PHYS 391 – Poisson Distribution

Derivation from probability for rare events

This follows the arguments I was presenting in class. In the following we can use ν and n to indicate the probability that ν events will occur in n trials (or n individual nuclei in the case of radioactive decay). Equivalently, we could make the substitution of $n \to t$ and think of this as describing the probability that some ν events will occur within a given time t. Both cases are governed by the Poisson distribution. The following will use the notation for probability where $P_n(\nu)$ is the probability of observing ν events out of n trials. Because n is very large, we can consider this a continuous variable.

Starting with the probability of observing one event in a small sample Δn,

$$P_{\Delta n}(1) = \lambda \Delta n, \tag{1}$$

with the assumption that Δn is small enough such that $P_{\Delta n}(2) \approx 0$, we can then write the probability for observing no events as the joint probability

$$\begin{aligned} P_{n+\Delta n}(0) &= P_n(0) P_{\Delta n}(0) \\ &= P_n(0)[1 - P_{\Delta n}(1)] \\ &= P_n(0)[1 - \lambda \Delta n]. \end{aligned}$$

Collecting terms leads to

$$P_{n+\Delta n}(0) - P_n(0) = -\lambda \Delta n P_n(0),$$

which in the limit of small Δn gives us the differential equation

$$\lim_{\Delta n \to 0} \frac{\Delta P_n(0)}{\Delta n} = \frac{dP_n(0)}{dn} = -\lambda P_n(0).$$

This differential equation has a well-known solution given by

$$P_n(0) = e^{-\lambda n}, \tag{2}$$

and we can identify the product λn as the mean number of events expected from n trials which we usually write as μ. In other words, $P_n(0) = e^{-\mu}$. In the notation typically used in the Binomial distribution, $\lambda = p$, or the probability of a given outcome occurring in one trial, which makes sense from the definition given in Equation 1.

To continue, we consider the probability of observing one event in a sample $n + \Delta n$, which can either occur by having one event in n and zero in Δn or vice versa:

$$\begin{aligned} P_{n+\Delta n}(1) &= P_n(1) P_{\Delta n}(0) + P_n(0) P_{\Delta n}(1) \\ &= P_n(1)(1 - \lambda \Delta n) + P_n(0) \lambda \Delta n. \end{aligned} \tag{3}$$

This can be rearranged as

$$P_{n+\Delta n}(1) - P_n(1) = -\lambda(P_n(1) - P_n(0))\Delta n,$$

or again in the limit of small Δn,

$$\frac{dP_n(1)}{dn} = -\lambda(P_n(1) - P_n(0)).$$

This recursive differential equation can be seen to be solved by $P_n(1) = \lambda n e^{-\lambda n}$ by explicitly taking the derivative with respect to n. Written in the more familiar form, this gives $P_n(1) = \mu e^{-\mu}$.

Equation 4 can be generalized to any number of observed events ν where there are two ways to achieve this outcome, either ν events in n trials followed by zero in Δn, or $\nu - 1$ events in n trials followed by one in Δn. Because the probability of two events in Δn is vanishingly small, we don't need to worry about any other terms. The equivalent of Equation 4 then can be written as

$$\begin{aligned} P_{n+\Delta n}(\nu) &= P_n(\nu)P_{\Delta n}(0) + P_n(\nu-1)P_{\Delta n}(1) \\ &= P_n(\nu)(1-\lambda\Delta n) + P_n(\mu-1)\lambda\Delta n. \end{aligned}$$

which leads to the general equation

$$\frac{dP_n(\nu)}{dn} = -\lambda(P_n(\nu) - P_n(\nu-1)). \tag{4}$$

The solution to this equation is the Poisson distribution

$$P_n(\nu) = \frac{n^\nu \lambda^\nu}{\nu!} e^{-n\lambda},$$

which gives the more familiar form with the replacement $\mu = n\lambda$.

Derivation from the Binomial distribution

Not surprisingly, the Poisson distribution can also be derived as a limiting case of the Binomial distribution, which can be written as

$$B_{n,p}(\nu) = \frac{n!}{\nu!(n-\nu)!}\, p^\nu (1-p)^{n-\nu}.$$

To show this, we need two results in the limit of large n and small p.

The first is to show that

$$(1-p)^{n-\nu} \approx e^{-np}. \tag{5}$$

This can be shown by taking the log of both sides and showing that they are approximately equal in the large n, small p limit:

$$\begin{aligned}
\ln[(1-p)^{n-\nu}] &= -np \\
(n-\nu)\ln(1-p) &= -np \\
(n-\nu)(-p) &= -np \;(\text{since}\, p \ll 1) \\
-np &= -np \;(\text{since}\, n \gg \nu)
\end{aligned}$$

The second is to show that

$$\frac{n!}{(n-\nu)!} \approx n^{\nu}. \tag{6}$$

This certainly makes sense, since for $n = 100, \nu = 2$, the value $100!/98! = 100 \times 99$ is very close to 100^2. To prove this we start with Stirlings approximation which says

$$\ln\left[\frac{n!}{(n-\nu)!}\right] = n\ln n - n - (n-\nu)\ln(n-\nu) + (n-\nu). \tag{7}$$

Because $n \gg \nu$, we can make the approximation that $\ln(n-\nu) = \ln n + \ln(1-\nu/n) = \ln n - \nu/n$. Making this replacement into Equation 7 gives

$$\begin{aligned}
\ln\left[\frac{n!}{(n-\nu)!}\right] &= n\ln n - n - (n-\nu)(\ln n - \nu/n) + (n-\nu) \\
&= \nu\ln n - \nu^2/n \\
&= \nu\ln n,
\end{aligned}$$

which confirms Equation 6.

So making these two replacements from Equation 5 and Equation 6 into the Binomial distribution gives

$$\begin{aligned}
B_{n,p}(\nu) &= \frac{n!}{\nu!(n-\nu)!}\, p^{\nu}(1-p)^{n-\nu} \\
&\approx n^{\nu}\frac{1}{\nu!}\, p^{\nu} e^{-np} \\
&\approx \frac{\mu^{\nu}}{\nu!}\, e^{-\mu},
\end{aligned}$$

where we have identified that $\mu = np$ from the original Binomial distribution.