Bifurcation effects in coupled Bose-Einstein condensates

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Bifurcation behavior of effective Hamiltonians is investigated for coupled systems of Bose-Einstein condensates. Phase-space structure mapped on the Bloch sphere shows a variety of Josephson-related behavior, systematically classified in the bifurcation analysis, leading to a phase diagram with prediction of some physical effects.

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When Bose-Einstein condensed (BEC) systems are coupled, dramatic phenomena occur, owing to macroscopic phase coherence. Effective second-quantized Hamiltonians are often used as simple models [1–3,5–7] to bring out the characteristic behavior of these systems. Very similar Hamiltonians are used [8–10] to model striking effects in vibrational spectra of polyatomic molecules, related to classical nonlinearity and chaos. This paper explores bifurcation phenomena and their classification as phases of BEC systems, and describes additional BEC effects, possibly accessible to experiment.

We start with a Hamiltonian for two states, e.g., two BEC traps or two vibrational degrees of freedom in a molecule:

$$\hat{H}=\omega_1(n_1+\tfrac{1}{2})+\omega_2(n_2+\tfrac{1}{2})+\alpha_{11}/2(n_1+\tfrac{1}{2})^2+\alpha_{22}/2(n_2+\tfrac{1}{2})^2+\alpha_{12}(n_1+\tfrac{1}{2})(n_2+\tfrac{1}{2})+\tfrac{1}{2}[\beta+\tfrac{1}{2}\epsilon(n_1+n_2+1)]\times(a_1^\dagger a_2+a_2^\dagger a_1)+\delta(a_1^\dagger a_1^\dagger a_2 a_2+a_2^\dagger a_2^\dagger a_1 a_1). \quad (1)$$

The Hamiltonian has linear and nonlinear terms, as well as coupling terms, the first-order one proportional to $(a_1^\dagger a_2+a_2^\dagger a_1)$ being the "Josephson coupling" in BEC systems. This Hamiltonian and special cases have been used [1–7] for coupled BEC systems. It has great familiarity in molecular spectroscopy [8,9]. Note that subsystems 1 and 2 in Eq. (1) are not necessarily taken to be identical, e.g., there can be different trap frequencies ω_1, ω_2.

The raising and lowering operators may be used to define operators that satisfy SU(2) commutation relations,

$$\hat{I}_x=\tfrac{1}{2}(a_1^\dagger a_2+a_2^\dagger a_1),$$

$$\hat{I}_y=-i/2(a_1^\dagger a_2-a_2^\dagger a_1),$$

$$\hat{I}_z=\tfrac{1}{2}(a_1^\dagger a_1-a_2^\dagger a_2)=(n_1-n_2)/2,$$

$$\hat{I}=\tfrac{1}{2}(a_1^\dagger a_1+a_2^\dagger a_2)=(n_1+n_2)/2. \quad (2)$$

The Hamiltonian (1) can be written in terms of the SU(2) operators as

$$\hat{H}=\omega_0(2\hat{I}+1)+[P+(1/2)Q(2\hat{I}+1)]\hat{I}_z+\alpha_1\hat{I}(\hat{I}+1)+\alpha_1/4+\alpha_2\hat{I}_z^2+[\beta+\tfrac{1}{2}\epsilon(2\hat{I}+1)]\hat{I}_x+\delta(\hat{I}_+^2+\hat{I}_-^2) \quad (3)$$

with

$$\omega_0=(\omega_1+\omega_2)/2,\quad P=\omega_1-\omega_2,\quad Q=\alpha_{11}-\alpha_{22},$$

$$\alpha_1=\alpha_{11}/2+\alpha_{22}/2+\alpha_{12},\quad \alpha_2=\alpha_{11}/2+\alpha_{22}/2-\alpha_{12}. \quad (4)$$

While the Hamiltonian (3) is defined in terms of states of the individual traps 1, 2, a representation in terms of symmetrized states is sometimes used [3,8,11], and will be considered later. It is easily shown [8] with the use of the SU(2) algebra that the two representations are equivalent.

In both the BEC and molecular systems, a semiclassical limit is very useful, but this limit arises in rather different ways. In a BEC system, each condensate $i=1$, 2 is described by a macroscopic wave function of form $\Phi=\sqrt{N_i}e^{i\phi_i(x_i,t)}$. Each phase ϕ_i can be thought of in terms of spontaneous breaking of a U(1) gauge symmetry [12,13], resulting in the "rigidity" of ϕ_i. Regarded as a dynamical variable, ϕ_i can be taken as the coordinate of the Goldstone boson corresponding to the broken U(1) symmetry [12].

