PHYS 611, Fall 2025, Homework set #1

1 Damped oscillator

Here I'm asking you to retrace the steps of **Fitz. 4.5**, but with different names for the variables. The goal is to solve the equation of motion of a damped harmonic oscillator,

$$m\ddot{x} = F = -m\omega_0^2 x - m\gamma\dot{x} \tag{1}$$

Here, $\omega_0 > 0$ is the natural frequency of oscillation in the absence of damping, and $\gamma > 0$ is the damping rate.

(a) (5 points) Make an *ansatz* with a complex-exponential time dependence,

$$x(t) = \Delta\, \mathrm{e}^{\mathrm{i}\omega t}. \tag{2}$$

where ω is unknown. Derive the condition for ω that must be satisfied if a solution with $\Delta \neq 0$ is to exist.

(b) (5 points) Find the solutions of the equation derived in **(a)**. How must the positive quantities γ and ω_0 be related in order for the solutions ω to be *purely imaginary*?
Hint: This should correspond to the inequality for the overdamped case, above (4.44) in **Fitz**, if you use $\nu = \gamma/2$ (but you're supposed to write it in my notation).

(c) (5 points) Under the conditions of part **(b)**, write down the general solution $x(t)$ in terms of two undetermined amplitudes a_+ and a_-. The form should be

$$x(t) = a_+\, \mathrm{e}^{-\kappa_+ t} + a_-\, \mathrm{e}^{-\kappa_- t} \tag{3}$$

Write down κ_+ and κ_-.
Hint: How are $\kappa_\pm$ related to the two solutions for ω? They should involve the real-valued parameter

$$\tilde{\omega} \equiv \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

(d) (10 points) Relate the unknowns $a_\pm$ to the initial position $x_0 = x(0)$ and velocity $v_0 = v(0)$ at $t = 0$. Then use the results to eliminate $a_\pm$ from (3), so $x(t)$ is expressed in terms of x_0 and v_0.

(e) (5 points) While holding x_0, v_0 and t constant, take the limit $\tilde{\omega} \to 0$ of the result in **(d)**.
Hint: According to the fundamental definition of the derivative,

$$\lim_{\tilde{\omega}\to 0} \frac{\mathrm{e}^{\tilde{\omega}t} - \mathrm{e}^{-\tilde{\omega}t}}{\tilde{\omega}} = 2\left[\frac{\mathrm{d}}{\mathrm{d}\tilde{\omega}}\mathrm{e}^{\tilde{\omega}t}\right]_{\tilde{\omega}=0}$$

(you can also see this from a Taylor expansion, but that's optional information). Your result should be writable in the form of **Fitz. (4.43)**.

2 Why the harmonic oscillator is special

Consider the one-dimensional motion of a point mass m at total energy $E > 0$. Its potential energy is

$$U(x) = A\,|x|^n$$

Here, $A > 0$ is real, and n is a positive integer.

(a) **(5 points)** Find the larger of the two turning points, x_{turn}, in terms of A and the total energy E.

(b) **(10 points)** Find the time τ it takes to go from the smaller to the larger turning point, i.e. from $-x_{\text{turn}}$ to x_{turn}.
Hint: Use equation **(4.9)** in Fitzpatrick. Use a substitution of variables to make the integral dimensionless:

$$u = \frac{x}{x_{\text{turn}}}$$

You *don't* need to evaluate the integral. You should find $\tau \propto E^{1/n-1/2}$.

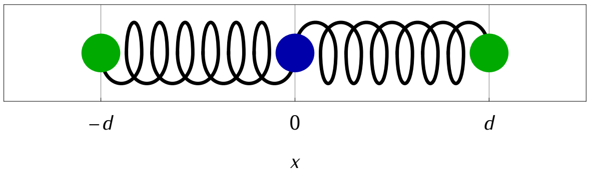
(c) **(5 points)** Calculate

$$\frac{1}{\tau}\frac{\mathrm{d}\tau}{\mathrm{d}E}$$

Note: the dimensionless integral that was left over in part **(b)** cancels out in the answer. The result is the rate at which the oscillation period $T = 2\tau$ changes if the total energy is varied, in relation to the period itself. You should find that there is exactly one value of n for which the answer is zero. This explains why the harmonic oscillator is extremely special in physics: it's the *only* system for which the period stays the same, no matter how much energy you put into it.

