PHYS 391 – Lab 3: Brownian Motion

Key Concepts

- Brownian Motion and Diffusion
- Video Analysis Techniques
- Statistical Description of Data
- Least-Squares Fitting

3.1 Introduction

The random motion of small particles when suspended in a liquid or gas has been recognized at least as long ago as 60 BC when the Roman Lucretius described this in the poem *On the Nature of Things* and used this as evidence for the existence of atoms. The name Brownian motion comes from the Scottish botanist Robert Brown, who made detailed observations of the motion of pollen in liquids under a microscope. In 1905, Einstein published a paper on Brownian motion (in addition to papers on special relativity and the photoelectric effect) providing a model for the phenomenon based on the kinetic theory of heat which tied together the Boltzmann constant, Avogadro's number, and gave some of the first hard evidence for the existence of atoms (in the modern sense of the word).

3.2 Goals of this Lab

This lab will analyze Brownian motion. By measuring the diffusion coefficient of micron-sized silicon spheres, a direct measurement of the Boltzmann constant will be inferred, thus testing the model of Brownian motion proposed by Einstein. You will hand in your own ipython notebook with markdown boxes for discussion prompted in the following sections.

3.3 Theory

3.3.1 A Statistical Description of a Random Walk

Our model for Brownian motion is the constant collisions of silicon micro-spheres with water molecules. In one dimension, each collision can be approximated as an impulse which causes the particle to either move left or right with equal probability. For a series of n random steps, the probability of observing k steps to the left (and $n-k$ steps to the right) is equivalent to the probability of flipping heads on a coin k times out of n trials. This is also sometimes called a Random Walk. The probability of observing k out of n discrete events is described by the Binomial distribution

$$\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k}$$

where p is the probability of a single discrete outcome and (1-p) is probability of the other discrete outcome. For an equally likely process ($p = 0.5$) like flipping a coin or a 1D random walk, this probability reduces to

$$\frac{n!}{k!(n-k)!}\frac{1}{2^n}$$

which in the limit of large n can be approximated very well by a Gaussian distribution with mean $\mu = n/2$ and $\sigma = \sqrt{n}/2$.

3.3.2 Relating Variance to the Diffusion Constant

If we want to think about the net displacement of our particle in one dimension $\Delta x = (x_2 - x_1)$ over some time interval $\Delta t = (t_2 - t_1)$, the model predicts this displacement will follow a Gaussian distribution with mean $\mu = 0$ and some characteristic width $\sigma_x^2 = 2D\Delta t$ where D is defined as the diffusion constant, which is simply a parameter which characterizes this motion for a given particle. The factor of 2 is included by convention. Even though in reality each collision imparts a varying amount of momentum to the object, the many collisions in any short time period can be viewed as an ensemble with an average value and variance, and the model of a random walk with a constant mean step size δ occurring every fixed time period τ still works well. Using the variance of the binomial distribution given above, one can then derive $D = \frac{\delta^2}{2\tau}$.

Observing Brownian motion, then, should lead to the following two results **which are to be verified in this lab.** The *average* displacement in any one dimension is $\overline{\Delta x} = 0$, which is to say on average the particle doesn't go anywhere, as it is equally probable to move in one direction or the other. The average squared displacement $\overline{(\Delta x)^2} = \sigma_x^2 = 2D\Delta t$ (see [1]) however is non-zero, which is to say that the particle does on average travel some *absolute* distance from where it started, and that distance, given by the average RMS displacement, increases with time as $\sqrt{2D\Delta t}$. The motion of any specific particle can not be predicted, but if you were to look at the average motion of a series of particles, you should observe this characteristic statistical behavior. This notion of the statistical characteristics of a large ensemble of particles is central to the entire premise of thermodynamics and statistical mechanics.

For more dimensions, we should analyze the displacement in r rather than the 1D displacement in x. Since the motion in each dimension is uncorrelated, and $r^2 = x^2 + y^2 + z^2$, we immediately arrive at $\overline{(\Delta r)^2} = 6D\Delta t$ in 3 dimensions, and $\overline{(\Delta r)^2} = 4D\Delta t$ in 2 dimensions.