In a molecular vibrational system each boson mode is an oscillator, rather than a field, unlike in the condensate. Rather than being a macroscopic wave function, $\sqrt{N_i}e^{i\phi_i(x_i,t)}$ for the oscillator is a dynamical quantity. It can be regarded as a semiclassical limit [8], obtained via the "Heisenberg correspondence," of an oscillator coherent state. The rigid semiclassical phase angle of the coherent state is the molecular correlate of the broken symmetry.

Systematic analysis of the bifurcation behavior of the semiclassical Hamiltonian reveals information about dynamical stabilities and instabilities. In molecular physics, these are reflected in the fully quantum dynamics, and in detailed (i.e., not statistical) patterns of the quantum spectrum [14,15]. In BEC systems, bifurcations are expected to have consequences both for classical behavior and semiclassical corrections.

Even though the physical meaning of $\sqrt{N_i}e^{i\phi_i(x_i,t)}$ is different in the BEC and molecular systems, the formal analogy, along with many of its consequences, is very close. The semiclassical correlate of the Hamiltonian (3) is

$$H=2\omega_0 I+[P+QI]I_z+\alpha_1 I^2+\alpha_2 I_z^2+[\beta+\epsilon I]\sqrt{I^2-I_z^2}\cos\psi+\delta(I^2-I_z^2)\cos 2\psi, \quad (5)$$

where $\psi=(\phi_1-\phi_2)$ and now the semiclassical total action I is understood as $I=(n_1+n_2+1)/2=(N+1)/2$. It is a constant of motion, conjugate to the phase angle $(\phi_1+\phi_2)$. In the BEC, this constant corresponds to the total particle number; in the molecule, to the total number of vibrational quanta.

The action $I_z=(n_1-n_2)/2$ is not conserved, so I_z and its conjugate angle $\psi=(\phi_1-\phi_2)$ are the significant dynamical variables of Hamiltonian (5). These variables are identified in the molecule [16] with the coset space SU(2)/U(1). In the condensate, by standard results in quantum field theory [12], the coset space SU(2)/U(1) parametrizes the Goldstone modes with coordinates I_z, ψ which remain after spontaneous breaking of the dual-condensate SU(2) symmetry to U(1).

States of a given energy can be plotted as (I_z,ψ) trajectories on the Bloch sphere [8,17,6], parametrized by ψ as the azimuthal angle and by the longitudinal angle α with $\sin\alpha=I_z/I$. Examples will be considered below.

We seek to classify and interpret the dynamics of the Hamiltonian (5), with attention to bifurcations and instabilities. A variety of special assumptions are commonly made: whether the traps are identical [1,3,5–7], or not [2]; whether the Josephson coupling has the number dependence given by the parameter ϵ; and whether the second-order coupling parameter δ is significant. This is often reasonable in molecules, but there are important exceptions [11]. The most common assumption in BEC systems is to take $\delta=0$, though this assumption is sometimes relaxed, e.g., in the work of Helmerson and You [18] on "massive entanglement" in Bose-Einstein systems, where an essential property of the coupling is the assumption that $\delta\neq 0$. The consequences for the semiclassical dynamics are important, and will be considered later for BEC systems.

We want to obtain as comprehensive a classification as possible, systematically encompassing the special cases. We first consider $\delta=0$, and within this approximation consider traps that may be identical or different. The Hamiltonian (5) is then classified [9,10] by just two reduced control parameters μ, Λ, distilled from the Hamiltonian parameters and the particle number $N=2I-1$:

$$\mu=(\beta+\epsilon I)/(2\alpha_2 I),\quad \Lambda=(P+QI)/(2\alpha_2 I). \tag{6}$$

The parameter μ gives the relative strength of the Josephson coupling term; Λ the asymmetry between the subsystems. The denominators in Eq. (6) take account of the nonlinearities in Eq. (5). In the BEC system this role is played by the chemical potentials in the traps; in the molecule system, by anharmonicity of the oscillators. Note that for a given set of Hamiltonian parameters, each total particle number $N=(2I-1)$ has its own dynamics on a distinct Bloch sphere, corresponding to a distinct pair of reduced control parameters (μ, Λ).

