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MATHEMATICS

Dijkgraaf–Witten Invariants over $\mathbb{Z}_2$ for 3-Manifolds

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1. DIJKGRAAF–WITTEN INVARIANTS

In 1990, Dijkgraaf and Witten [1] proposed a new approach to constructing invariants of closed topological manifolds; for our purposes, it is convenient to describe this approach as follows. Consider a finite group G and its classifying space $B = B(G)$. Let M be a closed connected 3-manifold of dimension $n \geq 1$. We choose two base points, in M and in B, and consider the set $S = S(M, B)$ of all pointed (i.e., preserving the base points) continuous maps $M \to B$. We consider these maps up to pointed homotopies. Note that S can be naturally identified with the set $\mathrm{Hom}(\pi_1(M), G)$ of all homomorphisms of the group $\pi_1(M)$ to G. Therefore, this set is finite.

Let U be a subgroup of the multiplicative group $U(1)$ of all complex numbers with absolute value 1. Choose any element h in the group $H^n(G; U) = H^n(B; U)$. If M is oriented, then to each map $f \in S$ we can assign the value $\langle f^*(h), [M]\rangle \in U$ of the element $f^*(h)$ of $H^n(M; U)$ at the fundamental class $[M]$ of the manifold M.

Definition 1. The Dijkgraaf–Witten invariant $Z(M, h)$ of M is defined by

$$Z(M, h) = \frac{1}{|G|}\sum_{f \in S(M, B)} \langle f^*(h), [M]\rangle.$$

The values of this invariant are complex numbers.

Determining the Dijkgraaf–Witten invariants requires calculating and summing many terms, the number of which exponentially increases with the first Betti number of M.

In this paper, we consider only the special case where $n = 3$ and both groups G and U have order 2 and are identified with the group $\mathbb{Z}_2$. For the classifying space B of $\mathbb{Z}_2$ we can take the infinite-dimensional projective space RP^∞. Its cohomology ring with coefficients in $U = \mathbb{Z}_2$ is very simple: this is the polynomial ring in a variable α, which represents the unique nontrivial element of the group $H^1(B; \mathbb{Z}_2) = \mathbb{Z}_2$. In particular, the group $H^3(B; \mathbb{Z}_2)$ contains only one nontrivial element $h = \alpha^3$, too. In this situation, the value $\langle f^*(h)$, $[M]\rangle$ belongs to $\mathbb{Z}_2$, and the formula given above acquires the form

$$Z(M) = \frac{1}{2}\sum_{f \in S(M, B)} (-1)^{\langle f^*(\alpha^3), [M]\rangle}$$

and becomes applicable to nonorientable manifolds. If $H^1(M; \mathbb{Z}_2) = 0$, then $Z(M) = \frac{1}{2}$. In all other cases, the number of terms in the above sum is even; therefore, $Z(M)$ is an integer.

2. THE ARF INVARIANT

Let V be a finite-dimensional vector space over the field $\mathbb{Z}_2$. Recall that a function $q: V \to \mathbb{Z}_2$ is said to be quadratic if the pairing $\ell_q: V \times V \to \mathbb{Z}_2$ defined by $\ell_q(x, y) = q(x + y) - q(x) - q(y)$, where $x, y \in V$, is bilinear. Certainly, ℓ_q is symmetric, and $\ell_q(0, 0) = 0$. A function q is said to be non-degenerate if the annihilator A of ℓ_q, which consists of all $x \in V$ such that $\ell_q(x, V) = \ell_q(V, x) = 0$, is trivial, i.e., contains only 0. Apropos, any linear function $q: V \to \mathbb{Z}_2$ is quadratic but degenerate, because the annihilator of ℓ_q coincides with the entire space V. This refers to any quadratic function on a one-dimensional space.

