Algebra and Geometry Throughout History: A Symbiotic Relationship

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Stages in the Development of Symbolic Algebra

- Rhetorical algebra: equations are written in full sentences like "the thing plus one equals two"; developed by ancient Babylonians
- Syncopated algebra: some symbolism was used, but not all characteristics of modern algebra; for example, Diophantus' *Arithmetica* (3rd century A.D.)
- Symbolic algebra - full symbolism is used; developed by François Viète (16th century)

Babylonian Algebra: Plimpton 322 ca. 1800 B.C.E.

Babylonians used cuneiform cut into a clay tablet with a blunt reed to record numbers and figures. They had a base 60 true place-value number system.
Plimpton 322 contains a table with 2 of 3 numbers of what are now called *Pythagorean triples*: integers $a, b,$ and c satisfying $a^2 + b^2 = c^2$.
These are integer length sides of a right triangle.

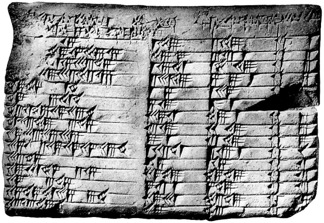


Figure: Plimpton 322

Babylonian Algebra: YBC 7289 ca. 1800 - 1600 B.C.E.

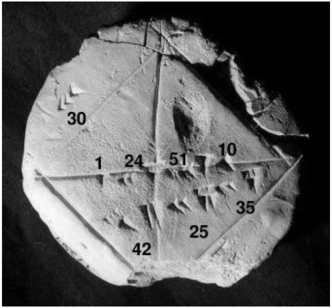


Figure: YBC 7289

YBC 7289 clay tablet illustrates "numbers" used in calculation of the square root of 2, the hypotenuse of a right triangle with two equal sides of length 1.

Rhind (Ahmes) Papyri ca. 1650 B.C.E.

Egyptians used hieroglyphs or hieratic on papyri to record numbers; they had base 10 system. Book I: 21 arithmetic and 20 algebraic problems.
Book II: geometric problems (volumes, areas, slopes of pyramids).
Copied by scribe Ahmes and purchased by Scottish antiquarian Rhind in 1858.

Figure: Rhind Papyrus

Problem 42 in Book II

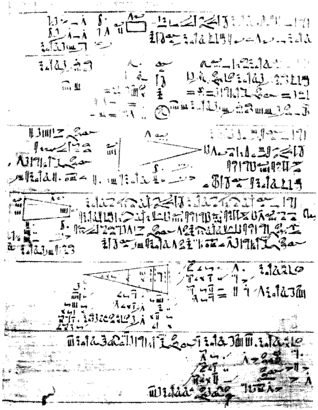


Figure: Rhind Papyrus

Compute the volume of a cylindrical granary:

$$V = [(1 - 1/9)\, d]^2\, h$$

Greek Origins ca. 600 B.C.E.

- To Greeks, algebra is essentially "geometric."
- Greek geometry and number theory were sophisticated but Greek algebra was weak.
- Calculations with magnitudes and their relations rather than numbers.
- They created and intersected auxiliary curves to solve algebraic problems.

Euclid of Alexandria

- Probably lived during reign of Ptolemy I (323 - 283 B.C.E.)
- Euclid is regarded as "father of geometry."
- His *Elements* is arguably the most successful textbook in history of mathematics.
- 14 propositions in Book II of Elements very significant for doing geometric algebra.

Persian Contributions

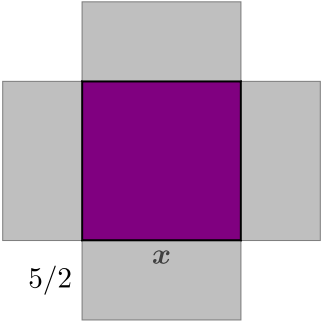
Figure: al-Khwārizmī

Muhammad bin Musa al-Khwārizmī (ca. 780 - 850 A.D.) was a Persian mathematician, astronomer and geographer from a district not far from Baghdad. Under the region of Caliph al-Ma'mūhn he became a member of the "House of Wisdom."

His book *al-jabr w al-muqabāla* describes techniques to solve quadratic equations by first reducing them to one of five standard forms.

