}|n\rangle = \hbar\omega(n + 1/2)|n\rangle.$$

2. But there is another set of states (not a basis!), which are very useful in practice, the so-called coherent states. For reasons soon to become clear, these states are sometimes called “quasi-classical” states. They can be defined as eigenstates of the lowering (or annihilation) operator \hat{a} . Since that operator is not hermitian, its eigenvalues do not have to be real. So let’s solve the equation

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle,$$

where α is a complex number, the eigenvalue of \hat{a} .

3. Expanding $|\alpha\rangle$ in the number states basis as

$$|\alpha\rangle = \sum_{n=0}^{\infty} a_n |n\rangle,$$

and substituting this in the eigenvalue equation gives

$$a_n \sqrt{n} = \alpha a_{n-1}.$$

for all $n > 0$. We can use this recursive relation to express all coefficients in terms of a_0 :

$$a_1 = \alpha a_0; \quad a_2 = \alpha^2 a_0 / \sqrt{2}; \quad a_3 = \alpha^3 a_0 / \sqrt{3!} \dots$$

That is, we find

$$a_n = \frac{\alpha^n}{\sqrt{n!}} a_0.$$

Now the state $|\alpha\rangle$ should be normalized:

$$1 = \sum_n |a_n|^2 = \sum_n \frac{(|\alpha|^2)^n}{n!} |a_0|^2 = \exp(|\alpha|^2) |a_0|^2,$$

and so we can choose

$$a_0 = \exp(-|\alpha|^2/2).$$

And thus we finally arrive at

$$|\alpha\rangle = \exp(-|\alpha|^2/2) \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

4. The coherent state evolves in time as

$$|\alpha\rangle_t = \exp(-|\alpha|^2/2) \sum_n \exp(-i(n+1/2)\omega t) \frac{\alpha^n}{\sqrt{n!}} |n\rangle = \exp(-i\omega t/2) |\alpha \exp(-i\omega t)\rangle.$$

That is, the coherent state stays a coherent state, but its eigenvalue, $\alpha(t) = \alpha \exp(-i\omega t)$, evolves in time.

5. Note the different meaning of the two phase factors appearing here: to measure the prefactor, $\exp(-i\omega t/2)$, we would need to create superpositions of different coherent states, with different prefactors. Superpositions of coherent states are *nonclassical* states. The phase of the amplitude $\alpha(t)$, on the other hand, can be measured by using just coherent states, by using beamsplitters, see below.

6. We like coherent states because the expectation values of position and momentum evolve in time in a nice, classical way (both oscillate at frequency ω):

$$\langle x \rangle_t = \langle \alpha(t) | \hat{x} | \alpha(t) \rangle = A \cos(\omega t - \phi)$$

and

$$\langle p \rangle_t = \langle \alpha(t) | \hat{p} | \alpha(t) \rangle = -m\omega A \sin(\omega t - \phi) = m d\langle x \rangle / dt,$$

where the amplitude of the oscillation is

$$A = \sqrt{\frac{2\hbar}{m\omega}} |\alpha|,$$

and $\alpha \equiv |\alpha| \exp(i\phi)$. All this follows from the expectation values of a and a^\dagger in a coherent state:

$$\langle \alpha(t) | \hat{a} | \alpha(t) \rangle = \alpha(t),$$

and

$$\langle \alpha(t) | \hat{a}^\dagger | \alpha(t) \rangle = (\alpha(t))^*.$$

7. For the same reason, the expectation values of the electric and magnetic field operators are identical to the classical solutions of Maxwell's equations, as both are just linear combinations of creation and annihilation operators, e.g.,

$$\hat{\vec{E}} = i \sum_{\vec{k}, \lambda} \sqrt{\frac{\hbar \omega}{2 \epsilon_0 V}} \left[\hat{a} \vec{\epsilon}_\lambda \exp(i \vec{k} \cdot \vec{r}) - \hat{a}^\dagger \vec{\epsilon}_\lambda^* \exp(-i \vec{k} \cdot \vec{r}) \right],$$

if we have quantized the field in terms of plane waves inside a (fictitious) cube of volume V , and $\omega = c|\vec{k}|$.

