Contemporary Mathematics

Weak normalization and weak subintegral closure

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Abstract. A few years ago Reid, Roberts, and Singh defined and developed an elementwise description of weak subintegrality. Building on this notion, Vitulli and Leahy defined the notion of the weak subintegral closure $^*I$ of an ideal I in a general commutative ring with identity. Reid and Vitulli then gave an algebraic characterization of the weak subintegral closure of a monomial ideal over a field of arbitrary characteristic and a geometric characterization in characteristic zero. In this note we sketch the history of the operations of weak normalization and seminormalization and recall the more recent notion of the weak subintegral closure of an ideal. After this historical journey we focus our attention on monomial ideals in a polynomial ring over a field of characteristic 0. We show that every integral point in the relative interior of the Newton polyhedron is the exponent of a monomial in $^*I$. We then prove that a conjecture of Lantz relating weak subintegral closure and Rees valuations is true for monomial ideals that are primary for the irrelevant maximal ideal. We give examples that illustrate our results.

1. Background

The operation of weak normalization has stimulated much work and interest since the late sixties when it was introduced in the complex-analytic setting by Andreotti and Norguet [**AN**] and in the scheme-theoretic setting by Andreotti and Bombieri [**AB**]. The notion arose from a classification problem. Often when one tries to parameterize algebraic objects associated with a space by an algebraic variety, the resulting variety may not be uniquely determined and may, for example, depend on the choice of coordinates. Under certain conditions the normalization of the parameter variety is unique. However, the normalization may no longer parameterize what it was intended to, since one point on the original parameter variety may split into several on the normalization. This problem is avoided by passing instead to the weak normalization of the parameter variety. One then obtains a variety homeomorphic to the original variety with universal mapping properties that guarantee uniqueness. Indeed, the author [**V**] showed that any birational morphism of varieties that is a homeomorphism of underlying topological spaces and whose target space is a weakly normal variety without one-dimensional components is an isomorphism of varieties. So the weak normalization of a variety

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X without one-dimensional components is the largest variety that is birationally equivalent and homeomorphic to X.

The closely related operation of seminormalization was introduced in an algebraic setting by Traverso [**T**] in 1970. The seminormalization of a commutative ring A in an integral extension B is a ring whose prime spectrum is in bijective correspondence with $\mathrm{Spec}(A)$ and with isomorphic residue class fields. Traverso's definition was local in nature and can be used to define the seminormalization of an algebraic variety or a noetherian scheme. Translating the scheme-theoretic definition in [**AB**] to a purely algebraic setting, we see that the weak normalization of a commutative ring A in an integral extension B is a ring whose prime spectrum is in bijective correspondence with $\mathrm{Spec}(A)$ and with purely inseparable residue class field extensions. As these characterizations suggest, the weak normalization and the seminormalization of a complex algebraic variety coincide; the common variety is homeomorphic to the original variety but its sheaf of regular functions is possibly enlarged. Varieties or rings that are equal to their weak normalizations (respectively, seminormalizations) are called weakly normal (respectively, seminormal).

In Traverso's seminal paper [**T**] it was shown that for a reduced Noetherian ring A with finite normalization, every finitely-generated projective $A[t]-$module of constant rank one is extended from a finitely-generated projective $A-$module of constant rank one if and only if A is seminormal. Swan [**S**] refined Traverso's definition of a seminormal ring and showed that A_{red} is seminormal if and only if the above-mentioned condition holds. Swan's definition of a seminormal ring is equivalent to Traverso's for rings whose total quotient rings are a product of fields.

Another important feature of weakly normal complex algebraic varieties is that their singularities are somewhat constrained. The singularities of a weakly normal complex algebraic curve are ordinary multiple points. In higher dimensions the most common singularity on a weakly normal complex algebraic variety is a "multicross" singularity; a point x on a variety X is a multicross if X at x is analytically isomorphic to a finite union of linearly disjoint linear subspaces of affine space at the origin. Vitulli and Leahy [**LV**] showed that the complement of the set of multicrosses on a weakly normal variety is a closed subvariety of codimension at least two.