Completing the Square 1

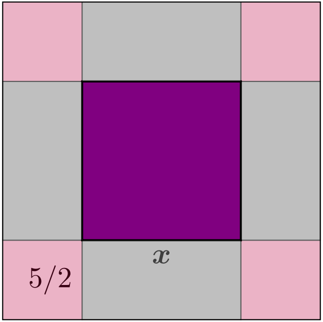
Solve (in modern notation) $x^2 + 10x = 39.$



Completing the Square 2

Adding 4 corner squares each of area 25/4, we end up solving

$$(x+5)^2 = 39 + 25 = 64 = 8^2 \Rightarrow x + 5 = \pm 8 \Rightarrow x = 3, -13.$$



François Viète (1540 - 1603 A.D.)

François Viète, a French lawyer and code-breaker who published under the latinized name Franciscus Vieta, made a critical contribution that was the first step in transforming algebra from a study of the specific to the general, where the equations need not have particular numerical coefficients.

Figure: François Viète

Viète's Algebraic Notation (1591)

In *Introduction to the Analytic Art* (1591) Viète introduced arbitrary parameters into an equation which were distinguished from variables occurring in equation.

- Used consonants (B, C, D, ...) to denote known parameters
- Used vowels (A, E, I, O, U) to denote variables
- Used "syncopated" (i.e., partly symbolic) notation
  Viète would express polynomial $A^3 + 3A^2B + 3AB^2 + B^3$ as

  A cubus, $+$ A quadrato in B ter, $+$ A in B quadratum ter, $+$ B cubo

- Equations Viète considered were *homogeneous*

Study of General Polynomial Equations

How do we find the roots of a polynomials, preferably exact solutions in term of radicals (square roots, cube roots, and so on)?

- Solution in radicals given for cubic (degree 3) equation published by Cardano in 1545; known earlier by del Ferro and Tartaglia.
- Formula in radicals for roots of quartic (degree 4) equations over the complex numbers published by Ferrari in 1545.
- In 1771 Joseph Lagrange gave a unifying method for producing roots of polynomials of degree at most 4.
- In 1824 Norwegian mathematician Niels Henrik Abel that the roots of a general quintic (degree 5) polynomial cannot be expressed as a finite number of radicals in the coefficients.

Descartes' Algebraic Notation (1637)

In his book "Geometry" (1637) René Descartes used fully symbolic notation.

- $x, y, z, \ldots$ were used to denote variables
- $a, b, c, \ldots$ were used to denote parameters
- Homogeneity of algebraic expressions was no longer needed

Cartesian Coordinates

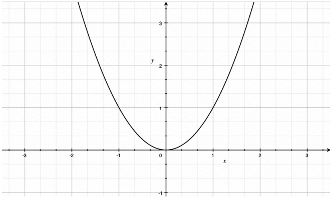


Figure: The Parabola $y = x^2$

Descartes introduced Cartesian coordinates into geometry, where each point in the plane is uniquely described by a pair of coordinates which are the signed distances to the point from two fixed perpendicular lines. This was the first systematic link between Euclidean geometry and algebra.

Carl Friedrich Gauss (1777 - 1855)

The German mathematician Carl Friedrich Gauss made significant contributions to many fields, including number theory, algebra, statistics, analysis, differential geometry, geophysics, astronomy, and optics. He is sometimes referred to as "The Prince of Mathematics."

Fundamental Theorem of Algebra

In his 1797 dissertation Gauss gave the first of four proofs of what is today known as the "Fundamental Theorem of Algebra (FTA)".

Theorem (FTA 1)

Every polynomial $f(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ *with real coefficients can be factored into linear and quadratic factors over the real numbers.*

Évariste Galois 1811- 1832

- During his teenage years Galois determined a necessary and sufficient condition for a polynomial to be solvable by radicals. Eg., any quadratic equation $ax^2 + bx + c = 0$ is solvable by radicals: $x = \left(-b \pm \sqrt{b^2 - 4ac}\right)/2$.
- Galois was first to use the word *group* (an algebraic structure with one operation).
- He realized that an algebraic solution to a polynomial equation is related to structure of a group of permutations associated to roots of the polynomial, now called the Galois group of the polynomial.

Throughout the rest of the talk, we will see frequently see the letters $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$.

