Math 246 (11-12am), Midterm I.

Show all your work! Use the most efficient method you know!

0. Write your name here:

1. Find the limit $\lim_{x \to 2} \frac{2x}{3x+4}$. How close the input must be to ~~3~~ 2 for the output to be within 0.05 of the limit?

The function $\frac{2x}{3x+4}$ is defined at $x=2$, so it is continuous at this point. Hence

$$\lim_{x \to 2} \frac{2x}{3x+4} = \frac{2 \cdot 2}{3 \cdot 2 + 4} = \frac{4}{10} = .4$$

To answer the second question we need to solve the inequalities $.35 \le \frac{2x}{3x+4} \le .45$. We have

$.35 \le \frac{2x}{3x+4}$, $.35(3x+4) \le 2x$, $1.05x + 1.4 \le 2x$, $.95x \ge 1.4$, $x \ge \frac{1.4}{.95} = 1.474$

$\frac{2x}{3x+4} \le .45$, $2x \le .45(3x+4)$, $2x \le 1.35x + 1.8$, $.65x \le 1.8$, $x \le \frac{1.8}{.65} = 2.769$

Answer: $\lim_{x \to 2} \frac{2x}{3x+4} = .4$; the input must be in the interval $[1.474, 2.769]$ for the output to be within $.05$ of the limit.

2. Find the second derivative of $g(x) = \cos(e^x)$.

For the first derivative we use Chain Rule:

$$g'(x) = -\sin(e^x) \cdot e^x = -e^x \sin(e^x)$$

For the second derivative we use Product Rule and Chain Rule:

$$g''(x) = -(e^x)' \sin(e^x) - e^x \cdot (\sin(e^x))' = -e^x \sin(e^x) - e^x \cdot \cos(e^x) \cdot e^x = -e^x \sin(e^x) - e^{2x} \cos(e^x).$$

Answer: $g''(x) = -e^x \sin(e^x) - e^{2x} \cos(e^x)$.

3. Find the derivative of $f(x) = \frac{x}{\sqrt{x^3+3}}$. What are the critical points of this function?

By algebra we have $f(x) = x\,(x^3+3)^{-1/2}$. Thus by Product Rule and Chain Rule we have

$f'(x) = (x^3+3)^{-1/2} + x\cdot\left(-\frac{1}{2}(x^3+3)^{-3/2}\right)\cdot 3x^2 =$

$= (x^3+3)^{-3/2}\left(x^3+3 - \frac{3}{2}x^3\right) = (x^3+3)^{-3/2}\left(3 - \frac{1}{2}x^3\right).$

To find critical points we solve equation $f'(x) = 0$

$(x^3+3)^{-3/2}\left(3 - \frac{1}{2}x^3\right) = 0$, $3 - \frac{1}{2}x^3 = 0$, $\frac{1}{2}x^3 = 3$, $x^3 = 6$, $x = \sqrt[3]{6}$

Answer: $f'(x) = (x^3+3)^{-3/2}\left(3 - \frac{1}{2}x^3\right)$; the critical point is $x = \sqrt[3]{6}$

Comment: $f'(x)$ is not defined at $x = -\sqrt[3]{3}$. However this is not a critical point, since it is not in the domain of $f(x)$

4. Find the equation of the tangent line to the curve $x^3 + 2y^3 = 17$ at the point $(1, 2)$.

The point $(1,2)$ is on the given curve since $1^3 + 2\cdot(2^3) = 1 + 2\cdot 8 = 17$. We use implicit differentiation to find the slope of the tangent line:

$3x^2 + 2\cdot 3y^2\cdot y' = 0$, $6y^2\cdot y' = -3x^2$, $y' = -\frac{3x^2}{6y^2} = -\frac{x^2}{2y^2}$

By plugging in the values $x=1$, $y=2$ we find $y' = -\frac{1}{2\cdot 2^2} = -\frac{1}{8}$; this is the slope of the tangent line

The equation of the tangent line is $y - 2 = -\frac{1}{8}(x-1)$. Equivalently $x + 8y = 17$.

Answer: the equation of the tangent line is $x + 8y = 17$.

5. The number of yeast cells in a laboratory culture is modeled by the function

$$n(t) = \frac{2000}{1+9e^{-0.1t}}$$

where the time t is measured in hours. What is (instantaneous) rate of change of the size of population at time $t = 0$? Answer the same question with $t = 10$ and $t = 100$.

We need to compute $n'(0), n'(10), n'(100)$. We use Quotient Rule and Chain Rule:

$$n'(t) = \frac{(2000)'(1+9e^{-.1t}) - 2000(1+9e^{-.1t})'}{(1+9e^{-.1t})^2} = \frac{-2000\cdot 9\cdot e^{-.1t}\cdot(-.1)}{(1+9e^{-.1t})^2} = \frac{1800e^{-.1t}}{(1+9e^{-.1t})^2}$$

Thus we have

$$n'(0) = \frac{1800\cdot 1}{(1+9\cdot 1)^2} = \frac{1800}{10^2} = \frac{1800}{100} = 18 \text{ cells/hour}$$

$$n'(10) = \frac{1800e^{-1}}{(1+9e^{-1})^2} \simeq 35.632 \text{ cells/hour}$$

$$n'(100) = \frac{1800e^{-10}}{(1+9e^{-10})^2} \simeq 0.082 \text{ cells/hour}$$

Answer: the rates of change are 18, 35.63, .082 cells/hour

Comment: from the numerical data it seems that the rate of change is increasing at first and after a while it starts decreasing. One can ask: when the growth is fastest? The answer is given by calculus (again): we have to find the critical point of $n'(t)$ (i.e. inflection point of $n(t)$). With some computations one finds that the maximal rate of change is 50 cells/hour, it occurs at $t = 21.972$ hours.