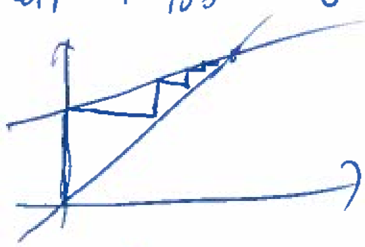
Math 246 (9-10am), Midterm II.

Show all your work! Use the most efficient method you know!

0. Write your name here:

1. Assume that the concentration of a medication in the bloodstream is described by the dynamical system $M_{t+1} = 0.45M_t + 3.3$ with initial condition $M_0 = 0$. What is M_{100} approximately? Explain your guess!

It seems reasonable to expect that $M_{100} \approx$ stable fixed point. Thus we find updating function $f(x) = 0.45x + 3.3$ and set up fixed point equation $x = 0.45x + 3.3$ whence $0.55x = 3.3$, $x = 3.3/0.55 = 6$. Since $f'(x) = 0.45 < 1$, this fixed point is stable. Thus my guess is that $M_{100} \approx 6$; cobwebbing strongly supports this:



Answer: $M_{100} \approx 6$.

2. Find the fixed points and determine their stability for the dynamical system $a_{t+1} = \frac{1}{4}(a_t^2 + 2)$.

The updating function is $f(x) = \frac{1}{4}(x^2+2)$; the fixed point equation is $x = \frac{1}{4}(x^2+2)$, equivalently $4x = x^2 + 2$, or $x^2 - 4x + 2 = 0$. Using quadratic formula we find two roots $x_{1,2} = 2 \pm \sqrt{2}$.

For stability we compute $f'(x) = \frac{1}{4} \cdot 2x = \frac{x}{2}$.

We have $f'(2+\sqrt{2}) = \frac{2+\sqrt{2}}{2} = 1 + \frac{\sqrt{2}}{2} > 1$, so $x = 2+\sqrt{2}$ is unstable; $f'(2-\sqrt{2}) = \frac{2-\sqrt{2}}{2} = 1 - \frac{\sqrt{2}}{2} \approx 0.293 < 1$, so $x = 2-\sqrt{2}$ is stable.

Answer: we have two fixed points $x = 2+\sqrt{2}$ and $x = 2-\sqrt{2}$; the equilibrium $x = 2+\sqrt{2}$ is unstable; the equilibrium $x = 2-\sqrt{2}$ is stable.

3. Find local maxima and minima of $f(x) = x^4 - 8x^2$.

First we find $f'(x) = 4x^3 - 8\cdot 2x = 4x(x^2-4) = 4x(x-2)(x+2)$.

Thus we have 3 critical points: $x=0$, $x=2$, $x=-2$.

Next we find $f''(x) = 4\cdot 3x^2 - 16 = 12x^2 - 16$ and compute

$f''(0) = -16 < 0$, so $x=0$ is a local maximum

$f''(2) = 12\cdot 2^2 - 16 = 48 - 16 = 32 > 0$, so $x=2$ is a local minimum

$f''(-2) = 12(-2)^2 - 16 = 48 - 16 = 32 > 0$, so $x=-2$ is a local minimum

Answer: we have one local maximum $x=0$ and two local minima: $x=2$ and $x=-2$.

4. Find the global maximum of $f(t) = t(2-\sqrt{t})$ on the interval $[0,4]$.

1) We compute f at the endpoints: $f(0)=0$, $f(4)=0$.

2) We find critical points: since $f(t) = 2t - t^{3/2}$, we have $f'(t) = 2 - \frac{3}{2}t^{1/2}$. The critical point equation is $2 - \frac{3}{2}\sqrt{t} = 0$, whence $\frac{3}{2}\sqrt{t} = 2$, $\sqrt{t} = \frac{4}{3}$, $t = \frac{16}{9}$

3) We compute f at the critical point
$f\left(\frac{16}{9}\right) = \frac{16}{9}\left(2 - \sqrt{\frac{16}{9}}\right) = \frac{16}{9}\left(2 - \frac{4}{3}\right) = \frac{16}{9}\cdot\frac{2}{3} = \frac{32}{27}$.

4) Out of values computed in 1) and 3) the largest is $\frac{32}{27}$, the smallest is 0.

Answer: the global maximum is $\frac{32}{27} \approx 1.185$, attained at $x = \frac{16}{9} \approx 1.778$

5. <u>Use calculus</u> to solve the following problem: There are 60 apple trees in an orchard. Each tree produces 800 apples. For each additional tree planted in the orchard, the output per tree drops by 10 apples. How many trees should be added to the existing orchard in order to maximize the total output of trees ?

Assume that the number of trees added is X. Then we have $60+X$ trees in total with productivity $800-10X$. Thus the total productivity is $P(X)=(60+X)(800-10X)$ and we want to maximize it. We find derivative:

$$P'(x) = 1\cdot(800-10x) + (60+x)\cdot(-10) = 800-10x-600-10x =$$
$$= 200-20x = 20(10-x).$$

Thus the only critical point is $x=10$; this is point of maximum since $P'(x)$ is positive for $x<10$ and negative for $x>10$.

Answer: we should add 10 trees.