Definition

- $\mathbb{N} = \{\ldots 0, 1, 2, 3, \ldots\}$ (the set of natural numbers)
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ (the ring of integers)
- $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$ (the field of rational numbers)
- $\mathbb{R}$ stands for the field of real numbers
- $\mathbb{C}$ stands for the field of complex numbers

More General Fields of Coefficients

Both Abel and Galois had an understanding of what we now call a *field*, that is, an algebraic structure in which you can add, subtract, multiply, and divide. Galois also talks about extending fields by adjoining elements. In his memoir Galois states:

"one can agree to consider as rational every rational function of a certain number of quantities regarded as known a priori..."

and he describes the process of adjoining a new quantity to a known field of quantities. These fields all contained the field of rational numbers, so had *characteristic 0*.

Galois also studied the finite field $F_p = \mathbb{Z}/p\mathbb{Z}$ and other finite fields containing F_p. These are fields of *characteristic p*.

Groups, Rings, and Fields

Abstract algebra, with its origins in the late 1800s, deals with "abstract" algebraic structures rather than the usual number systems.

- *Groups* are algebraic structures with one operation. The collection of all permutations on a set with two elements under composition is an example of a group.
- *Rings* are algebraic structures with two operations, usually referred to as addition and multiplication. The set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ under ordinary addition and muliplication is an example of a ring.
- *Fields* are special examples of rings, in which you can divide by nonzero elements. The set of rational numbers $\mathbb{Q}$ under ordinary addition and muliplication is an example of a field.

Emmy Noether, 1882 - 1935

Figure: Emmy Noether

Emmy Noether was an important Jewish German mathematician. Noether's work in abstract algebra began around 1913. N. Jacobson writes in his introduction to Noether's Collected Papers "The development of abstract algebra, which is one of the most distinctive innovations of twentieth century mathematics, is largely due to her - in published papers, in lectures, and in personal influence on her contemporaries."

Emmy Noether, 1882 - 1935

Shortly after 1913, Noether was invited to Göttingen by David Hilbert but never became a regular faculty member (*privatdozent*) there because she was a woman. Hilbert protested saying
"I do not see that the sex of the candidate is an argument against her admission as privatdozent. After all, we are a university, not a bath house."
In 1921 she published *Idealtheorie in Ringbereichen*, in which she analyzed ascending chain conditions with regard to ideals; rings with this conditions are called *Noetherian* rings in her honor.
In 1933 Noether and many other Jewish instructors were dismissed when the Nazis came to power in Germany. She landed at Bryn Mawr College. She died there in 1935, shortly after an operation to remove a very large ovarian cyst.

Figure: Fasten Your Seat Belt!

We now take a leap in the level of abstraction so I can describe some of the problems I have worked on.

Next up on our agenda is the notion of an *integral extension* of rings. The famous Noether Normalization Lemma of Emmy Noether talks about an important example of this concept.

Integral Extensions of Rings

We now consider commutative rings (so $ab = ba$) with (multiplicative) identity 1. The ring of integers $\mathbb{Z}$ and the polynomial ring over the rational numbers $\mathbb{Q}[X]$ are examples of commutative rings with 1.

Definition

Let $A \subset B$ be an extension of rings and $x \in B$. We say x is *integral* over A if there exist a positive integer n and $a_1, \ldots, a_n \in A$ such that

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0.$$

We say that $A \subset B$ is *integral extension* if every $x \in B$ is integral over A.

Examples of Integral and Nonintegral Extensions

The following are integral extensions of rings.

- $\mathbb{R} \subset \mathbb{R}[X]/(X^2+1)$
- $\mathbb{Z} \subset \mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$

The following extensions of rings are not integral.

- $\mathbb{Z} \subset \mathbb{Z}[\pi]$
- $\mathbb{C} \subset \mathbb{C}[X]$

Noether Normalization Lemma

Theorem

Let A be an integrally domain that is finitely generated over a field K. Then, there is a set of algebraically independent elements $\{x_1, \ldots, x_n\} \subset A$ such that $K[x_1, \ldots, x_n] \subset A$ is an integral extension.

Integral Closure of A in B

Definition

Suppose that $A \subset B$ is an extension of rings. The set

$$\overline{A}_B = \{x \in B \mid x \text{ is integral over } A\}$$

is called *integral closure of A in B*. We say that A is *integrally closed in B* if $A = \overline{A}_B$.