We now look for the basic periodic orbits of the Hamiltonian, i.e., the normal modes and the new modes born in their bifurcations. Since the action I is fixed, and the Hamiltonian does not depend on the conjugate angle $(\phi_1+\phi_2)$, these are determined by the fixed points of the two remaining dynamical variables I_z, ψ, which satisfy

$$\dot{\psi}(I_z,\psi)=\partial H/\partial I_z=0,\quad \dot{I}_z(I_z,\psi)=-\partial H/\partial\psi=0. \tag{7}$$

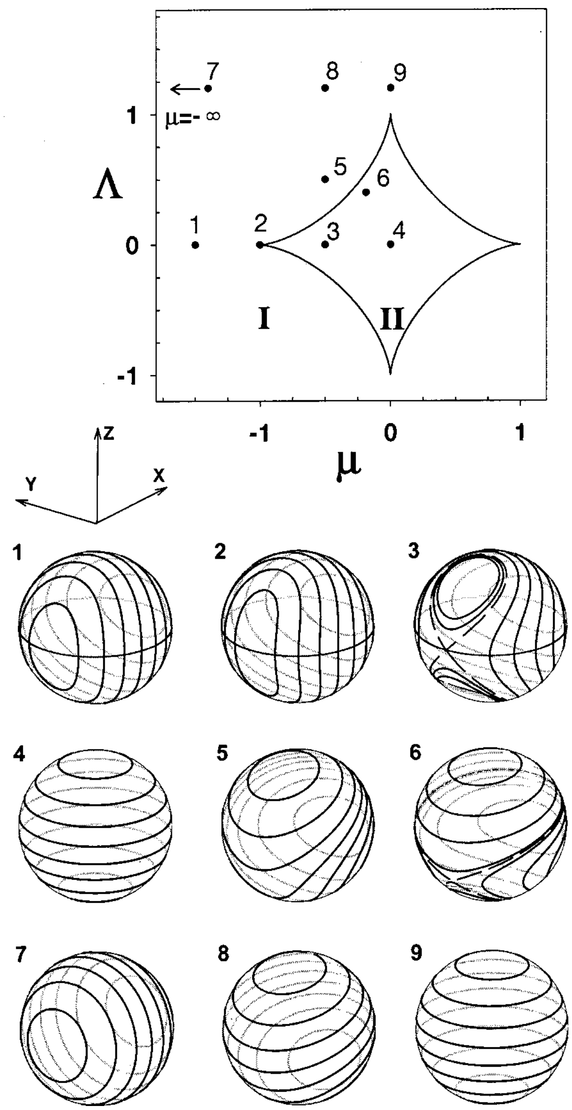

FIG. 1. Catastrophe map of Hamiltonian (5). Bloch spheres are shown for indicated points.

An important consequence of Eqs. (5) and (7) is that all fixed points lie on the great circle lying in the I_x, I_z plane, with $I_y=0$, as can be seen from the spheres in Fig. 1. From Eq. (7) the fixed points are the critical points of the Hamiltonian evaluated on the great circle. We call this the "pseudopotential" $H(\alpha)$, where the range of α is taken as $(0,\ 2\pi)$. The sphere has symmetry across the $I_y=0$ plane. When the asymmetry Λ is 0, there is also symmetry across the $I_z=0$ plane. (As will be seen later, the asymmetric situation $\Lambda\neq 0$ has important consequences.) However, the sphere is not symmetrical across the $I_y=0$ plane. This accounts for the distinction between "$\phi=0$ modes" and "$\phi=\pi$ modes" [2].

A very useful analytical device is to map the bifurcations as a function of the control parameter space. This gives the catastrophe map [10,19]. The map gives a complete classification of the bifurcation behavior of the Hamiltonian. This classification can be thought of as a kind of phase diagram. The catastrophe map for the Hamiltonian system (5) (with $\delta=0$) is shown in Fig. 1. A particular experimental arrangement is represented as a point on the map, defined by the values of the control parameters (μ, Λ). As seen in Fig. 1, the catastrophe map has a ''diamond cusp'' shape, dividing the parameter space into two zones, or phases, labeled I and II. Bifurcations on the sphere occur as zone boundaries are crossed.

We examine the structure of the sphere and the bifurcations it undergoes as the control parameters are varied, and the physical meaning in the BEC system. First, we consider the symmetric system, i.e., with asymmetry parameter $\Lambda=0$. This corresponds to two identical traps (though *not* identical numbers of particles.)

In zone I, the sphere 1 has an undivided phase space. It has been argued [5] that in the BEC system, this corresponds to a single trap. The sphere has stable fixed points on the equator. One fixed point corresponds to all the particles in the ground vibrational state of the trap, with symmetric single-particle state. The other fixed point corresponds to all the particles in the first excited vibrational state. (In molecules, the fixed points are the symmetric and antisymmetric normal modes.) The ground state has a fixed angle $\psi=(\phi_1-\phi_2)=0$.

