TOPOLOGICAL FIELD THEORIES AND TENSOR CATEGORIES. HOMEWORK #3.

Target day for this homework: November 11

Submit (in class or by email) any of the following problems.

1. Write a complete proof of the identity $\mathrm{Tr}(fg) = \mathrm{Tr}(gf)$.

2. Assume that the base field is of characteristic zero. Recall that we defined the exterior power $\wedge^n(X)$ as the image of the idempotent $\frac{1}{n!}\sum_{\pi\in S_n}(-1)^\pi\pi$ acting on the tensor power $X^{\otimes n}$ (so this definition makes sense in any Karoubian category). Prove the formula

$$\wedge^n(X\oplus Y) = \bigoplus_{i=0}^{n} \wedge^i(X)\otimes\wedge^{n-i}(Y).$$

3. (a) Let X be a dualizable object. Prove that braiding $c_{X^*,Y}$ equals to composition

$$X^*\otimes Y \xrightarrow{coev_X} X^*\otimes Y\otimes X\otimes X^* \xrightarrow{c_{Y,X}} X^*\otimes X\otimes Y\otimes X^* \xrightarrow{ev_X} Y\otimes X^*$$

(b) Let X be a dualizable object such that $c_{X,X} = \mathrm{id}$. Prove that X is invertible.

(c) Give a counterexample to (b) when X is not dualizable.

(d)* Prove or disprove the following Conjecture: if X is dualizable and non-invertible then S_n-action on $X^{\otimes n}$ is faithful (I don't know a complete solution for this problem; feel free to assume extra things about the category, e.g. that it is abelian).

4. (a) Let $\mathcal{C}$ be a Karoubian rigid symmetric tensor category with finite dimensional Hom spaces. Prove that non-isomorphic non-negligible indecomposable objects of $\mathcal{C}$ are non-isomorphic in the gligible quotient $\bar{\mathcal{C}}$.

(b) Let U, V, W be non-negligible indecomposable objects of $\mathcal{C}$. Prove that if U is a direct summand of $V\otimes W$ then V^* is a direct summand of $U^*\otimes W$. Give a counterexample when negligibility assumption is dropped (Hint: what if $V = \mathbf{1}$?).

5. Let k be a field of characteristic $p > 0$ and let C_p be a cyclic group of order p. We compute tensor multiplication rules for $Rep(C_p)$ (and hence for its gligible quotient). Recall that the indecomposable objects are $L_1, L_2, \ldots, L_p$.

(a) Compute directly $L_{p-1}\otimes L_{p-1}$ (Hint: what is the total number of summands?)

(b) Prove that for any i with $1 < i < p$ we have $L_2\otimes L_i = L_{i-1} + L_{i+1}$. What about $L_2\otimes L_p$? (Hint: do induction in i using Problem 4 (b)).

(c) Prove "Verlinde formula" for tensor product in the gligible quotient of $Rep(C_p)$: for $1 \le m, n < p$ we have

$$L_m\otimes L_n = \bigoplus_{i=1}^{min(m,n,p-m,p-n)} L_{|m-n|+2i-1}.$$