The original definitions of the weak normalization and seminormalization of an affine domain involved collecting all elements of the normalization that satisfied certain local conditions. There were global elementwise criteria (due to Hamann [**H**] for seminormality and Yanagihara [**Y**] for weak normality) for determining when a ring is seminormal or weakly normal but no analagous characterization of which elements are included in the seminormalization or weak normalization. Hamann [**H**] proved that a ring A is seminormal in an overring B if and only if A contains each element of B whose square and cube are in A. Yanagihara [**Y**] showed that a ring A is weakly normal in an overring B if and only if A contains each element b of B such that either b^2 and b^3 are in A or b^p and pb are in A for some rational prime p. Recently, Roberts and Singh [**RS**] gave an elementwise characterization of those elements which comprise the weak normalization of a $\mathbb{Q}-$subalgebra A in an overring B. In a later paper [**RRS**] Reid, Roberts, and Singh eliminated the assumption that the base ring is a $\mathbb{Q}-$algebra and showed that an element is in the weak normalization of A in B if and only if the element satisfies a highly-structured sequence of equations of integral dependence.

Suppose one blows up a weakly normal complex algebraic variety along a closed subvariety. One might ask when is the blow up again weakly normal or what one can say about the blow up. From an algebraic viewpoint, the analogous questions are when is the Rees algebra of a weakly normal affine ring A with respect to an ideal I again weakly normal or what can one say about the weak normalization of the Rees algebra of a weakly normal affine ring A with respect to an ideal I of A. D.F. Anderson [**A**] showed that the Rees algebra of a seminormal domain with respect to an invertible ideal is again seminormal. Leahy and Vitulli [**LV**] observed that the Rees algebra of a seminormal domain with respect to a reduced ideal that is generated by a regular sequence is again seminormal. Until recently no general answers for these questions were known. In their 1999 paper [**VL**], Vitulli and Leahy adapted the Roberts-Singh element-wise characterization of the weak normalization of a ring in an overring to define what they termed the weak subintegral closure of an ideal $I \subseteq A$ in an extension ring B. Leahy and Vitulli showed that the direct sum of the weak subintegral closures of the powers of the ideal I in the original ring A is the weak normalization of the Rees ring $A[It]$. Thus the weak normalization of the Rees algebra of a ring A with respect to an ideal I is again weakly normal if and only if all powers of I are weakly subintegrally closed; we say such an ideal is weakly normal. The aforementioned result and subsequent terminology parallel the normal case. We recall that a normal ideal is one all of whose powers are integrally closed. It is well known that I is a normal ideal in a domain A if and only if the Rees algebra of A with respect to I is normal. Similarly, it is known that I is a weakly normal ideal of a domain A if and only if $A[It]$ is weakly normal (see [**VL**, Corollary 3.5]).

Searching for computable examples, Reid and Vitulli [**RV**] turned their attention to the class of monomial ideals in a polynomial ring over a field. It is well known that the integral closure of a monomial ideal I is again a monomial ideal whose exponent set is the set of integral points of the convex hull of the set of exponents of I; we believe that Tessier was the first to record this result. Reid and Vitulli showed that the weak subintegral closure of a monomial ideal is again monomial by first establishing that the weak normalization of a multi-graded ring is again multi-graded. They then gave algebraic characterizations of the weak subintegral closure of a monomial ideal both in characteristic 0 and characteristic p. Additionally, they gave a geometric characterization of the weak subintegral closure of a monomial ideal in characteristic 0. This geometric characterization is like the one known for the integral closure of a monomial ideal, but also has an “arithmetic” component to it. Using the geometric description we are able to calculate some examples of the weak subintegral closure of a monomial ideal in polynomial rings in a small number of indeterminates.

We now give a brief explanation of the content of this article. In Section 2 we establish some conventions and give the basic definition of the weak subintegral closure of an ideal. We state some basic properties of this construction. In Section 3 we recall the algebraic and geometric characterizations of the weak subintegral closure of a monomial ideal. Then we review the connection between the bounded facets of the Newton polyhedron of a monomial ideal that is primary for the irrelevant maximal ideal and Rees valuations of the ideal. Next we prove that a conjecture of D. Lantz is true in our setting. In Section 4 we provide several examples that illustrate our results.

2. Preliminaries

Conventions. All rings are assumed to be commutative with identity. If a set S is properly contained in a set T we write $S \subset T$. We let $\mathbb{N}$ denote the set of nonnegative integers, $\mathbb{R}_+$ the set of nonnegative real numbers, and $\mathbf{e}_1, \ldots, \mathbf{e}_n$ the standard basis vectors in $\mathbb{R}^n$.

We now recall the definition of the weak subintegral closure of an ideal introduced by the current author and J.V. Leahy in [**VL**]. One should consult their original paper for several equivalent characterizations of the weak subintegral closure of an ideal.

