Ray Optics via Fermat's Principle

Now that you've seen ray optics as originating from Huygens' construction (i.e. by thinking about the propagation of wavefronts with rays following the radial directions to the fronts [Lec20]) - you might find it interesting to learn that there is a geometrically ***equivalent*** perspective that can be stated very simply:

Fermat's Principle: *The geometric path that a ray of light takes between two points is the path which takes it the least time to travel along.*

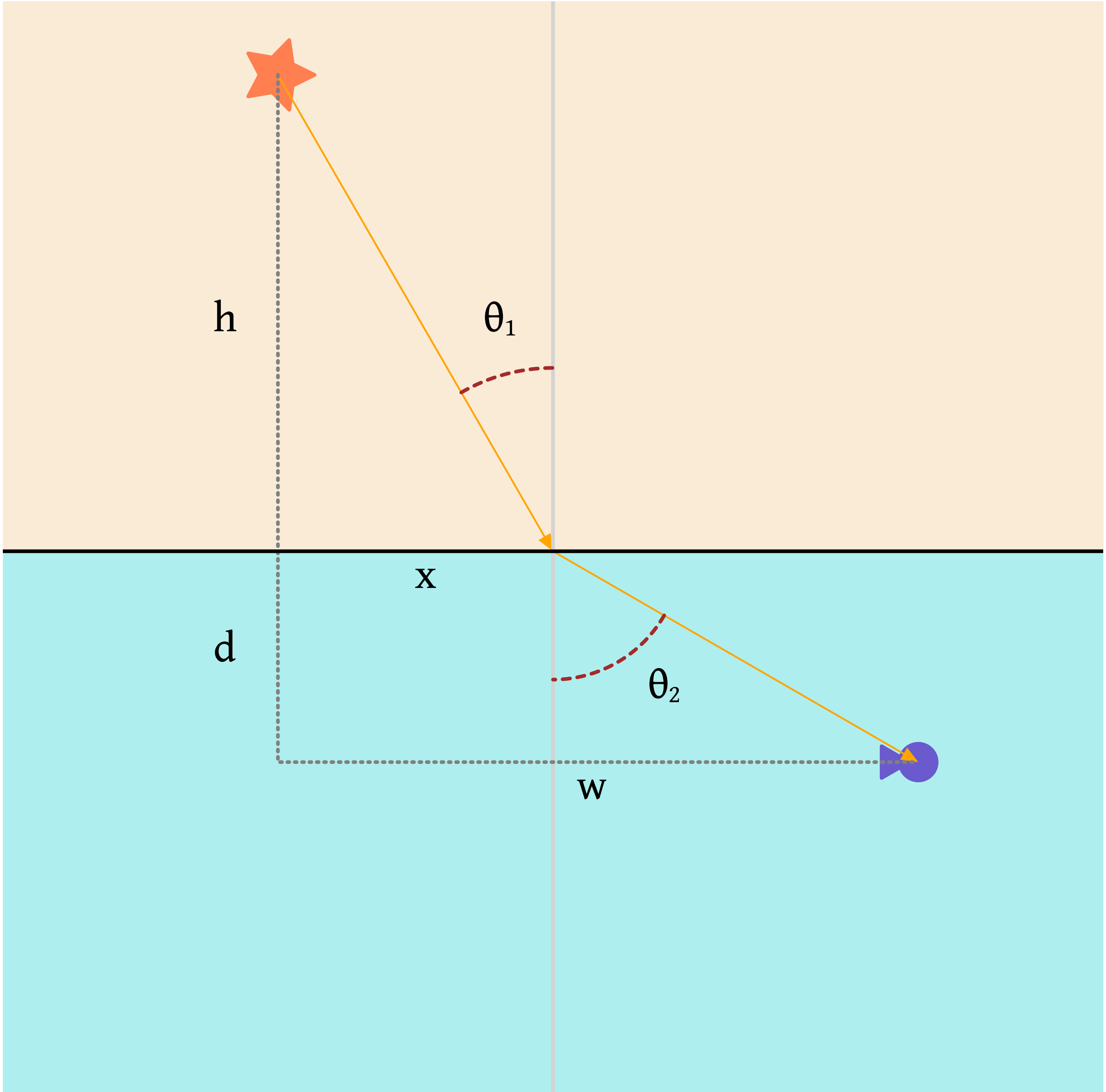
This is an example of an *action principle* - a statement about physics that is phrased in terms of an *optimization* (minimization/maximization - i.e. extremization) of a global property instead of in terms of local coordinates and/or rates of change. The physics you've seen so far has mostly presented physical laws as the latter kind of local statements (e.g. the Huygens' construction describes how small parts of wavefronts act as local point sources that determine the propagation of the front nearby) - but it turns out that most of it can also be rephrased in terms of action principles! Such global approaches provide powerful and profound analytic tools that lie at the root of most modern physics.