It is well known that $\overline{A}_B$ is a subring of B. Also, $A \subset \overline{A}_B$ is an integral extension whereas $\overline{A}_B \subset B$ and is integrally closed in B.

Normal Domains and Normalization

An integral domain is a ring in which $ab = 0$ implies $a = 0$ or $b = 0$.

Definition

We say that an integral domain A is *normal* provided that A is integrally closed in quotient field K. We define the *normalization* $\overline{A}$ of A to be the integral closure of A in its quotient field K.

We point out that any unique factorization domain is normal. In particular, $\mathbb{Z}$ and $\mathbb{Q}[X]$ are normal integral domains.

A Domain That Isn't Normal

$\mathbb{C}[X,Y]/(Y^2 - X^3 - 3X^2) =: \mathbb{C}[x,y]$ is an integral domain that is not normal. Notice that $x = (y/x)^2 - 3$ and hence $\overline{\mathbb{C}[x,y]} = \mathbb{C}[y/x]$.

Looking at the zeros of the equation $Y^2 - X^3 - 3X^2 = 0$ we obtain a plane curve that has a singularity at the origin, called an *ordinary double point* or *node*. By passing to the normalization we remove the singularity.

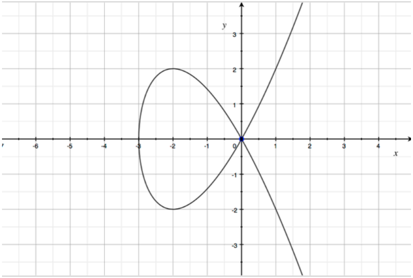


Figure: $Z(Y^2 - X^3 - 3X^2)$

Another Non-Normal Domain

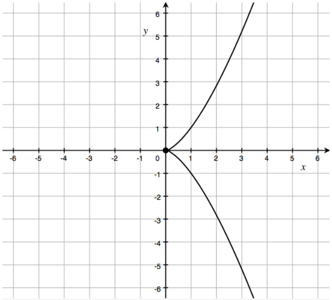


Figure: $Z(X^3 - Y^2)$

Consider the integral domain $\mathbb{C}[X, Y]/(X^3 - Y^2) =: \mathbb{C}[x, y]$ and notice that in its quotient field $x = (y/x)^2$ Again, the normalization of $\mathbb{C}[x, y]$ is $\mathbb{C}[y/x]$. Looking at the zeros of $X^3 - Y^2 = 0$ we obtain a plane curve that has a singularity at the origin called a *cusp*.

We will later see that algebraically we can distinguish the node from the cusp by a condition called seminormality.

Integral Closure of an Ideal

Definition

Let $I \subset A$ be an ideal of a ring A. We say an element $x \in A$ is *integral over* I provided that there exist a positive integer n and elements $a_k \in I^k (k = 1, \ldots, n)$ such that

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0.$$

We define the *integral closure of* I to be the set

$$\overline{I} = \{x \in A \mid x \text{ is integral over } I\}.$$

We say that I is *integrally closed* if $I = \overline{I}$.

It is well known that $\overline{I}$ is an integrally closed ideal of A.

The Exponent Set of a Monomial Ideal

Suppose that $I \subset K[X_1, \ldots, X_n]$ is a *monomial ideal*, that is, I is generated by products of powers of the variables.

Define the *exponent set of I* to be the set

$$\Gamma(I) = \{\alpha \in \mathbb{N}^n \mid X^\alpha := X_1^{a_1} \cdots X_n^{a_n} \in I\},$$

where $\mathbb{N} = \{0, 1, 2, \ldots\}$ and the *Newton Polyhedron of I* to be

$$\mathrm{NP}(I) = \mathrm{conv}(\Gamma(I)),$$

where $\mathrm{conv}(S)$ stands for convex hull of a subset $S \subset \mathbb{R}^n$, that is, the smallest convex set containing S.