Definition 2.1. Let $A \subseteq B$ be an extension of rings. For an ideal I of A we say $b \in B$ is *weakly subintegral over* I if there exists an integer $q \in \mathbb{N}$ and elements $a_i \in I^i$ $(1 \leq i \leq 2q+1)$ such that

$$b^n + \sum_{i=1}^{n} \binom{n}{i} a_i b^{n-i} = 0 \quad (q+1 \leq n \leq 2q+1).$$

We let ${}_B^*I$ denote the set of all elements of B that are weakly subintegral over I. We call ${}_B^*I$ the *weak subintegral closure of* I *in* B. We define the *weak subintegral closure of* I to be the weak subintegral closure of I in A and denote this by ${}^*I$. We say I is *weakly subintegrally closed in* B if I is equal to ${}_B^*I$. We say I is *weakly subintegrally closed* if I is equal to ${}^*I$.

It was shown in [**VL**] that ${}_B^*I$ is an ideal in ${}_B^*A$, the weak normalization of A in B. In particular if A is a weakly normal integral domain, then ${}^*I$ is again an ideal of A. This is the case when dealing with ideals in a polynomial ring over a field. We always have containments

$${}^*I \subseteq \overline{I} \subseteq \sqrt{I},$$

where $\overline{I}$ denotes the integral closure of I. One useful consequence of the notion of the weak subintegral closure of an ideal is that the direct sum of the weak subintegral closures of the powers of an ideal I in a ring A describes the weak normalization of the Rees ring $A[IT]$. For the precise statement and proof one should consult [**VL**, Theorem 3.3]. This result was a motivating force behind the definition of the weak subintegral closure of an ideal.

3. Results for Monomial Ideals

In this section we describe the weak subintegral closure of a monomial ideal. We also discuss a conjecture by D. Lantz concerning Rees valuations and monomial ideals. For the remainder of this paper we will assume that k denotes a field, that $R = k[x_1, \ldots, x_n]$ denotes the polynomial ring in n indeterminates over k, and that $\mathfrak{m} = (x_1, \ldots, x_n)$.

To start we recall some results and notation that were established in [**RV**]. The first result gives an algebraic characterization of the weak subintegral closure of a monomial ideal in a polynomial ring over a field. The second result gives a geometric characterization when the coefficient field has characteristic 0. For definitions and fundamental results on polyhedra the reader can consult [**Z**]. By a monomial ideal we mean a nonzero ideal of the polynomial ring that can be generated by monomials.

PROPOSITION 3.1. *Let I be a monomial ideal in a polynomial ring $k[x_1, \dots, x_n]$.*

(a) *If $\operatorname{char}(k) = 0$, then ${}^*I$ is the monomial ideal generated by all monomials x^γ such that $x^{m\gamma} \in I^m$ for all sufficiently large positive integers m.*
(b) *If $\operatorname{char}(k) = p > 0$, then ${}^*I$ is the monomial ideal generated by all monomials x^γ such that $x^{p^m\gamma} \in I^{p^m}$ for some nonnegative integer m.*

We remind the reader that a monomial $x^\gamma \in \overline{I}$ if and only if $x^{m\gamma} \in I^m$ for some positive integer m. Thus over a field of characteristic 0 a monomial is in the integral closure of I if and only if some power falls in the corresponding power of I whereas a monomial is in the weak subintegral closure of I precisely when all sufficiently large powers are in the corresponding powers of I. Before we can state the geometric characterization we need to recall some notation.

NOTATION 3.2. For an arbitrary subset Γ of $\mathbb{N}^n$ we let

$$m \cdot \Gamma = \{\gamma_1 + \cdots + \gamma_m \ \mid \gamma_i \in \Gamma \ \ (i = 1, \dots, m)\}.$$

To a monomial ideal $I \subseteq k[x_1, \dots, x_n]$ we associate various subsets as follows. Let

$$\begin{aligned} \Gamma(I) &= \{\gamma \in \mathbb{N}^n \mid x^\gamma \in I\}; \\ \overline{\Gamma}(I) &= \{\gamma \in \mathbb{N}^n \mid m\gamma \in m \cdot \Gamma(I),\ \exists m \geq 1\}; \quad \text{and} \\ {}^*\Gamma(I) &= \{\gamma \in \mathbb{N}^n \mid m\gamma \in m \cdot \Gamma(I),\ \forall m \gg 0\}. \end{aligned}$$

