Math 246 (9-10am), Midterm I.

Show all your work! Use the most efficient method you know!

0. Write your name here:

1. Find the derivative of $f(x) = \frac{x}{\sqrt{x^3+1}}$. What are the critical points of this function?

By algebra we have $f(x) = x(x^3+1)^{-1/2}$. Hence by Product Rule and Chain Rule:

$$f'(x) = (x^3+1)^{-1/2} + x \cdot \left(-\frac{1}{2}(x^3+1)^{-3/2}\right) \cdot 3x^2 =$$

$$= (x^3+1)^{-3/2}\left((x^3+1) + \left(-\frac{1}{2}\right)\cdot 3x^3\right) = (x^3+1)^{-3/2}\left(1 - \frac{1}{2}x^3\right).$$

To find critical points we solve equation $f'(x) = 0$:

$$(x^3+1)^{-3/2}\left(1 - \frac{1}{2}x^3\right) = 0, \quad 1 - \frac{1}{2}x^3 = 0, \quad 1 = \frac{1}{2}x^3, \quad x^3 = 2, \quad x = \sqrt[3]{2}$$

Answer: $f'(x) = (x^3+1)^{-3/2}\left(1 - \frac{x^3}{2}\right)$; the critical point $x = \sqrt[3]{2}$.

Comment: ~~As~~ the derivative $f'(x)$ is not defined at $x=-1$. However this is not critical point as $x=-1$ is not in the domain of $f(x)$.

2. Find the second derivative of $g(x) = \sin(x^2)$.

For the first derivative we use Chain Rule:

$$g'(x) = \cos(x^2) \cdot 2x = 2x\cos(x^2).$$

For the second derivative we use Product Rule and Chain Rule:

$$g(x)'' = (2x\cos(x^2))' = 2\cos(x^2) + 2x(\cos(x^2))' =$$

$$= 2\cos(x^2) + 2x(-\sin(x^2)) \cdot 2x =$$

$$= 2\cos(x^2) - 4x^2\sin(x^2).$$

Answer: $g''(x) = 2\cos(x^2) - 4x^2\sin(x^2)$.

3. Find the equation of the tangent line to the curve $x^3 + 2y^3 = 10$ at the point $(2, 1)$.

The point $(2,1)$ is on the given curve since $2^3 + 2\cdot(1)^3 = 8+2=10$. We use implicit differentiation to find the slope of the tangent line:

$3x^2 + 2\cdot 3y^2 \cdot y' = 0$, $6y^2\cdot y' = -3x^2$, $y' = -\frac{3x^2}{6y^2} = -\frac{x^2}{2y^2}$.

By plugging in the values $x=2$, $y=1$ we find:

$y' = -\frac{4}{2} = -2$; this is the slope of the tangent line.

The equation of the tangent line is $y-1=-2(x-2)$, equivalently $y+2x=5$.

Answer: the equation of the tangent line is $y+2x=5$.

4. Find the limit $\lim_{x\to 3} \frac{6x}{4x+3}$. How close the input must be to 3 for the output to be within 0.05 of the limit?

The function $\frac{6x}{4x+3}$ is defined at $x=3$, so it is continuous at this point. Hence

$$\lim_{x\to 3} \frac{6x}{4x+3} = \frac{6\cdot 3}{4\cdot 3+3} = \frac{18}{15} = \frac{6}{5} = 1.2$$

To answer the 2nd question we need to solve the inequalities $1.15 \le \frac{6x}{4x+3} \le 1.25$. We have:

$\frac{6x}{4x+3} \le 1.25$, $6x \le 1.25(4x+3)$, $6x \le 5x+3.75$, $x \le 3.75$

$1.15 \le \frac{6x}{4x+3}$, $1.15(4x+3) \le 6x$, $4.6x + 3.45 \le 6x$, $3.45 \le 1.4x$, $x \ge \frac{3.45}{1.4} = 2.464$

Answer: $\lim_{x\to 3} \frac{6x}{4x+3} = 1.2$; the input must be in the interval $[2.464, 3.75]$ for the output to be within .05 of the limit.

5. The number of yeast cells in a laboratory culture is modeled by the function

$$n(t) = \frac{2000}{1 + 19e^{-0.1t}}$$

where the time t is measured in hours. What is (instantaneous) rate of change of the size of population at time $t = 0$? Answer the same question with $t = 10$ and $t = 100$.

We need to find $n'(0)$, $n'(10)$ and $n'(100)$.

We compute by the Quotient Rule and Chain Rule:

$$n'(t) = \frac{(2000)'(1+19e^{-.1t}) - 2000(1+19e^{-.1t})'}{(1+19e^{-.1t})^2} = \frac{-2000 \cdot 19 \cdot e^{-.1t} \cdot (-.1)}{(1+19e^{-.1t})^2} = \frac{380e^{-.1t}}{(1+19e^{-.1t})^2}$$

Thus we have:

$$n'(0) = \frac{380 \cdot 1}{(1+19 \cdot 1)^2} = \frac{380}{20^2} = \frac{380}{400} = 9.5 \text{ cells/hour}$$

$$n'(10) = \frac{3800e^{-1}}{(1+19e^{-1})^2} \approx 21.899 \text{ cells/hour}$$

$$n'(100) = \frac{3800e^{-10}}{(1+19e^{-10})^2} \approx 0.172 \text{ cells/hour}$$

Answer: $n'(0) = 9.5$ cells/hour, $n'(10) = 21.9$ cells/hour, $n'(100) = .17$ cells/hour

<u>Comment</u>: from the numerical data it seems that the rate of change is increasing at first and then it starts to decrease. One can ask: what is the maximal rate of change? The answer is given by calculus (again): we have to find the critical point of $n'(t)$ (i.e. inflection point of $n(t)$). With some computations one finds that the maximal rate of change is 50 cells/hour, it occurs at $t = 29.444$ hours.