(an important caveat: certain non-conservative systems such as those with friction or drag cannot be rephrased this way)

The math required to understand that translation in general is beyond a reasonable scope for this course, but the relevant action principle for ray optics **is** simple enough we can see that it works and spookily reproduces now-familiar results - and so get a hint at things to come:

Snell's Law

Consider rays of light which leave a source at height h above the surface of a pond, refract through the surface, and are incident on a fish that's at a depth d - traveling an overall horizontal distance w:



The total travel time of the ray along a particular path can be written in terms of the geometry and the (unknown) point of incidence:

$$T = \int dt = \int \frac{dt}{dl} dl = \frac{l_1}{v_1} + \frac{l_2}{v_2} = \frac{\sqrt{x^2 + h^2}}{v_1} + \frac{\sqrt{(w - x)^2 + d^2}}{v_2}$$

where the light velocities in each region are related to the indices of refraction and the speed of light:

$$v_1 = c/n_1$$

$$v_2 = c/n_2$$

Fermat's principle says that the actual ray of light that goes from the sun to the fish will pass through the surface at the point which leads to a minimal travel time. To ask for such a point we can extremize T, demanding that the point of incidence be such that:

$$\frac{dT}{dx} = \frac{n_1 x}{c\sqrt{x^2 + h^2}} - \frac{n_2(w - x)}{c\sqrt{(w - x)^2 + d^2}} = 0$$

Notice, now, that in terms of the incident and refracted angles we can write things trigonometrically and get a much simpler form:

$$\frac{dT}{dx} = \frac{n_1 \sin\theta_1}{c} - \frac{n_2 \sin\theta_2}{c} = 0$$

which is just Snell's law! Rearranging:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

That is - the point of incidence on the surface that minimizes the total travel time is **exactly** the point that geometrically obeys Snell's law!

Law of Reflection

Challenge: Can you use a similar argument to find that the *Law of Reflection* ($\theta_1 = \theta_2$ for a reflection) also obeys Fermat's principle? i.e. that a ray which leaves a source, reflects off a surface, and is incident on some target will strike the surface at a point which minimizes the time it takes to reach the target via reflection, and such a point is one where $\sin\theta_1 = \sin\theta_2$ - implying the law.

0. Draw a diagram:

1. Write the total travel time $T = \ ?$

Use only the point of incidence on the mirror x and known distances like the vertical and horizontal distances between the source and target, and/or the height of each from the mirror - **don't** use any angles *yet*. Remember that we'll want these to result in nice trigonometric ($\sin$) quantities when we differentiate in the next part, so you should think long and hard about how you choose to parameterize the geometry - otherwise you might need a couple of back-and-forth tries between this part and the next to work things into a tidy form, or will need to do a lot of trig to show the resulting expression is what we expect to find.

2. Differentiate w.r.t. the point of reflection x; $\frac{dT}{dx} = \ ?$

3. Extremize and simplify the resulting condition

Now, using your expression from (2) demand that $\frac{dT}{dx} = 0$ by setting it so - then do some algebra

References

[1]: Feynman, Leighton, Sands; *The Feynman Lectures on Physics*, Vol. 1. Ch. 26 accessed 2024-03-04