We refer to $\Gamma(I)$ as the *exponent set of I*. We suppress I in the notation for each of these subsets if we are working with a fixed monomial ideal. From our earlier algebraic characterization of ${}^*I$ and remarks it is clear that ${}^*\Gamma$ is the exponent set of the monomial ideal ${}^*I$ and that $\overline{\Gamma}$ is the exponent set of $\overline{I}$. Considering Γ as a subset of $\mathbb{R}^n$ we let $\Sigma = \operatorname{conv}(\Gamma)$ denote the convex hull of Γ. This set is referred to as the *Newton polyhedron of I*.

DEFINITION 3.3. For a polyhedron P we define the relative interior of P by

$$\operatorname{relint}(P) := P - \cup E,$$

where the union is taken over all facets E of P.

We introduce one more definition and a lemma before we give the geometric characterization of the weak subintegral closure of a monomial ideal in characteristic 0.

DEFINITION 3.4. For a face F of the Newton polyhedron $\Sigma = \operatorname{conv}(\Gamma)$ of a monomial ideal, define

$${}^*F = \left\{ \mathbf{x} \in \operatorname{relint}(F) \ \mid \ \mathbf{x} = \sum n_i\gamma_i,\ \gamma_i \in F \cap \Gamma\ , n_i \in \mathbb{Z} \right\}.$$

That is, ${}^*F$ is the intersection of the lattice generated by $F \cap \Gamma$ with the relative interior of F.

LEMMA 3.5. *Let $\Sigma = \operatorname{conv}(\Gamma)$ be the Newton polyhedron of a monomial ideal and F be a face of Σ. Then,*

(a) *${}^*\Sigma = \operatorname{relint}(\Sigma) \cap \mathbb{Z}^n$, and*
(b) *for each $\mathbf{x} \in {}^*F$, there exist vectors $\gamma_i \in F \cap \Gamma$ and integers n_i such that $\mathbf{x} = \sum n_i\gamma_i$ and $\sum n_i = 1$.*

PROOF. The first assertion is priori true if we are working in a polynomial ring in one indeterminate. So assume $n \geq 2$. Notice that we can always find a $\mathbb{Z}-$basis for $\mathbb{Z}^n$ consisting of vectors in Γ. Start with $\gamma_1 = (a, c, *, \ldots, *) \in \Gamma$ such that $\gcd(a, c) = 1 = ax - cy$ for some $x, y \in \mathbb{N}$. Then set $\gamma_2 = (b, d, *, \ldots, *) = (x, y, 0, \ldots, 0) + m\gamma_1$ where m is chosen large enough so that $\gamma_2 \in \Gamma$. Now consider the matrix with columns $\gamma_1, \gamma_2, \mathbf{e}_3, \ldots, \mathbf{e}_n$. It has determinant 1 so its columns form a $\mathbb{Z}-$basis for $\mathbb{Z}^n$. If we add large enough integer multiples of γ_1 to the last $(n-3)$ columns we get a $\mathbb{Z}-$basis $\gamma_1, \ldots, \gamma_n$ for $\mathbb{Z}^n$ consisting of vectors in Γ. In particular, ${}^*\Sigma = \mathrm{relint}(\Sigma)$. Let F be a proper face of Σ cut out by a supporting hyperplane $H_{\omega,c} = \{\mathbf{x} \in \mathbb{R}^n \mid \omega \cdot \mathbf{x} = c\}$, with normal vector $\omega \in \mathbb{N}^n$; this is always possible by [**RV**, Lemma 4.5]. If $c > 0$ then whenever we write $\gamma \in F$ as an integral linear combination $\sum n_i\gamma_i$ of vectors $\gamma_i \in F \cap \Gamma$ then $c = (\sum n_i)c$ forces $\sum n_i = 1$. If $c = 0$ we can choose vectors $\delta_1, \delta_2 \in F \cap \Gamma$ and set $\delta_3 := \delta_1 + \delta_2 \in F \cap \Gamma$. We can add suitable integral multiples of $\delta_1 + \delta_2 - \delta_3$ to our representation $\mathbf{x} = \sum n_i\gamma_i$ so that the sum of the coefficients becomes 1. □

REMARK 3.6. In [**RV**] the additional requirement that $\sum n_i = 1$ was part of the definition of ${}^*F$. The above lemma shows that this condition may and shall be deleted.