For a nonsingular quadratic function $q: V \to \mathbb{Z}_2$, there always exists a basis $a_1, b_1, \ldots, a_k, b_k$ such that $\ell_q(a_i, a_j) = \ell_q(b_i, b_j) = 0$ and $\ell_q(a_i, b_j) = \delta_{i,j}$ for all $i, j = 1, 2, \ldots, k$; here, $\delta_{i,j}$ is the Kronecker symbol.

Definition 2. The element $\mathrm{Arf}(q)$ of the group $\mathbb{Z}_2$ defined by

$$\mathrm{Arf}(q) = \sum_{i=1}^{k} q(a_i)q(b_i)$$

is called the Arf invariant of q.

It is well known that the Arf invariant does not depend on the choice of the basis $a_1, b_1, \ldots, a_k, b_k$. We

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need a slight generalization of this definition to the case of a quadratic function q: $V \to \mathbb{Z}_2$ such that the annihilator $A \subset V$ of its bilinear pairing ℓ_q is nontrivial and the restriction $q|_A$ of q is identically zero. If $A \neq V$, then any basis $c_1, \ldots, c_m$ of A can be extended to a basis $a_1, b_1, \ldots, a_k, b_k, c_1, \ldots, c_m$ of V so that $\ell_q(a_i, a_j) = \ell_q(b_i, b_j) = 0$ and $\ell_q(a_i, b_j) = \delta_{i,j}$ for all $i, j = 1, 2, \ldots, k$. Then q induces a nondegenerate quadratic function q': $V/A \to \mathbb{Z}_2$, and we set $\mathrm{Arf}(q) = \mathrm{Arf}(q')$, i.e., use the same formula $\mathrm{Arf}(q) = \sum_{i=1}^{k} q(a_i)q(b_i)$. If $A = V$, then we set $\mathrm{Arf}(q) = 0$. Note that if $q|_A$ is not identically zero, then the value of $\sum_{i=1}^{k} q(a_i)q(b_i)$ may depend on the choice of the basis $a_1, b_1, \ldots, a_k, b_k, c_1, \ldots, c_m$, but we do not need it in this case.

3. MAIN THEOREM

Let M be a closed 3-manifold. Consider the function Q_M: $H^1(M; \mathbb{Z}_2) \to \mathbb{Z}_2$ defined by $Q_M(x) = \langle x^3, [M]\rangle$, where $x \in H^1(M; \mathbb{Z}_2)$, $[M]$ is the fundamental class of M, and $x^3 \in H^3(M; \mathbb{Z}_2)$ denotes the cube of x in the sense of multiplication in cohomology. We claim that Q_M is quadratic. To prove this, it suffices to note that, for any $x, y \in H^1(M; \mathbb{Z}_2)$, we have

$$Q_M(x+y) - Q_M(x) - Q_M(y) = \langle x^2y + xy^2, [M]\rangle$$

and the rule $\ell_M(x, y) = \langle x^2y + xy^2, [M]\rangle$ defines a bilinear pairing on $H^1(M; \mathbb{Z}_2)$. This follows from the obvious relation

$$\begin{aligned} &x^2(y_1+y_2) + x(y_1+y_2)^2 \\ &= (x^2y_1 + xy_1^2) + (x^2y_2 + xy_2^2), \end{aligned}$$

which means linearity in the second argument and, by symmetry, in the first.

Theorem 1. *Let M be a closed connected 3-manifold, and let $A \subset H^1(M; \mathbb{Z}_2)$ be the annihilator of the bilinear pairing ℓ_M corresponding to the function Q_M. If there exists an element $x \in A$ such that $x^3 \neq 0$, then $Z(M) = 0$. If there are no such elements, then $Z(M) = 2^{k+m-1}(-1)^{\mathrm{Arf}(Q_M)}$, where m is the dimension of A and k equals half the dimension of the quotient space $H^1(M; \mathbb{Z}_2)/A$.*