The Integral Closure of a Monomial Ideal

It is well known that the integral closure $\overline{I}$ of a monomial ideal I is again a monomial ideal with exponent set

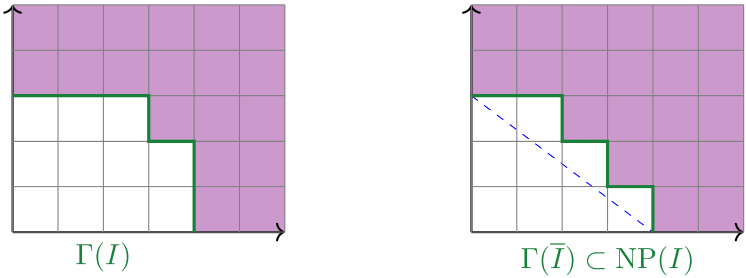
$$\Gamma(\overline{I}) = \overline{\Gamma}(I) := \{\alpha \in \mathbb{N}^n \mid m\alpha \in \sum_{i=1}^{m} \Gamma(I) \ \ \exists m \geq 1\}.$$

Furthermore, we know that

$$\Gamma(\overline{I}) = \mathrm{NP}(I) \cap \mathbb{N}^n.$$

Example of Integral Closure of a Monomial Ideal

In the polynomial ring $K[X,Y]$ over the field K consider the monomial ideal $I = (X^4, X^3Y^2, Y^3)$ and its integral closure $\overline{I} = (X^4, X^3Y, X^2Y^2, Y^3)$.



Normal Ideals

Definition

An ideal I of a ring A is said to be *normal* if all positive powers of I are integrally closed.

Notice that $\mathfrak{m} = (X, Y) \subset K[X, Y]$ is a normal ideal (think of the Newton Polyhedra of powers of $\mathfrak{m}$).
It turns out that one doesn't have to test all positive powers of a monomial ideal to establish normality.

Theorem (Reid-Roberts-Vitulli, 2003)

Let $I \subset K[X_1, \ldots, X_n]$ be a monomial ideal. If I^k is integrally closed for $k = 1, \ldots, n-1$, then I is normal.

Some Classes of Normal Ideals

This result enables us to produce classes of normal and non-normal monomial ideals in $K[X_1, X_2]$ and $K[X_1, X_2, X_3]$.

Theorem (Reid-Roberts-Vitulli, 2003)

Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ *be an ordered triple of positive integers. If* $\gcd(\lambda_1, \lambda_2, \lambda_3) > 1$, *then* $\overline{(X_1^{\lambda_1}, X_2^{\lambda_2}, X_3^{\lambda_3})} \subset K[X_1, X_2, X_3]$ *is normal.*

Here is another class found by a student of mine.

Theorem (Heather Coughlin, 2004)

Let $\boldsymbol{\lambda} = (j, j+1, j+2)$ *for* $j \geq 2$. *The monomial ideal* $\overline{(X_1^{\lambda_1}, X_2^{\lambda_2}, X_3^{\lambda_3})} \subset K[X_1, X_2, X_3]$ *is normal if and only if* j *is even.*

Affine Algebraic Sets and Affine Coordinate Rings

To an affine algebraic set

$$V = Z(F_1(X_1, \ldots, X_n), \ldots, F_s(X_1, \ldots, X_n)) \subset K^n$$

we associate its *affine coordinate ring*

$$\Gamma(V) = K[X_1, \ldots, X_n]/I(V) =: K[x_1, \ldots, x_n], \text{ where}$$

$$I(V) = \{F(X_1, \ldots, X_n) \in K[X_1, \ldots, X_n] \mid F(a) = 0 \ \ \forall a \in V\}$$

and $x_i = X_i + I(V)$.

The Ordinary Double Point

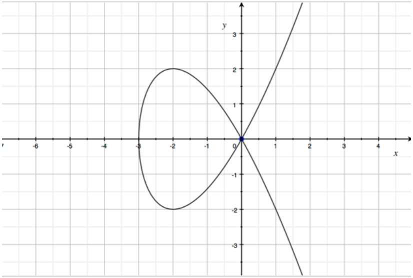


Figure: $Z(Y^2 - X^3 - 3X^2)$

This plane curve has a singularity at the origin, called an *ordinary double point* or *node*.

Its affine coordinate ring $\mathbb{C}[X,Y]/(Y^2 - X^3 - 3X^2) =: \mathbb{C}[x,y]$ is an integral domain that is not normal. Notice that $x = (y/x)^2 - 3$ and hence $\overline{\mathbb{C}[x,y]} = \mathbb{C}[y/x]$.