THEOREM 3.7. *Let $\Gamma = \Gamma(I)$ be the exponent set of a monomial ideal I in a polynomial ring $k[x_1, \ldots, x_n]$ over a field k of characteristic 0. Then, an integral point $\gamma \in \mathrm{conv}(\Gamma)$ is in the exponent set of ${}^*I$ if and only if $\gamma \in \bigcup {}^*F$, where the union is taken over all faces F of $\mathrm{conv}(\Gamma)$.*

We will use this geometric characterization to construct examples in Section 4.

D. Lantz asked whether a monomial whose exponent is "strictly inside" the Newton polyhedron of a monomial ideal is in the weak subintegral closure of the ideal. We now show that his question has an affirmative answer.

COROLLARY 3.8. *Let $\Gamma = \Gamma(I)$ be the exponent set of a monomial ideal I in a polynomial ring $k[x_1, \ldots, x_n]$ over a field k of characteristic zero. Then, any monomial whose exponent is in the relative interior of the Newton polyhedron Σ of I is in ${}^*I$.*

PROOF. This follows immediately from Theorem 3.7 and Lemma 3.5. □

We now recall the notion of a Rees valuation.

DEFINITION 3.9. For an ideal I of a ring S define the function v_I on an element x of S by

$$\mathrm{v}_I(x) = \sup\{m \in \mathbb{N} \mid x \in I^m\}.$$

The asymptotic Samuel function $\overline{\mathrm{v}}_I$ is defined by

$$\overline{\mathrm{v}}_I(x) = \lim_{n\to\infty} \frac{\mathrm{v}_I(x^n)}{n}.$$

For justification that the latter function is well defined the reader can consult [**Sa**].

Rees [**R**] established the following relationship between the asymptotic Samuel function of an $\mathfrak{n}-$primary ideal I in a local ring $(S, \mathfrak{n})$ and a family of discrete rank one valuations.

THEOREM 3.10 (Rees). *Let $(S, \mathfrak{n})$ be a local ring and I an $\mathfrak{n}-$primary ideal. Then, there is a finite set $\{\mathrm{v}_1, \ldots, \mathrm{v}_s\}$ of discrete rank one valuations centered on S, such that*

$$\overline{\mathrm{v}}_I(x) = \min\left\{\frac{\mathrm{v}_i(x)}{\mathrm{v}_i(I)}\right\}_{i=1}^{s} \quad \textit{for any } x \in S,$$

where $\mathrm{v}(I) = \min\{\mathrm{v}(x) \mid x \in I\}$.

If this representation is irredundant, the valuations v_i *are unique up to equivalence; they are called the Rees valuations of the ideal* I.

For the remainder of this section we will further assume that $\mathrm{char}(k) = 0$ and that I is an $\mathfrak{m}-$primary monomial ideal. In this setting, by abuse of language, we refer to the Rees valuations of $IR_{\mathfrak{m}}$ as the Rees valuations of I.

In a recent talk D. Lantz conjectured that for an $\mathfrak{n}-$primary ideal in a 2-dimensional regular local ring $(S, \mathfrak{n})$, if $\mathrm{v}(a) > \mathrm{v}(I)$ for all Rees valuations v of I then $a \in {}^*I$. We will show that for an $\mathfrak{m}-$primanry monomial ideal I, Lantz's conjecture is true in arbitrary dimension.

In order to address Lantz's conjecture we need to review the connection between the bounded facets of the Newton polyhedron of a $\mathfrak{m}-$primary monomial ideal and the Rees valuations of the ideal. This relationship was first established in a general setting in [**KKMD**]. For a self-contained discussion of this relationship in the case at hand the reader can consult J. Gately's thesis [**G**].

Let $F = H_{\omega,c} \cap \Sigma$ be a facet of the Newton polyhedron $\Sigma = \mathrm{conv}(\Gamma)$ of I. We may assume that all the components of the normal vector ω for the supporting hyperplane are nonnegative integers by [**RV**, Lemma 4.5]. We use the normal vector ω to define a monomial valuation v on the quotient field K of $R := k[x_1, \ldots, x_n]$ as follows. First define v on monomials by

$$\mathrm{v}(x^\alpha) = \omega \cdot \alpha.$$

Then define v on $R \setminus \{0\}$ by

$$\mathrm{v}\left(\sum c_\alpha x^\alpha\right) = \min\{\mathrm{v}(x^\alpha)\},$$

where we take the minimum over all monomials occurring with nonzero coefficient. Finally, extend to $K \setminus \{0\}$ by defining

$$\mathrm{v}(f/g) = \mathrm{v}(f) - \mathrm{v}(g),$$

whenever $f, g \in R \setminus \{0\}$.