8. If we measure the number of photons in a coherent state with amplitude α , we get a Poissonian distribution:

$$P(n) = |a_n|^2 = \exp(-|\alpha|^2) \frac{(|\alpha|^2)^n}{n!},$$

with $|\alpha|^2$ the average number of photons.

9. Inside a laser we expect to find a Poissonian distribution for photon number, as well as for the number of photons that leaks out of the laser cavity in a fixed amount of time. A simple estimate for the average number of photons inside a HeNe laser can be obtained as follows: A typical output power of about 10mW at a visible wavelength, corresponds to a rate of

$$R \approx \frac{10^{-2} \text{Js}^{-1}}{3 \times 10^{-19} \text{J}} \approx 3 \times 10^{16} \text{photons/sec.}$$

An output mirror with transmission probability of 1%, and a laser cavity of length $L \approx 15\text{cm}$, implies an output rate per photon of

$$R_1 \approx 1\% \times \frac{c}{2L} \approx 10^7 \text{s}^{-1},$$

so that the average number of photons inside the laser cavity is

$$N = \frac{R}{R_1} \approx 3 \times 10^9.$$

10. Just like we have

$$\hat{I} = \sum_n |n\rangle \langle n|,$$

we may get something similar when calculating

$$\int d^2\alpha |\alpha\rangle \langle \alpha|,$$

where the integral is over the 2D complex plane: $d^2\alpha = d\text{Re}(\alpha)d\text{Im}(\alpha)$. Substituting the number-state expansion for $|\alpha\rangle$, and then switching to polar coordinates in the complex plane, $\alpha = r \exp(i\phi)$, gives

$$\int d^2\alpha |\alpha\rangle\langle\alpha| = \sum_n \sum_m \int_0^{2\pi} d\phi \int_0^\infty dr r \frac{r^{n+m} \exp(-r^2) \exp(i(n-m)\phi)}{\sqrt{n!m!}} |n\rangle\langle m|.$$

The integral over ϕ is easy, and we get:

$$\int d^2\alpha |\alpha\rangle\langle\alpha| = \sum_n 2\pi \int_0^\infty dr \frac{r^{2n+1} \exp(-r^2)}{n!} |n\rangle\langle n| = \pi \sum_n |n\rangle\langle n| = \pi \hat{I}.$$

This can be used to write any (pure) state as a linear superposition of coherent states,

$$|\psi\rangle = \hat{I}|\psi\rangle = \pi^{-1} \int d^2\alpha |\alpha\rangle\langle\alpha|\psi\rangle.$$

11. No two coherent states are orthogonal:

$$\langle\alpha|\beta\rangle = \exp(-|\alpha|^2/2 - |\beta|^2/2) \sum_n \sum_m \frac{(\alpha^*)^n \beta^m}{\sqrt{n!m!}} \langle n|m\rangle = \exp(-|\alpha|^2/2 - |\beta|^2/2 + \alpha^*\beta),$$

so that

$$|\langle\alpha|\beta\rangle|^2 = \exp(-|\alpha - \beta|^2).$$

12. Now consider the following mixed state:

$$\hat{\rho} = \frac{1}{2\pi} \int_0^{2\pi} d\phi |\alpha \exp(i\phi)\rangle\langle\alpha \exp(i\phi)|.$$

We can view this as a mixture of coherent states with random phase and fixed amplitude $|\alpha|$. We can perform the integration over ϕ , if we first expand the coherent states in number states. The result is that we can rewrite the same mixed state as

$$\hat{\rho} = \exp(-|\alpha|^2) \sum_n \frac{|\alpha|^{2n}}{n!} |n\rangle\langle n|.$$

That is, the mixture of coherent states with random phase but fixed amplitude is equivalent to a mixture of number states with a Poissonian distribution, with average number of photons equal to $|\alpha|^2$.