3.3.3 Relating the Diffusion Constant to Thermal Energy

The diffusion constant D depends on the size and shape of the diffusing particle, plus the nature of the medium it is suspended in. By thinking about drag forces, Einstein derived the relationship

$$Df = k_B T$$

where f if the drag coefficient relating the drag force to the velocity through the fluid ($F_{\mathrm{d}rag} = fv$) and $k_B T$ is the usual Boltzmann factor from statistical mechanics. One might think of this expression as relating the thermal kinetic energy of the particles to the energy lost to diffusion through the medium (likely through collisions with it). The Stokes formula gives the drag coefficient for a sphere in terms of the viscosity of the fluid and the radius of the sphere, R, as

$$f = 6\pi\eta R$$

where η is the viscosity of the fluid at temperature T. At room temperature, water has a nominal viscosity of $\eta = 1.0 \times 10^{-3}$ Pa s in SI units[2]. In practice, then, measuring the diffusion of a particle of known size from Brownian motion allows for a direct measurement of the Boltzmann constant k_B.

3.3.4 Measure Long-term Motion of a Single Particle (instead)

To observe and quantitatively characterize Brownian motion in the lab, we would really like to measure the average RMS motion of many different particles as a function of time. That is to say, we want to measure a large number of random walks and calculate the RMS displacement *on average* since any one random walk may deviate significantly from the average. In practice, this approach is rather tedious and involves a huge amount of time and labor collecting data. Since the net displacement of a single particle over two different time intervals is independent, an equivalent prescription is to track a single particle for a long time and compare the size of the individual displacements over each time slice Δt. The use of a computer-controlled camera to take pictures at regular time intervals makes this technique particularly convenient.

[1] Remember, $\sigma_x^2 = \overline{(\Delta x)^2} - (\overline{\Delta x})^2$, but in this case the mean displacement $\overline{\Delta x} = 0$.

[2] The units are Pascal seconds, or pressure multiplied by time

3.4 Observing Brownian Motion

Your TA will show you a video captured with an OMAX microscope displaying a slide containing deionized water with silicon microspheres in suspension. The *diameter* of these microspheres will be posted on the lab website, along with the frame rate of the video. You can assume that the solution was at room temperature when the data was recorded. There should be enough spheres that you can easily see several in the microscope frame, but not so many that they can't be distinguished. The visible motion in the video is in the horizontal plane. Vertical diffusion will cause the small beads to move out of the focal plane, and this does happen, causing individual particles to disappear and reappear over time. The focal depth of our microscope is wide enough that this isn't a significant problem.

The general procedure for recording data is to start with a lower magnification (x10) to get things mostly centered before switching to the higher magnification (x40 or x100) that will be used. The microscope has an adjustable light along with a translational stage and a focus mechanism that has been used to find the beads and generate a reasonable contrast in the video.

Discuss with your TA and peers ideas for what kind of systematic effects may have impacted the video capture.

3.4.1 Video Capture

The video capture was performed using a command-line tool `imagesnap` that takes pictures automatically at a given interval, and then these images are merged into a movie using `tlassemble`. Several time-lapse movies with reasonable periods (data was taken at both $\Delta t = 1$ and 5 seconds) and magnifications were taken over a long enough time to get at least 100 frames. Note that the frame rate in the final movie is not necessarily the actual data sampling rate. In many cases, the movie was sped up so that the movie could be watched in a shorter amount of time. Your TA will assist you in analyzing one of the data videos. Make sure to check the resolution – it should be 640x480, which is the default camera resolution used by `imagesnap`.

3.4.2 Data Extraction

To do any meaningful statistical analysis, we need to measure the position of a particle undergoing Brownian motion. To convert the movie from the microscope into position measurements, we will use the `trackpy` package in python.

Your TA will take you through the provided Jupyter notebook with the basics of this process. Together as a group you will adjust the particle tracking parameters to identify the microspheres (and not just noise) and finally extract the trajectory of one (or a few) microspheres to analyze further into a text file. Once you have vectors of x and y positions (or Δx and Δy) your TA will save these data to a file and provide it to you so you can read it back for further analysis later.

Do make sure to include a description of the relevant parameter settings for the video you used, along with a figure of the final particle trajectory which you analyzed.

3.4.3 Calibration Slide

In order to determine the length scale of the movie taken by the camera, a calibration image needs to be taken of some object of known length. A video capture of a Motic calibration slide was also taken; the length scale is 10 microns / division. There isn't any information on the accuracy of this calibration target, and we will assume the uncertainty is negligible.