Notice that each bounded facet F of Σ has a normal vector ω with positive integer components (or else $\mathbf{x} + m\mathbf{e}_i \in F$, $\exists i, \forall \mathbf{x} \in F, m \gg 0$, contradicting the assumption that F is bounded) and F is the intersection of Σ with the supporting hyperplane $H_{\omega,c} = \{\mathbf{x} \in \mathbb{R}^n \mid \omega \cdot \mathbf{x} = c\}$, where the constant c is a postive integer. Recall that the unbounded facets are the intersections of Σ with the coordinate hyperplanes $H_i = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \cdot \mathbf{e}_i = 0\}$. The following representation of the Newton polyhedron as an intersection of half-spaces appeared in Gately's thesis [**G**, Theorem 3.30]. As explicated by Gately [**G**, Theorem 3.68] and stated formally below, if we only consider the bounded facets of the Newton polyhedron of a monomial ideal I, the above association defines a bijective correspondence between the bounded facets of Σ and the Rees valuations of I. We point out to the reader that Gately also describes an algorithmic procedure for determining the bounded facets and for calculating the Rees valuations. The bounded facets arise from certain vector space bases for $\mathbb{R}^n$ consisting of vectors from $\mathbb{N}^n$; Gately dubs these bases facet sets.

THEOREM 3.11 (Gately). *Let H_{ω_i,c_i} $(i = 1, \ldots, s)$ denote the bounded facets of Σ. With notation and assumptions as above,*

$$\Sigma = \bigcap_{i=1}^{s} (H_{\omega_i,c_i})_+ \cap \bigcap_{j=1}^{n} (H_j)_+ ,$$

where for a hyperplane $H = H_{\omega,c} = \{\mathbf{x} \in \mathbb{R}^n \mid \omega \cdot \mathbf{x} = c\}$ we let H_+ denote the half-space defined by $H_+ = \{\mathbf{x} \in \mathbb{R}^n \mid \omega \cdot \mathbf{x} \geq c\}$. Furthermore, if we let v_i denote the monomial valuation associated with ω_i $(i = 1, \ldots, s)$ then, $\mathrm{v}_1, \ldots, \mathrm{v}_s$ are the Rees valuations of I.

From this description of the Newton polyhdedron as the intersection of half-spaces and the correspondence between the Rees valuations of I and the bounded facets of Σ we obtain the following description of Σ.

LEMMA 3.12. *Let $I \subset R$ be an $\mathfrak{m}-$primary monomial ideal, $f \in R$, and let $\mathfrak{V}_R(I)$ denote the set of Rees valuations of I. If $\mathrm{v}(f) \geq \mathrm{v}(I)\ \forall\, \mathrm{v} \in \mathfrak{V}_R(I)$, then $f \in \overline{I}$.*

Notice that if we intersect the Newton polyhedron Σ with a coordinate hyperplane H_j we get the Newton polyhedron Σ_j of the intersection of the original monomial ideal I with the subring $R_j = k[x_1, \ldots, \hat{x}_j, \ldots, x_n]$. Let $I_j = I \cap R_j$. Notice that I_j is generated by the monomials in I that do not involve x_j. Also notice that

$$\Sigma \cap H_j = \bigcap_{i=1}^{s} \Big((H_{\omega_i,c_i})_+ \cap H_j \Big) \cap \bigcap_{i \neq j} \left((H_i)_+ \cap H_j \right) .$$

If we write $\overline{\mathbf{x}}$ for the orthogonal projection of $\mathbf{x}$ onto the coordinate hyperplane H_j then we have $(H_{\omega_i,c_i})_+ \cap H_j = (H_{\overline{\omega}_i,c_i})_+$, where we view the right-hand side as a half-space in $\mathbb{R}^{n-1}$. After we eliminate redundancies we have the irredundant expression of Σ as an intersection of half-spaces. Further note that if $\Sigma_j \cap H_{\overline{\omega}_i,c_i}$ is again a bounded facet of Σ_j, then regarding R_j as a subring of R, we see that the Rees valuation corresponding to this bounded facet is simply the restriction of the Rees valuation corresponding to $\Sigma \cap H_{\omega_i,c_i}$ to the subring R_j.