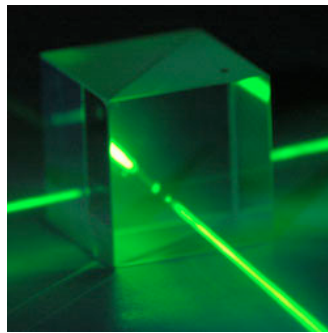
13. The mixed state $\hat{\rho}$ is a very good description of the state of the field inside a laser cavity. Physically, this can be explained by a hand-waving argument as follows: even if we expect a coherent state to be generated inside the cavity, the phase of that coherent state amplitude is not predictable at all. Namely, the process starts by a spontaneous emission event that happens to deposit a photon in just the right mode (the laser

mode). If subsequent stimulated emissions happen before the photons leak out of the cavity, then the process of lasing starts. However, the time at which this process starts is certainly not determined to an accuracy of $1/\omega$. On that short time scale, then, the start of the lasing process, and the buildup of the coherent state is random. So, whereas we do know the coherent-state amplitude will evolve in time as $\alpha(t) = \alpha \exp(-i\omega t)$, we do not know in advance what the initial phase is of α .

14. But a laser is very “phase-coherent” . . . what does that mean then? Answer: if you take the output of a laser and split it on a 50/50 beamsplitter (see below), then each of the two parts will have the *same*, albeit unknown, phase. Those two parts then will display perfect interference (see below).

Alternatively, if you take the outputs from a laser at two different times, the phases will still be (almost) the same. I write “almost”, because the phase correlation does decay at some characteristic rate. The coherence time of a laser is much longer than that of a lightbulb. For example, for a HeNe laser it is about 10ns, for a lightbulb about a femtosecond (and for LEDs about 10 fs).

15. Classically, a 50/50 beamsplitter splits the intensity of an incoming beam in two. Quantum-mechanically, it will not split each photon in two, but it will transmit or reflect each photon with 50% probability (see picture of an actual beamsplitter).



16. More precisely, a beamsplitter contains two input ports and two output ports. Thus, consider two classical fields, with the same polarization and same frequency, entering the two input ports of a beamsplitter. Suppose the complex amplitudes of the (classical) fields are α_1 and α_2 . The beamsplitter produces two output fields with the same polarizations and frequencies, but with different amplitudes:

$$\alpha_3 = \frac{\alpha_1 + i\alpha_2}{\sqrt{2}}$$

$$\alpha_4 = \frac{i\alpha_1 + \alpha_2}{\sqrt{2}}.$$

The factor i arises because a reflection adds a phase of π .

17. Examples: (i) if there is only one nonzero input field, then the output contains equally intense light beams: $|\alpha_3| = |\alpha_4|$, (ii) if the two input beams are out of phase, then in one output we'll have destructive interference, in the other constructive, (iii) if the input beams are in phase, then destructive and constructive interference occur in the opposite output ports (**P**).
18. In the simplest picture, a quantum beamsplitter acts in the same way on annihilation operators of the two input and two output beams:

$$\begin{aligned}\hat{a}_3 &= \frac{\hat{a}_1 + i\hat{a}_2}{\sqrt{2}} \\ \hat{a}_4 &= \frac{i\hat{a}_1 + \hat{a}_2}{\sqrt{2}}.\end{aligned}$$

The inverse of this transformation is even easier to use:

$$\begin{aligned}\hat{a}_1 &= \frac{\hat{a}_3 - i\hat{a}_4}{\sqrt{2}} \\ \hat{a}_2 &= \frac{-i\hat{a}_3 + \hat{a}_4}{\sqrt{2}}.\end{aligned}$$

Namely, if we have an input state

$$|\psi\rangle_{\text{in}} = |n\rangle_1 |m\rangle_2,$$

then we construct the output state by first rewriting

$$|\psi\rangle_{\text{in}} = \frac{(\hat{a}_1^+)^n}{\sqrt{n!}} \frac{(\hat{a}_2^+)^m}{\sqrt{m!}} |0\rangle_1 |0\rangle_2,$$

and then using the inverse transformation for the creation operators (i.e., we need to take the hermitian conjugate of the inverse transformations for the annihilation operators):

$$|\psi\rangle_{\text{out}} = \frac{(\frac{\hat{a}_3^+ + i\hat{a}_4^+}{\sqrt{2}})^n}{\sqrt{n!}} \frac{(\frac{i\hat{a}_3^+ + \hat{a}_4^+}{\sqrt{2}})^m}{\sqrt{m!}} |0\rangle_3 |0\rangle_4.$$