3 Linear two-spring oscillator

You'll find a discussion of this system in Fitzpatrick section 12.8, "Triatomic molecule". But the calculation in the text uses more formal matrix algebra, so I'm not assigning that as reading. I want you to do the calculation slightly differently. You can use Fitzpatrick to check that you got the correct frequencies in part **(d)**. But to get these frequencies, *follow the instructions in this problem* and *not Fitzpatrick*.



Three point masses are constrained to move along the x axis. The mass in the middle is M, the outer masses are m. The masses are connected by ideal springs of spring constant k and equilibrium length d. The equilibrium configuration is shown in the figure.

In an inertial reference frame, the coordinates of the outer masses m are x_1 and x_2; the coordinate of the mass M is X. There are no external forces on the system, so it's center of mass does not accelerate.

(a) **(2 points)** Assume that the coordinate system has been chosen such that the center of mass is fixed at $x_{\mathrm{COM}} = 0$. Express X in terms of x_1 and x_2. The equation must hold for any values of x_1, x_2 – not just for the equilibrium configuration shown in the sketch.

(b) **(2 points)** Write down the equations of motion (Newton's Second Law) for x_1 and x_2. Eliminate X from the forces using the result of **(a)**.

(c) **(2 points)** Find the equilibrium values of x_1 and x_2 by setting $\ddot{x}_1 = \ddot{x}_2 = 0$ in your answer to **(b)**. Confirm that the result is $x_{e1} = -d$, $x_{e2} = d$.

(d) **(10 points)** Use the *ansatz* $x_n(t) = x_{en} + a_n e^{i\omega t}$ $(n = 1, 2)$. The system of equations in **(b)** has a solution if you try the following two possibilities: Set $a_1 = a_2$ and find the resulting value of ω^2; we'll call it ω_+^2. Then set $a_1 = -a_2$ and find the corresponding value of ω^2; we'll call it ω_-^2.

(e) **(10 points)** Assume that the initial conditions at $t = 0$ are

$$x_1(0) = -d, \quad x_2(0) = d, \quad \dot{x}_1(0) = 0, \quad \dot{x}_2(0) = v$$

where $v > 0$. Find $x_1(t)$ and $x_2(t)$ for $t > 0$ in terms of v.

(f) **(2 points)** The central mass M may also be moving if the COM is not at X. Write down the position $X(t)$ of the central atom in terms of v, assuming the conditions in **(e)**. Unlike $x_1$and x_2, it will be a simple harmonic oscillation.

(g) **(2 points)** The distance between the outer masses is $\delta x = x_2 - x_1$. For the initial conditions in **(e)**, δx also performs a simple harmonic oscillation. Is its frequency larger or smaller than that of $X(t)$?

(h) **(5 points)** We'll choose the units such that $d = 1$, $\omega_- = 2$, $m/M = 1/20$, $v = 1$. Make a plot of x_1, X and x_2 for $0 \le t \le 40\pi$, for the initial conditions in **(e)**.
Hint: two of the masses show a periodic exchange of energy (beats).

(i) **(5 points)** Now we'll prove that δx and X *always* perform simple harmonic oscillations, no matter what the initial conditions are. To do that, use the equations of motion from part **(b)** to find the equations for $\delta\ddot{x}$ and $\ddot{X}$. Express equation for $\delta\ddot{x}$ in terms of δx and no other coordinates. Likewise, express the equation for $\ddot{X}$ in terms only of X. This means both equations should have the form of a simple harmonic oscillator.