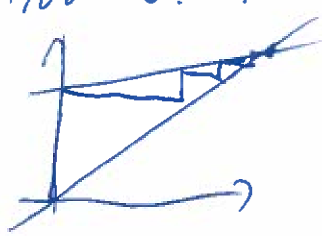
# Math 246 (11-12am), Midterm II.

*Show all your work! Use the most efficient method you know!*

0. Write your name here:

1. Assume that the concentration of a medication in the bloodstream is described by the dynamical system $M_{t+1} = 0.55M_t + 2.7$ with initial condition $M_0 = 0$. What is $M_{100}$ approximately? Explain your guess!

It seems reasonable to expect that $M_{100} \approx$ stable equilibrium. We find updating function $f(x) = 0.55x + 2.7$ and set up fixed point equation $x = 0.55x + 2.7$ whence $0.45x = 2.7$, $x = 2.7/0.45 = 6$. Note that $f'(x) = 0.55$, so the ~~stable~~ equilibrium is stable, and we expect $M_{100} \approx 6$. This is also confirmed by cobwebbing:



Answer: $M_{100} \approx 6$

2. Find the global maximum of $f(t) = (2-t)\sqrt{t}$ on the interval $[0,2]$.

1) We compute $f$ at endpoints: $f(0) = 0$, $f(2) = 0$.

2) We find critical points: $f(t) = 2\sqrt{t} - t\sqrt{t} = 2t^{1/2} - t^{3/2}$, so $f'(t) = 2 \cdot \frac{1}{2} t^{-1/2} - \frac{3}{2} t^{1/2} = t^{-1/2} - \frac{3}{2} t^{1/2}$. Critical points equation: $t^{-1/2} - \frac{3}{2} t^{1/2} = 0$, $t^{-1/2} = \frac{3}{2} t^{1/2}$, $1 = \frac{3}{2} t$, $t = \frac{2}{3}$.

3) We compute $f$ at the critical points:

$f(\frac{2}{3}) = (2 - \frac{2}{3})\sqrt{\frac{2}{3}} = \frac{4}{3}\sqrt{\frac{2}{3}}$

$t=0$ is also a critical point as $f'(0)$ is undefined, but this was covered in 1).

4) Clearly the largest of the numbers computed in 1) and 3) is $\frac{4}{3}\sqrt{\frac{2}{3}}$.

Answer: the global maximum is $\frac{4}{3}\sqrt{\frac{2}{3}}$, attained at $t = \frac{2}{3}$.

3. Find the fixed points and determine their stability for the dynamical system $a_{t+1} = \frac{1}{4}(a_t^2 + 1)$.

The updating function is $f(x) = \frac{1}{4}(x^2+1)$; the fixed point equation $x = \frac{1}{4}(x^2+1)$ or $4x = x^2+1$, or $x^2 - 4x + 1 = 0$. Using quadratic formula we find $x_{1,2} = 2 \pm \sqrt{3}$.

For stability we find $f'(x) = \frac{1}{4} \cdot 2x = \frac{x}{2}$.

Since $f'(2+\sqrt{3}) = \frac{2+\sqrt{3}}{2} = 1 + \frac{\sqrt{3}}{2} > 1$, the equilibrium $x = 2+\sqrt{3}$ is unstable

Since $f'(2-\sqrt{3}) = \frac{2-\sqrt{3}}{2} = 1 - \frac{\sqrt{3}}{2} \simeq 0.134 < 1$, the equilibrium $x = 2-\sqrt{3}$ is stable.

Answer: we have 2 fixed points: unstable $x = 2+\sqrt{3}$ and stable $x = 2-\sqrt{3}$.

4. Find local maxima and minima of $f(x) = x^4 - 2x^2$.

We compute $f'(x) = 4x^3 - 2 \cdot 2x = 4x(x^2-1) = 4x(x-1)(x+1)$.

Thus the critical points are $x=0$, $x=1$, $x=-1$

We compute $f''(x) = 4 \cdot 3x^2 - 4 = 12x^2 - 4$.

Since $f''(0) = -4 < 0$, we see that $x=0$ is a local maximum

Since $f''(1) = 12 - 4 = 8 > 0$, we see that $x=1$ is a local minimum

Since $f''(-1) = 12(-1)^2 - 4 = 8 > 0$, we see that $x=-1$ is a local minimum

Answer: we have two local minima $x=1$ and $x=-1$ and one local maximum $x=0$.

5. Use calculus to solve the following problem: There are 60 apple trees in an orchard. Each tree produces 900 apples. For each additional tree planted in the orchard, the output per tree drops by 10 apples. How many trees should be added to the existing orchard in order to maximize the total output of trees ?

Let us denote the number of trees added by $x$. Then we will have $60+x$ trees, and productivity per tree is $900-10x$. The total output is given by $P(x) = (60+x)(900-10x)$. We want to maximize $P(x)$. We have

$$P'(x) = 1\cdot(900-10x) + (60+x)\cdot(-10) = 900-10x-600-10x =$$
$$= 300-20x = 20(15-x).$$

Thus the only critical point is $x=15$. Since $P'(x)$ is positive for $x<15$ and negative for $x>15$, we see that $x=15$ is the maximum.

Answer: we should add 15 trees.