19. Example (i): one photon on a beam splitter, say, $|\psi\rangle_{\text{in}} = |0\rangle_1 |1\rangle_2$. The output state is

$$|\psi\rangle_{\text{out}} = \frac{i|1\rangle_3 |0\rangle_4 + |0\rangle_3 |1\rangle_4}{\sqrt{2}}.$$

If we detect light with two photodetectors, then we'll either get one click on detector 3, or one click in detector 4. Classically, in contrast, we would get half the intensity in each. An experiment with an appropriate nonclassical light source (not a laser!) and a 50/50 beamsplitter can thus “prove” the existence of photons: no two clicks are simultaneously observed (see Kimble, Dagenais, and Mandel, Phys Rev Lett **39**, 691 (1977).). (Note that the photoelectric effect does not require quantization of the field: quantization of the atoms is enough: think of perturbation theory with a sinusoidally varying perturbation (**P**)! Nor does the derivation of the blackbody radiation spectrum require the concept of photons: in fact, Planck didn't use the photon concept. Instead he assumed the “black body” consists of oscillators that take up or lose energies only in certain discrete amounts.).

20. Example (ii): we take the output from the previous item, and recombine two paths onto a second beamsplitter. Then, if we assume we add an extra phase ϕ shift to the

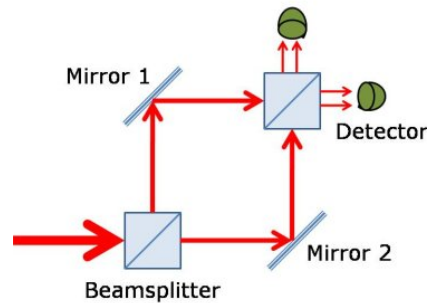


FIG. 1: A so-called *Mach-Zehnder* interferometer.

photon (or field) in path '3' (say, the one passing mirror 1), then we get as input to the second beamsplitter this state (up to an irrelevant overall phase factor):

$$|\psi\rangle_{\text{in}} = \frac{i \exp(i\phi)|1\rangle_3|0\rangle_4 + |0\rangle_3|1\rangle_4}{\sqrt{2}}.$$

This yields an output state (**P**!)

$$|\psi\rangle_{\text{out}} = -\sin(\phi/2)|0\rangle_5|1\rangle_6 + i \cos(\phi/2)|1\rangle_5|0\rangle_6.$$

The probability to detect a photon in detector 5 varies sinusoidally with ϕ : $P_5 = \sin^2(\phi/2) = (1 - \cos \phi)/2$.

21. In the Mach-Zehnder interferometer there is only one photon: *the photon interferes with itself*. More precisely, the interference is between the two different paths the photon can take to detector 5.

22. Example (iii): two coherent states as input. Since the input states are eigenstates of the input annihilation operators, the beamsplitter transformation will lead to eigenstates of the output annihilation operators, i.e., coherent states. The amplitudes are related just as the classical amplitudes. (And that's why you can't use coherent states to prove the existence of photons this way!).

There is interference between coherent states. That interference can be understood classically, but, quantum-mechanically, is still between the different paths each photon in the coherent state can take.

23. Example (iv): one photon in each input port: $|\psi\rangle_{\text{in}} = |1\rangle_1|1\rangle_2$. The output state is (**P**)

$$|\psi\rangle_{\text{out}} = \frac{-i|2\rangle_3|0\rangle_4 - i|0\rangle_3|2\rangle_4}{\sqrt{2}}.$$

That is, surprisingly, the two photons always exit in the same (albeit undetermined) output port, never one in each! This is called the Hong-Ou-Mandel effect (Hong, Ou, and Mandel, Phys. Rev. Lett. **59**, 2044 (1987)). This effect reflects the bosonic character of photons (2 fermions on a beamsplitter would never end up in the same state!).

24. Example (v): thermal light in one or both input ports...that sounds like a homework question.

25. Example (vi): the Elitzur-Vaidman paradox. Suppose we have a very sensitive device that either contains a mirror whose recoil, upon reflecting just a single photon, will induce an explosion, or does not contain such a mirror, so that any photon will just propagate through without doing anything harmful. Now we have such a device and would like to know if it would explode or not, without actually exploding it. Can we achieve this, at least with some nonzero probability? Classically, no, but using the Mach-Zehnder interferometer, yes! Namely, put the device in one arm of the interferometer, say arm "3", after having made sure that a setting of $\phi = 0$ sends the photon always to detector 6. Now if there is no mirror in the device, then the photon will still always come out in output 6. If there is a mirror, then we either get an explosion,

with 50% probability, and with 50% probability we get a photon entering the second beamsplitter in the state $|0\rangle_3|1\rangle_4$. That state leads to the photon being detected in output 5 with 50% probability. Thus, if we detect a photon in mode 5 we will know the device would have exploded, even though it actually did not explode. This occurs with 25% probability.