4. GAUSSIAN SUMS

Let V be a finite-dimensional vector space over $\mathbb{Z}_2$, and let q: $V \to \mathbb{Z}_2$ be a quadratic function. To this function we assign the integer $\sigma(q) = \sum_{x \in V} (-1)^{q(x)}$. This expression is a special case of a nonnormalized Gaussian sum; see, e.g., [2]. The main property of such sums is their multiplicativity. Suppose that a quadratic function q: $V_1 \oplus V_2 \to \mathbb{Z}_2$ is composed of two quadratic functions q_1: $V_1 \to \mathbb{Z}_2$ and q_2: $V_2 \to \mathbb{Z}_2$ by the rule $q(x_1, x_2) = q_1(x_1) + q_2(x_2)$. Then $\sigma(q) = \sigma(q_1)\sigma(q_2)$. For our purposes, Gaussian sums are useful for two reasons. First, they can be calculated in terms of the Arf invariants of the corresponding quadratic functions. Secondly, the Gaussian sum of the above quadratic function Q_M: $H^1(M; \mathbb{Z}_2) \to \mathbb{Z}_2$ on the cohomology of any closed connected 3-manifold M is directly related to the invariant $Z(M)$.

Lemma 1. *Let q: $V \to \mathbb{Z}_2$ be a quadratic function on a vector space V over $\mathbb{Z}_2$, and let $a_1, b_1, \ldots, a_k, b_k, c_1, \ldots, c_m$ be a basis of V such that the vectors $c_1, \ldots, c_m$ generate the annihilator of the pairing ℓ_q and, moreover, $\ell_q(a_i, a_j) = \ell_q(b_i, b_j) = 0$ and $\ell_q(a_i, b_j) = \delta_{i,j}$ for all $i, j = 1, 2, \ldots, k$. Then $\sigma(q) = 2^{k+m}(-1)^{\mathrm{Arf}(q)}\prod_{j=1}^{m}(1 - q(c_j))$.*

The proof is based on the fact that the pair (V, q) is the direct sum of $k + m$ quadratic functions obtained by restricting q to the 2-spaces $V_i = \mathbb{Z}_2a_i \oplus \mathbb{Z}_2b_i$ for $1 \le i \le k$ and to the 1-spaces $C_j = \mathbb{Z}_2c_j$ for $1 \le j \le m$.

Let us show that $\sigma(q|_{V_i}) = 2 \cdot (-1)^{q(a_i)q(b_i)}$. Indeed, the space V_i consists of the vectors $0, a_i, b_i$, and $a_i + b_i$. Therefore, $\sigma(q|_{V_i}) = (-1)^0 + (-1)^{q(a_i)} + (-1)^{q(b_i)} + (-1)^{q(a_i+b_i)}$, which, together with the relations $q(a_i + b_i) - q(a_i) - q(b_i) = \ell(a_i, b_i) = 1$, gives

$$\begin{aligned} \sigma(q|_{V_i}) &= 1 + (-1)^{q(a_i)} + (-1)^{q(b_i)} + (-1)^{1+q(a_i)+q(b_i)} \\ &= 2 \cdot (-1)^{q(a_i)q(b_i)}. \end{aligned}$$

The last relation is easy to verify: if at least one of the values $q(a_i)$ and $q(b_i)$ equals 0, then both sides of the relation are equal to 2, and if $q(a_i) = q(b_i) = 1$, then both sides are equal to -2. Similarly, for the 1-space $C_j = \{0, c_j\}$, we have $\sigma(q|_{C_j}) = 1 + (-1)^{q(c_j)} = 2 \cdot (1 - q(c_j))$, because both sides of the last equality equal 2 if $q(c_j) = 0$ and 0 if $q(c_j) = 1$. Using the multiplicativity of Gaussian sums mentioned above, we obtain

$$\begin{aligned} \sigma(q) &= \prod_{i=1}^{k}\sigma(q|_{V_i})\prod_{j=1}^{m}\sigma(q|_{C_j}) \\ &= 2^{k+m}(-1)^{\mathrm{Arf}(q)}\prod_{j=1}^{m}(1 - q(c_j)). \end{aligned}$$