Normalization of a Curve

Suppose that C is an irreducible affine curve and let A denote its affine coordinate ring. Then, $\overline{A}$ is the affine coordinate ring of an irreducible affine curve $\widetilde{C}$ and the associated map $\pi\colon \widetilde{C} \to C$ is a surjective closed mapping with finite fibers (i.e., only finitely many points in $\widetilde{C}$ map to the same point in C). This map is called the *normalization* of C. The curve $\widetilde{C}$ is nonsingular.

Example

Let $C = Z(Y^2 - X^3 - 3X^2) \subset \mathbb{C}^2$ so that $\widetilde{C} = \mathbb{C}^1$ and

$$\pi\colon \widetilde{C} \to C \text{ is given by } \pi(t) = (t^2 - 3, t(t^2 - 3)).$$

Notice that π is 1-1 except that $\pi^{-1}(0,0) = \{\pm\sqrt{3}\}$.

Traverso's Local Gluing, 1970

Suppose that $A \subset B$ is an integral extension of rings and that A is local with unique maximal ideal $\mathfrak{m}$. Then, all of the maximal ideals of B lie over $\mathfrak{m}$. Let $R(B)$ denote the *Jacobson radical* of B, that is, the intersection of the maximal ideals of B. Consider the subring $A' = A + R(B)$. This ring is called the *gluing of A in B over* $\mathfrak{m}$.

Theorem (Traverso, 1970)

Let $A \subset A' \subset B$ be as above. Then,

- *A' is local with unique maximal ideal $\mathfrak{m}' = R(B)$; and*
- *the induced map $A/\mathfrak{m} \to A'/\mathfrak{m}'$ is an isomorphism.*

A notion of *weak gluing* was introduced in 1969 by Andreotti and Bombieri in the context of schemes, which are generalizations of algebraic varieties.

Seminormalization

Definition

For an integral extension of rings $A \subset B$ we define the *seminormalization* ${}^{+}_{B}A$ of A in B by

$${}^{+}_{B}A = \{b \in B \mid b_P \in A_P + R(B_P), \forall \text{ primes } P \subset A\}.$$

By the *seminormalization* ${}^{+}A$ of an integral domain A we mean its seminormalization in $\overline{A}$, the normalization of A.

Notice that the seminormalization of A in B is obtained by gluing over *all* the prime ideals of A. This concept was introduced by Carlo Traverso in 1970.

Seminormal Extension

Definition

Let $A \subset B$ be an integral extension of rings.
We say that A is *seminormal* in B (or that the extension is seminormal) provided that $A = {}^{+}_{B}A$. We say an integral domain is *seminormal* if $A = {}^{+}A$.

Hamann's Criterion

Theorem (Eloise Hamann, 1975)

Let $A \subset B$ be an arbitrary integral extension.
A is seminormal in B if and only if

$$b \in B, b^2, b^3 \in A \Rightarrow b \in A$$

Example

Let K be a field and X an indeterminate. Then,

- $K[X^2, X^3]$ is not seminormal in $K[X]$.
- $K[X^2]$ is seminormal in $K[X]$.

The Ring of the Ordinary Double Point is Seminormal

Example

Consider $\mathbb{C}[X,Y]/(Y^2 - X^3 - 3X^2) =: \mathbb{C}[x,y]$ and its normalization $\overline{\mathbb{C}[x,y]} = \mathbb{C}[y/x]$. The maximal ideals of $\mathbb{C}[x,y]$ are of the form $(x-a, y-b)$, where $b^2 = a^3 + 3a^2$.
Note that $(y/x - \sqrt{3})$ and $(y/x + \sqrt{3})$ in $\mathbb{C}[y/x]$ both lie over (x,y) in $\mathbb{C}[x,y]$. Since

$$(y/x - \sqrt{3}) \cap (y/x + \sqrt{3}) = ((y/x)^2 - 3)\mathbb{C}[x/y]$$

and

$$((y/x)^2 - 3)\mathbb{C}[x/y] = (x,y)\mathbb{C}[x,y] \subset \mathbb{C}[x,y]$$

and all other points on the curve are nonsingular, $\mathbb{C}[x,y]$ is seminormal.