When the condensate is separated into two traps, e.g., by ''splitting'' the single trap with a laser [7], effectively the coupling μ between condensates 1 and 2 is reduced. On the catastrophe map, there is a cusp at $(\mu,\Lambda)=(\pm1,0)$. A pitchfork bifurcation takes place at the cusp, sphere 2, and the system crosses into zone II. This bifurcation is important in molecules [9], and has been noted in model BEC systems [1,6]. The bifurcated sphere 3 now has a phase space divided by a separatrix. One of the original fixed points, at the back of sphere 3 on the equator, remains stable. In BEC systems [6], this is the one with phase $\psi=0$. In this ''resonance region'' of the sphere, Josephson tunneling occurs as oscillation around the stable equatorial fixed point. The other equatorial fixed point becomes unstable, with the birth in the bifurcation of two new stable modes, which correspond to ''macroscopic self-trapped'' [2] conditions. As the coupling μ between traps approaches 0, the self-trapped regions grow. At $\mu=0$ the resonance region closes, leaving stable fixed points on the sphere 4 at the north and south poles. These represent systems with all the particles in one trap or the other. Trajectories away from the poles represent states with definite $I_z=(n_1-n_2)/2$, hence the numbers of particles in each trap, so there is no Josephson tunneling. Conversely, the phase $\psi=(\phi_1-\phi_2)$ is completely indefinite.

These bifurcations and associated changes in the phase-space structure of the sphere are associated with definite physical effects. For example, the pitchfork bifurcation and the resonance-closing bifurcation correspond to the parameter values at which qualitative changes occur in the atom number quantum fluctuations between the left and right sides of a ''split trap'' (see Fig. 1 of Ref. [3]).

We now consider asymmetric systems, i.e., nonidentical traps, with $\Lambda\neq0$ in Eq. (3). Asymmetric traps have been considered in Ref. [2]. With $|\Lambda|<1$, there are still distinct catastrophe map zones I and II. Sphere 5 in zone I in Fig. 1 has an undivided phase space. The fixed points are again on the great circle, but asymmetrically situated. At the boundary of zones I and II a tangent bifurcation takes place. In the interior of zone II, the sphere 6 is asymmetric and divided by a separatrix.

With $|\Lambda|>1$, a much different situation results. At $\mu=0$, $\Lambda=\pm1$, zone I ends in a cusp. Above the cusp, the sphere evolves continuously, without bifurcations, from the limit of strongly coupled traps ($\mu=-\infty$, sphere 7), to intermediate μ with an asymmetric, undivided sphere 8, to the uncoupled limit ($\mu=0$, sphere 9, identical in appearance to sphere 4).

This continuous change has an interesting physical meaning. When the fixed point is at the equator, there is minimum uncertainty in the relative phase ψ (classically, there is definite phase ψ); when it is at the north pole, there is minimum uncertainty in the particle number I_z (classically, there is definite particle number). The continuous transition between these limits corresponds to a continuous evolution of I_z and ψ, with $\Delta I_z \Delta\psi$ fixed, behavior absent in systems with asymmetry $|\Lambda|<1$.

Now we let the higher-order Josephson coupling $\delta\neq0$ in Eq. (1). This is relevant in molecular systems [11], with pronounced effects on the dynamics and phase-space structure. The coupling $\delta\neq0$ has received attention in BEC systems as creating the possibility of ''massive entanglement'' [18]. We