5. IDEA OF THE PROOF OF THE MAIN THEOREM

First, we show that $Z(M) = \frac{1}{2}\sigma(Q_M)$. Let f: $M \to B$ be any map. Then

$$\langle f^*(\alpha^3), [M]\rangle = \langle (f^*(\alpha))^3, [M]\rangle = Q_M(f^*(\alpha)),$$

where the first relation holds because f^* preserves multiplication in cohomology and the second, by the definition of Q_M. Note that the sets $S(M, B)$ and $H^1(M; \mathbb{Z}_2)$, over which the summation in the expressions for $Z(M)$ and $\frac{1}{2}\sigma(Q_M)$ is performed, admit a natural identification by taking each map $f \in S(M, B)$ to the elements $x = f^*(\alpha)$ of the group $H^1(M; \mathbb{Z}_2)$. Thus, the expressions

$$Z(M) = \frac{1}{2}\sum_{f \in S(M, B)} (-1)^{\langle f^*(\alpha^3), [M]\rangle},$$

$$\frac{1}{2}\sigma(Q_M) = \frac{1}{2}\sum_{x \in H^1(M; \mathbb{Z}_2)} (-1)^{Q_M(x)}$$

are the sums of the same terms, and therefore they are equal.

Now, suppose that there exists an element x of the annihilator A of the pairing ℓ_M such that $Q_M(x) = 1$. Then we can take this element for, say, the first basis vector c_1 of A. Since $Q_M(c_1) = 1$ and the difference $(1 - Q_M(c_1)) = 0$ is contained in the expression for $\sigma(Q_M)$ as one of the factors, it follows that $Z(M) = 0$ by Lemma 1. If there are no such elements, then $Z(M) = \frac{1}{2}\sigma(Q_M) = 2^{k+m-1}(-1)^{\mathrm{Arf}(Q_M)}$, where $m = \dim_{\mathbb{Z}_2}(A)$ and k equals half the dimension of the coset space $H^1(M; \mathbb{Z}_2)/A$.

Let us say a few words about the practical calculation of the Dijkgraaf–Witten invariants. Let M be a closed orientable 3-manifold. It is well known (see, e.g., [3]) that, for all $x, y \in H^1(M; \mathbb{Z}_2)$, we have $x^2y + xy^2 = wxy$, where $w = w^1(M) \in H^1(M; \mathbb{Z}_2)$ is the first Stiefel–Whitney class of M. Since M is orientable, it follows that $w = 0$. Therefore, $\ell_M(x, y) \equiv 0$ and the annihilator A of ℓ_M coincides with the group $H^1(M; \mathbb{Z}_2)$; thus, $\mathrm{Arf}(Q_M) = 0$, $k = 0$, and $m = \dim_{\mathbb{Z}_2}(H^1(M; \mathbb{Z}_2))$. By Theorem 1 we conclude that if there exists $x \in H^1(M; \mathbb{Z}_2)$ such that $x^3 \neq 0$, then $Z(M) = 0$, and if there are no such elements, then $Z(M) = 2^{m-1}$.

Consider an example. It is known that the cohomology group $H^1(L_{p,q}; \mathbb{Z}_2)$ of the lens space $L_{p,q}$ contains an element with nonzero cube if and only if p is even but not divisible by 4; see [4, 5]. Therefore, it follows from the above considerations that

$$Z(L_{p,q}) = \begin{cases} 0 & \text{if} \quad p \equiv 2 \bmod 4 \\ \frac{1}{2} & \text{if} \quad p \quad \text{is odd} \\ 1 & \text{if} \quad p \quad \text{is divisible by } 4. \end{cases}$$

This extends the results of [6] concerning the calculation of Dijkgraaf–Witten invariants for lens spaces of the type $L(p, 1)$ to the case of any lens spaces.

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