26. Example (vii): suppose someone gives us either a coherent state $|\alpha\rangle$, with some fixed and known amplitude α , or a coherent state $|-\alpha\rangle$. These states are not orthogonal and so cannot be perfectly distinguished. There is, nevertheless, a measurement that sometimes succeeds in unambiguously identifying the state. Namely, put the unknown state on a 50/50 beamsplitter, in input port 1, and combine it with a coherent state $|i\alpha\rangle$ in the other input port, 2. Then we either get $\alpha_3 = 0$ and $\alpha_4 = \sqrt{2}i\alpha$ (if $\alpha_1 = \alpha$) or $\alpha_3 = -\sqrt{2}i\alpha$ and $\alpha_4 = 0$ (if $\alpha_1 = -\alpha$). If we then detect a photon in output 3 (4), the state must have been $|-\alpha\rangle$ ($|\alpha\rangle$), and if we do not detect a photon at all, we have learned nothing.

Note: this measurement really has 3 different outcomes that tell us different things about the input state, which is chosen from a 2-dimensional Hilbert space: thus, this measurement is an example of a more general sort of measurement than the ideal von Neumann measurement we have considered so far. More about this later!

27. Example (viii): Let us do a Hong-Ou-Mandel experiment with 2 input photons in *mixed* states. A mixed state for a single photon, means this: we do know we have exactly one photon, but we do not know in which mode that photon is. The density matrix is hermitian, and so can be diagonalized. In its diagonalized form the state will look like

$$\rho_1 = \sum_i p_i |1\rangle_i \langle 1|_i,$$

with $p_i > 0$ and $\sum_i p_i = 1$, and the index i refers to a set of orthogonal modes (namely, the states $\{|1\rangle_i\}$ are the eigenstates of ρ_1 with eigenvalues $\{p_i\}$).

Now we have 2 such input states, and those states are not necessarily diagonal in the same basis (after we have rotated the \vec{k}_2 vectors to align with \vec{k}_1 : so think of the indices i and j as referring only to frequency and polarization) . Let's write the other input state (entering input port 2) as

$$\rho_2 = \sum_j q_j |1\rangle_j \langle 1|_j,$$

where $\{|1\rangle_j\}$ is a different set of single-photon states. In order to calculate how the two photons interfere, we rewrite ρ_2 in terms of basis $\{|1\rangle_i\}$ (the other way works too).

First consider what happens when we put two pure states in the inputs, $|1\rangle_i$ into 1, and $|1\rangle_j$ into 2. We write $|1\rangle_j = \alpha|1\rangle_i + \beta|1\rangle_\perp$, where the latter notation indicates all modes orthogonal to mode i . That's sufficient for our purposes, as that part will not interfere with $|1\rangle_i$ in the output. The probability to find 1 photon in each output detector, P_{11} , in fact can arise only from that orthogonal part (thanks to the HOMi effect). That probability is, in fact,

$$P_{11} = |\beta|^2/2 = (1 - |\alpha|^2)/2.$$

Now averaging over all possibilities we get

$$P_{11} = \sum_i \sum_j p_i q_j (1 - |\langle 1_i | 1_j \rangle|^2)/2.$$

We can rewrite this in terms of

$$\text{Tr}(\rho_1 \rho_2) = \sum_i \sum_j p_i q_j |\langle 1_i | 1_j \rangle|^2,$$

as

$$P_{11} = 1/2 - \text{Tr}(\rho_1 \rho_2)/2.$$

28. Example (ix): could one measure the purity of a quantum state, $\text{Tr}\rho^2$? Answer: yes, provided we know how to make prepare a system identically: then, as the previous item shows, there is a measurement that determines the purity directly, namely $1 - 2P_{11}$.