A Jupyter notebook is provided that demonstrates how to read in this calibration image and display it in `matplotlib`. The images are naturally recorded by pixels, but we need to convert each pixel to a physical location with a known length scale. Use the calibration image to find a calibration factor to convert pixels into a physically meaningful length. Because we are looking at small objects, microns (μm) are probably the most convenient length unit to use. It is best to take data that will allow you to check the x and y calibration independently, so you should have at least two points separated by a known distance in x, and two other points separated by a known distance in y.

Record the distances both in pixels (from Jupyter) and in actual length units. Make sure you have an estimate for the uncertainty in your calibration procedure. Check whether your calibration (best is to report this in units of microns/pixel) appears to be the same in both directions. Make sure to provide this information and any relevant discussion in your final Jupyter notebook.

3.5 Data Analysis

Everything up to this point you will do as a lab section, with your TA. From here on, everything can be done in python. Students may collaborate, but the discussion in your notebook should be unique. It is easiest if your analysis starts with the CSV file you exported above.

3.5.1 Data Calibration

The data you have collected is recorded as positions (in pixels) and time by the frame number. You will need to calibrate this data into a proper length, and also make sure to keep track of the Δt between frames from the time-lapse video capture settings.

Write down clearly what calibration parameters you are using for positions and how you determined these. Discuss how uncertain you think these length-scale calibrations might be, and explain whether you have used a common calibration parameter for x and y, or separate calibrations for each.

3.5.2 Dispersion

We want to quantitatively measure the dispersion relation for Brownian motion. Read your raw data into python, calibrate the data as necessary, and write a script which turns the N position values in x and y into $N-1$ displacement values where each displacement is $\Delta x_i = x_i - x_{i-1}$. A handy trick to do this without looping over all values is to make two copies of the same data, remove the first element from one and the last element from the other, then subtract the two arrays. Make displacement arrays for both x and y separately.

Make histograms of these arrays (Δx and Δy), and analyze this data to find the mean and standard deviation for each. For each direction, is the mean statistically consistent with zero? Is the standard deviation for Δx consistent with Δy? Use what we have learned in class to be as quantitative here as possible. In other words, how probable is it that the measured widths in x and y correspond to the same fundamental value?

From your data, evaluate your best measured value for the dispersion constant D (including uncertainty). Explain in detail how you arrived at this result from your experimental data (describing the combination procedure and error propagation, for instance), and discuss which uncertainties dominate the final result. Remember that your calibration scale uncertainty may also effect the uncertainty on D.

3.5.3 Boltzmann Constant

Using what we know from Stokes Law (given in the introduction) convert your measurement of D into a measurement of the Boltzmann constant k_B. To estimate the uncertainty on k_B, explicitly write down each value which is needed to compute k_B (with uncertainty) and explicitly write out the error propagation formula for δk_B. In your error analysis, include any reasonable estimates of systematic uncertainties which may be important (or argue why they are not important). Quantitatively compare your measured value to the accepted value of $k_B = 1.38 \times 10^{-23}$ J/K. You should quote how many standard deviations away your result is from the accepted value, and you should also indicate how probable that discrepancy is.

3.5.4 Time Evolution

We also want to empirically verify the time dependence predicted by the dispersion relation given above. In particular, we would like to see that the RMS displacement grows as $\sqrt{\Delta t}$. To avoid having to take more movies with different Δt intervals, instead we can analyze our data by making displacement measurements over longer time intervals. If instead of calculating $\Delta x_i = x_i - x_{i-1}$ we calculate $\Delta x_i = x_i - x_{i-2}$, we have effectively doubled the interval Δt.

Measure the RMS displacement as you did before, but now do this separately for several different Δt intervals (at least four). Put these values into a table and include the uncertainties on σ^2 in each case. Note that in principle these data for different Δt intervals are not strictly independent, but I suspect you won't notice the difference. You should be able to demonstrate a linear relationship between the variance of your displacement data and the time interval according to $\sigma^2 = 2D\Delta t$.

Make a scatter plot of your data and perform a linear fit to the data. How does the value of D found here (including an uncertainty) compare to the value of D that you measured in Section 3.5.2? Is there any evidence of deviations from the expected linear behavior? Make sure you are really plotting the correct things.

3.6 Final Thoughts

Brownian motion was one of the first phenomena which gave direct evidence for the atomic nature of matter. By combining measurements of the Boltzmann constant in Brownian motion with PV measurements using the ideal gas law, physicists were able for the first time to get a direct estimate of Avogadro's number. Even knowing the order of magnitude of Avogadro's number was a major achievement at the time, and opened the door to many modern concepts which all derived from the understanding of the atomic nature of matter.