We now show that the appropriate generalization of Lantz's conjecture is true for $\mathfrak{m}-$primary monomial ideals.

PROPOSITION 3.13. *Let $I \subseteq R$ be an $\mathfrak{m}-$primary monomial ideal, $f \in R$, and let $\mathfrak{V}_R(I)$ denote the set of Rees valuations of I. If $\mathrm{v}(f) > \mathrm{v}(I)\ \forall\, \mathrm{v} \in \mathfrak{V}_R(I)$, then $f \in {}^*I$.*

PROOF. It suffices to show that the theorem is true for monomials. For this assertion we proceed by induction of n, the case $n = 1$ being clear. Suppose that the integer $n > 1$ and the result is true for polynomial rings in fewer than n indeterminates. Assume that $\mathrm{v}(x^\alpha) > \mathrm{v}(I)\ \forall \mathrm{v} \in \mathfrak{V}_R(I)$. Then $\alpha \in \Sigma \setminus \bigcup_{i=1}^{s} \left(H_{\omega_i,c_i} \cap \Sigma\right)$, where $\{H_{\omega_i,c_i} \cap \Sigma\}_{i=1}^{s}$ denotes the bounded facets of Σ. If $\alpha \in \mathrm{relint}(\Sigma)$, then $\alpha \in {}^*I$ by Lemma 3.5. So assume that $\alpha \notin \mathrm{relint}(\Sigma)$, that is, $\alpha \in \Sigma \cap H_j$ for some coordinate hyperplane H_j. Then we have $\mathrm{v}(x^\alpha) > \mathrm{v}(I \cap R_j)$ for all Rees valuations v of $I \cap R_j$ so that $x^\alpha \in {}^*I \cap R_j$ by our induction hypothesis. $\square$

4. Examples

Thoroughout this section I denotes a monomial ideal in a polynomial ring over a field of characteristic 0, $\mathfrak{m}$ denotes the irrelevant maximal ideal, $\Gamma = \Gamma(I)$, and $\Sigma = \mathrm{conv}(\Gamma)$.

EXAMPLE 4.1. Let $I = (x^6, x^2y^4, y^6) \subseteq \mathfrak{m} = (x, y) \subseteq k[x, y]$. The 2-dimensional face Σ, the edges E_1, E_2, E_3, and vertices V_1, V_2 of Σ are depicted below. The thick-lined staircase figure bounds Γ and points depicted by open circles are integral points in $\Sigma \setminus \Gamma$. By Theorem 3.7 we know that $\overline{\Gamma}$ and ${}^*\Gamma$ agree except possibly along the edge E_2. Only those integral points in E_2 that are in the lattice generated by $\Gamma \cap E_2$ are included in ${}^*\Gamma$. Thus we have

$$^*I = (x^6, x^2y^4, x^4y^2, y^6) \subset \overline{I} = \mathfrak{m}^6.$$

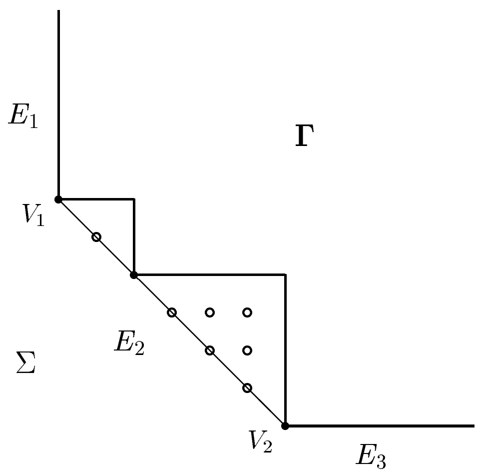
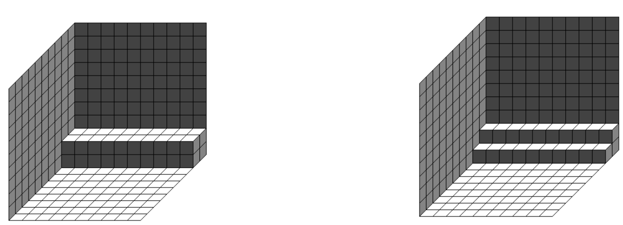


Figure 1.

