A FORMULA FOR THE JONES-WENZL PROJECTIONS

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Abstract. I present a method of calculating the coefficients appearing in the Jones-Wenzl projections in the Temperley-Lieb algebras. It essentially repeats the approach of Frenkel and Khovanov in [FK97] published in 1997. I wrote this article mid-2002, not knowing about their work, but then set it aside upon discovering their article.

Recently I decided to dust it off and place it on the arXiv — hoping the self-contained and detailed proof I give here may be useful. It's also been cited a number of times [Han10, Pet10, Eli11, BPMS12, Lev13], so I thought it best to give it a permanent home.

The proof is based upon a simplification of the Wenzl recurrence relation. I give an example calculation, and compare this method to the formula announced by Ocneanu [Ocn02] and partially proved by Reznikoff [Rez07]. I also describe certain moves on diagrams which modify their coefficients in a simple way.

1. Basic Definitions

The **quantum integers** are denoted by $[n]$, and are given in terms of the formal **quantum parameter** q by the formula

$$[n] = q^{n-1} + q^{n-3} + \cdots + q^{-(n-1)} = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

The quantum integers satisfy many relations, all of which reduce to simple arithmetic relations when evaluated at $q = 1$. For example, a simple result we will need later is

Lemma 1.1. *If $m \geq a$, then $[m-a] + [m+1][a] = [m][a+1]$.*

An n **strand Temperley-Lieb diagram** is a diagram drawn inside a rectangle with n marked points on both the upper and lower edges, with non-intersecting arcs joining these points. We consider isotopic diagrams as equivalent. A **through strand** is an arc joining a point on the upper edge of a diagram to the lower edge. A **cup** joins a point on the upper edge with another point on the upper edge, and similarly a **cap** joins the lower edge to itself. A cap or cup is called **innermost** if it is exactly that — there are no nested caps or cups inside it. This terminology is illustrated in Figure 1.

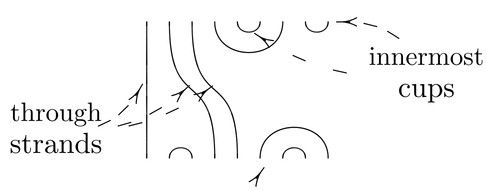


Figure 1.

The n **strand Temperley-Lieb algebra**, denoted TL_n, is the algebra over $\mathbb{C}(q)$ spanned by the Temperley-Lieb diagrams, with multiplication defined on this basis by stacking diagrams. In such a product of diagrams closed loops may appear, each of which we remove while inserting an additional factor of $-[2]$. Two quite different sign conventions appear in the literature. Generally, in topological applications loops are given the value $-[2]$, but in the theory of subfactors the value $[2]$. I have employed the present convention, because it results in simpler formulas, with all coefficients positive. To pass between the two conventions, replace everywhere $[i]$ with $(-1)^{i+1}[i]$, or equivalently q with $-q$.

Figure 2 illustrates multiplication in the 5 strand algebra.

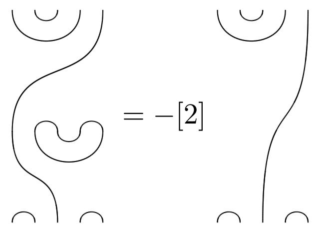


FIGURE 2. A calculation in the 5 strand Temperley-Lieb algebra.

We can also define vector spaces $TL_{n,m}$, spanned by isotopy classes of diagrams with m points on the lower boundary of the rectangle, and n along the top. These fit together into a monoidal category [CFS95, CP94] over $\mathbb{C}(q)$, with objects in $\mathbb{N}$, and $TL_{n,m}$ giving the morphisms from m to n.

Equivalently, we can give a definition of the Temperley-Lieb algebra in terms of generators and relations [Jon91].[1] Define the **multiplicative generator** e_i $(i = 1, \ldots, n-1)$ as the diagram with $i-1$ vertical strands, a cap-cup pair, then $n-i-1$ more vertical strands. Figure 3 illustrates the multiplicative generators in the 5 strand algebra.

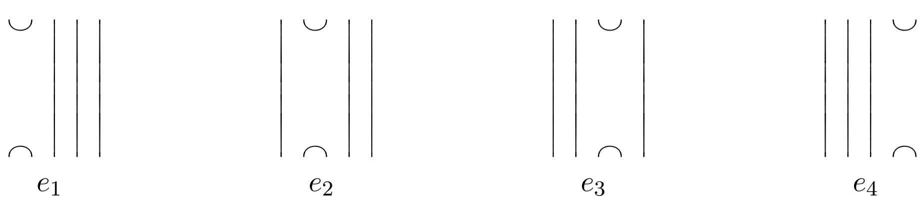


FIGURE 3. The multiplicative generators in the 5 strand Temperley-Lieb algebra.