Subintegral Extensions

Definition

An extension of rings $A \subset B$ is *subintegral* if

1. B is integral over A;
2. for each prime ideal of A there is a unique prime ideal of B lying over it; and
3. the residue field extensions are isomorphisms.

These were first studied and called *quasi-isomorphisms* by Silvio Greco and Carlo Traverso in 1980. Richard Swan was first to call them subintegral extensions in the same year.

The Cusp

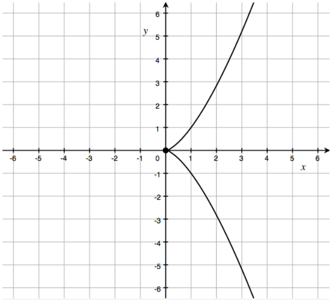


Figure: $Z(X^3 - Y^2)$

Recall that this plane curve has a singularity at the origin called a cusp. Consider the affine coordinate ring and its normalization $\mathbb{C}[X,Y]/(X^3 - Y^2) =: \mathbb{C}[x,y] \subset \mathbb{C}[y/x] = \overline{\mathbb{C}[x,y]}$. This is a subintegral extension, since the nonzero primes of $\mathbb{C}[x,y]$ are of the form $(x-a, y-b)$, where $a^3 = b^2$, and the unique prime of $\mathbb{C}[y/x]$ lying over $(x-a, y-b)$ is (y/x) if $a = 0$ and $(y/x - b/a)$ if $a \neq 0$.

Clarification

Let $A \subset B$ be an integral extension of rings.

- A is seminormal in $B \Leftrightarrow A$ DOES NOT admit any proper subintegral extension in B (so A is "subintegrally closed" in B).

Traverso's Characterization

Theorem (Traverso, 1970)

Let $A \subset B$ be an integral extension of rings. Then,

1. $A \subset {}^{+}_{B}A$ *is a subintegral extension.*
2. ${}^{+}_{B}A$ *is seminormal in* B.
3. ${}^{+}_{B}A$ *is the unique largest subintegral extension of* A *in* B.

Theorem (Leahy-Vitulli, 1981)

- *If $A \subset B$ is a seminormal extension and S is any multiplicative subset of A, then $S^{-1}A \subset S^{-1}B$ is seminormal.*
- *The operations of seminormalization and localization commute.*
- *A is seminormal in $B \Leftrightarrow A_{\mathfrak{m}}$ is seminormal in $B_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of A.*
- *The local ring of an algebraic variety at a point is seminormal $\Leftrightarrow$ its completion is seminormal.*

Weakly Subintegral Extensions

Notice that if $K \subset L$ is an algebraic extension of fields, then ${}^+_L K = K$ from the original definition. Alternately, K is seminormal in L by Hamann's criterion. Since there are no nontrivial subintegral extensions of fields, there is little hope of giving an elementwise characterization of subintegrality. Thus we switch gears at this point and talk about weakly subintegral extensions.

Definition

An extension of rings $A \subset B$ is *weakly subintegral* if

1. B is integral over A;
2. for each prime ideal of A there is a unique prime ideal of B lying over it; and
3. the residue field extensions are purely inseparable.

Provisional Elementwise Definition

Example

Recall that $F_p(X^p) \subset F_p(X)$ is a purely inseparable extension of fields. Thus $F_p(X^p) \subset F_p(X)$ is a weakly subintegral extension of rings.

Definition

Let $A \subset B$ be an extension of rings and $b \in B$.
We say b is *weakly subintegral* over A provided that
$A \subset A[b]$ is a weakly subintegral extension.

New Criterion for Weak Subintegrality

Lemma (Vitulli, 2011)

Suppose that $K \subset L$ is an extension of fields and $x \in L$. Then, x is weakly subintegral over K $\Leftrightarrow$ x is a common root of some monic polynomial $F(T) \in K[T]$ of positive degree n and its first $\lfloor \frac{n}{2} \rfloor$ derivatives.

Theorem (Vitulli, 2011)

Let $A \subset B$ be an extension of rings and $x \in B$. Then, x is weakly subintegral over A $\Leftrightarrow$ there is some monic polynomial $F(T) \in A[T]$ of positive degree n such that x is a common root of $F(T)$ and its first $\lfloor \frac{n}{2} \rfloor$ derivatives.

Thanks!

I'm happy to answer questions after the talk.