EXAMPLE 4.2. Let $I = (x^2, z^2) \subseteq k[x, y, z]$. Note that Γ looks like the positive octant in 3-space (with coordinates a, b, c) with an infinite (in the y-direction) rectangular solid removed. The convex hull adds the positive lattice points in the plane $P = \{(a, b, c)|a + c = 2\}$. Thus we have $\overline{I} = (x^2, xz, z^2)$. As above, $\overline{\Gamma}$ and ${}^*\Gamma$ agree except possibly along the facet $P \cap \Sigma$, where ${}^*\Gamma$ only includes those integral points in the group generated by $P \cap \Gamma$. Hence we don't get any new points in the weak subintegral closure of this ideal. We illustrate this below.



$\Gamma = {}^*\Gamma$ $\overline{\Gamma}$

Figure 2.

These truncated "staircase" diagrams of unit cubes represent monomials in the complement of the exponent set of the monomial ideal; when we say the cube is in the complement we mean its vertex $(a_{\min}, b_{\min}, c_{\min})$ is in the complement.

EXAMPLE 4.3. Let $I = (x_1^4, x_2^4, x_3^4, x_1x_2x_3) \subseteq k[x_1, x_2, x_3]$. It is easy to see that $x_1x_2x_3$ is the only degree 3 monomial in $\overline{I}$. Also observe that if $\mu = x_1^{a_1}x_2^{a_2}x_3^{a_3}$ is any degree 4 monomial then $\mu^4 = (x_1^4)^{a_1}(x_2^4)^{a_2}(x_3^4)^{a_3} \in I^4$ and hence $\mu \in \overline{I}$. Thus $\overline{I} = (x_1x_2x_3) + \mathfrak{m}^4$. Now let us calculate ${}^*I$. Each degree 4 monomial that involves all the indeterminates is a multiple of $x_1x_2x_3$ and hence is in ${}^*I$. The only other degree 4 monomials that are in $\overline{I} \setminus I$ involve exactly 2 of the indeterminates. None of them is in ${}^*I$ since ${}^*(x_i^4, x_j^4) = (x_i^4, x_j^4) + (x_i, x_j)^5$ by Theorem 3.7 and a straightforward calculation.

According to Gately's calculations (see [**G**, Example 3.70]), there are three Rees valuations of I corresponding to the three bounded facets of Σ. These are as follows:

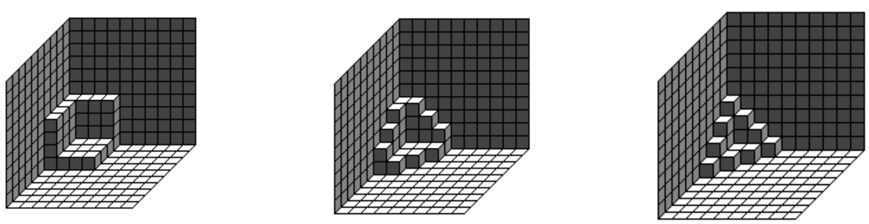
$$\mathrm{v}_i(x_1^{a_1}x_2^{a_2}x_3^{a_3}) = 2a_i + a_{i+1} + a_{i+2}\ (i = 1, 2, 3),$$

where the indices $i+1$ and $i+2$ are to be read modulo 3. The minimum value of each of these valuations on I is 4. The equations of the corresponding supporting hyperplanes are

$$2a_i + a_{i+1} + a_{i+2} = 4\ (i = 1, 2, 3).$$

The only integral points that lie on a bounded facet of Σ are the vertices of that facet and the points that are exponents of monomials of degree 4 involving exactly two indeterminates. As mentioned above, none of the latter points is in ${}^*I$. Now consider the unbounded facets $\Sigma \cap H_j\ (j = 1, 2, 3)$. Applying Theorem 3.7 to $I \cap R_1$ we see that the only integral points in ${}^*\Gamma \cap H_1 \setminus \Gamma \cap H_1$ are $(0, 2, 3), (0, 3, 3), (0, 3, 2)$. By symmetry we pick up three additional points in ${}^*\Gamma$ for each unbounded facet of Σ. Thus ${}^*I = I + \mathfrak{m}^5$.

Notice that if $\mathrm{v}_i(x^\alpha) > 4\ (i = 1, 2, 3)$ then either $x_1x_2x_3 | x^\alpha$ or $\deg(x^\alpha) \geq 5$. In either case $x^\alpha \in {}^*I$. We illustrate the Newton polyhedra of all three ideals below.



Γ $\quad$ ${}^*\Gamma$ $\quad$ $\overline{\Gamma}$

Figure 3.

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