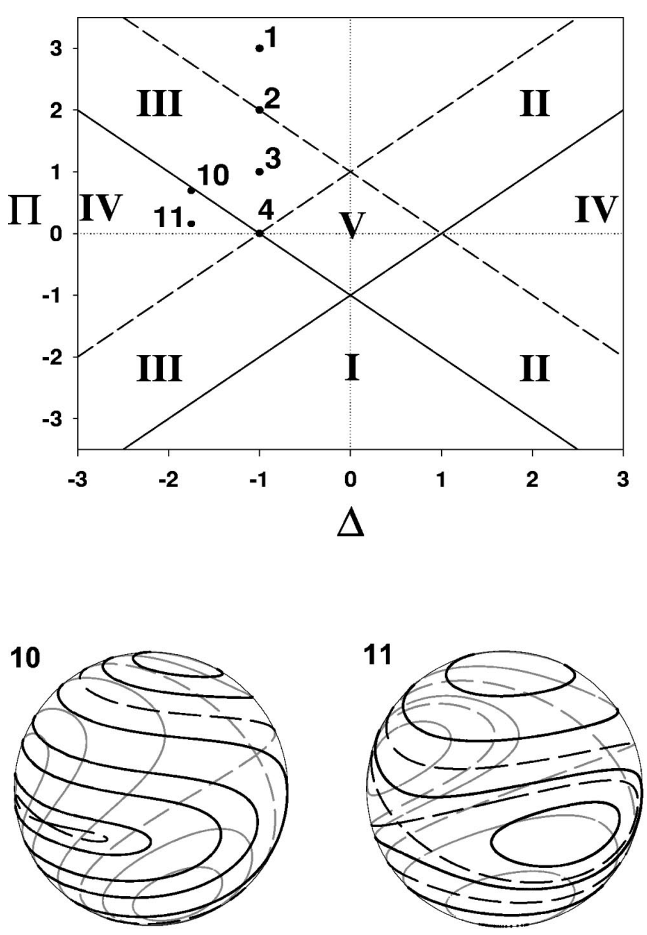

FIG. 2. Catastrophe map of Hamiltonian (8). Bloch spheres are shown for points 10 and 11. (These are rotated by $\approx\pi$ about the vertical axis from the orientation of the spheres of Fig. 1.)

restrict ourselves to symmetric systems with $|\Lambda|=0$.

The catastrophe map for $\delta\neq 0$ is more conveniently analyzed [11] in the symmetrized representation with single-particle operators $a_s^\dagger = 1/\sqrt{2}(a_1^\dagger + a_2^\dagger)$, $a_a^\dagger = 1/\sqrt{2}(a_1^\dagger - a_2^\dagger)$, and number variables and conjugate angles n_s, n_a, ϕ_s, ϕ_a. This representation has been used for BEC systems [3]. It is obtained by an SU(2) transformation in which the operator $\hat{I}_x$ becomes diagonal [8]. The semiclassical Hamiltonian (5) becomes

$$H=\Omega+\gamma_1 I_x+\gamma_2 I_x^2+\kappa(I^2-I_x^2)\cos 2\Psi, \tag{8}$$

where now $\Psi=(\phi_s-\phi_a)$ is the angle conjugate to I_x. The relevant control parameters in this symmetrized representation are

$$\Delta=\kappa/2\gamma_2, \quad \Pi=\gamma_1/2I\gamma_2 \tag{9}$$

(there being no third control parameter because of the restriction to symmetric systems).

The catastrophe map is shown in Fig. 2. It has an elaborate division into five zones, each having a different kind of sphere structure. The connection with the earlier catastrophe map of Fig. 1 of the unsymmetrized representation is made by noting that points 1–4 of Fig. 1 correspond to points 1–4 of Fig. 2.

The characteristic behavior associated to $\delta\neq 0$ is seen with spheres 10 and 11. (In order to show their most salient features, these spheres are rotated by $\approx\pi$ about the vertical axis from the orientation of the spheres of Fig. 1.) These spheres lie in zone IV, which has different sphere structure than anything seen previously in Fig. 1. With sphere 10, $\delta\neq 0$ has caused the stable fixed point on the equator of sphere 3 at $\psi=0$ to undergo a pitchfork bifurcation, in which the stable fixed point becomes unstable, and two new stable modes are born on the equator. Sphere 11 shows the latter when they have moved away from $\psi=0$. In the $\delta=\infty$ limit, they move to $\psi=\pi/2$, $3\pi/2$. Physically, this bifurcation represents the onset of instability in the BEC and its gradual division into two subsystems with phases locked at new values $\psi\neq\pi$; in the limit, at $\pi/2$, $3\pi/2$.

Finally, we note that Hamiltonians analogous to Eq. (1) but with coupling like $(a^\dagger a^\dagger b + aab^\dagger)$ have been considered [20,21] for photoassociation of atoms to form molecular condensates. The bifurcation and catastrophe map classification of systems with this coupling has been extensively studied in molecular spectra [9,10,14,15]. The bifurcation classification and sphere structures are distinctly different from either of the catastrophe maps of Figs. 1 and 2.

This paper has systematically analyzed bifurcation behavior, represented by dynamics on the Bloch sphere, for model systems of coupled Bose-Einstein condensates. The catastrophe map phase classification gives a complete analysis of the dynamics possible for the model Hamiltonian. Additional phenomena, possibly accessible to experiment, are predicted for asymmetric systems, and systems with higher-order Josephson coupling. It is noteworthy that the reduced control parameters of the catastrophe map depend on the total particle number, suggesting experiments that tune through various bifurcation scenarios by variation of system size, as well as variation of the Hamiltonian parameters.

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