The Temperley-Lieb algebra is generated by these diagrams along with the identity diagram, denoted $\mathbf{1}$, subject to the relations

$$\begin{aligned} e_i e_i &= -[2] e_i \\ e_i e_{i\pm 1} e_i &= e_i \\ e_i e_j &= e_j e_i \qquad \text{if } |i-j| \geq 2. \end{aligned}$$

Inside the Temperley-Lieb algebra TL_n we have the two-sided ideal $\mathcal{I}_n$, generated by the elements $\{e_1, \ldots, e_{n-1}\}$. This ideal has codimension 1; it is spanned by diagrams with $n-2$ or fewer through strands, that is, every diagram except the identity diagram.

[1]For the relationship between the diagrammatic algebra and 'generators and relations' algebra when the formal parameter q has been evaluated at a complex root of unity, see [Fre03, JR06].

2. The Jones-Wenzl idempotent

Inside the n strand Temperley-Lieb algebra there is a special element called the **Jones-Wenzl idempotent**, denoted $f^{(n)}$. It is characterised by the properties

$$\begin{aligned} f^{(n)} &\neq 0 \\ f^{(n)} f^{(n)} &= f^{(n)} \\ e_i f^{(n)} = f^{(n)} e_i &= 0 \qquad \forall i \in \{1, \dots, n-1\}\,. \end{aligned} \tag{2.1}$$

The second equation could be equivalently stated as $\mathcal{I}_n f^{(n)} = f^{(n)} \mathcal{I}_n = 0$.

The aim of this work is to present new methods for calculating the coefficients for each diagram appearing in the Jones-Wenzl idempotent. The starting point will be the Wenzl recurrence formula, allowing us to calculate $f^{(n+1)}$ in terms of $f^{(n)}$.

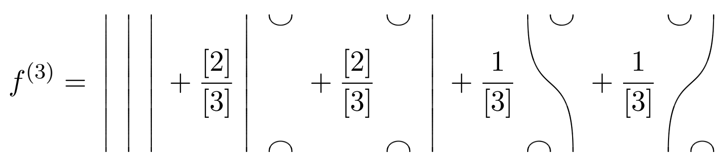
Lemma 2.1. *The coefficient of the identity diagram in a Jones-Wenzl idempotent is always* 1*.*

Proof. Write $f^{(n)} = \alpha \mathbf{1} + g$, with $\alpha \in \mathbb{C}$ and $g \in \mathcal{I}_n$. We want to see that $\alpha = 1$. This follows from $f^{(n)} f^{(n)} = f^{(n)} \alpha \mathbf{1} + f^{(n)} g = \alpha f^{(n)} + 0$, so $\alpha = 1$. □

Lemma 2.2. *The Jones-Wenzl idempotent, characterised by Equation 2.1, is unique.*

Proof. Suppose both f_1^n and f_2^n satisfy Equation 2.1. Write $f_1^n = \mathbf{1} + g_1$ and $f_2^n = \mathbf{1} + g_2$, where $g_1, g_2 \in \mathcal{I}_n$. Then $f_1^n f_2^n = f_1^n(\mathbf{1} + g_2) = f_1^n$ and similarly $f_1^n f_2^n = (\mathbf{1} + g_1) f_2^n = f_2^n$. Thus $f_1^n = f_2^n$. □

For example, the 3 strand idempotent is

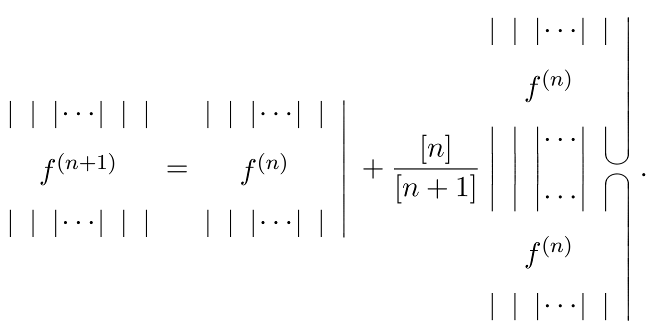


The n strand Temperley-Lieb algebra naturally includes into the $n+1$ strand algebra, by adding a vertical strand to the right side of the diagram. Taking advantage of this, we abuse notation and write $f^{(n)} \in TL_{n+1}$ to mean the n strand Jones-Wenzl idempotent, with a vertical strand added to the right, living in the $n+1$ strand algebra.

Proposition 2.3 (Wenzl recurrence formula)**.** *The Jones-Wenzl idempotent satisfies*

$$f^{(n+1)} = f^{(n)} + \frac{[n]}{[n+1]} f^{(n)} e_n f^{(n)}, \tag{2.2}$$

or, diagrammatically,

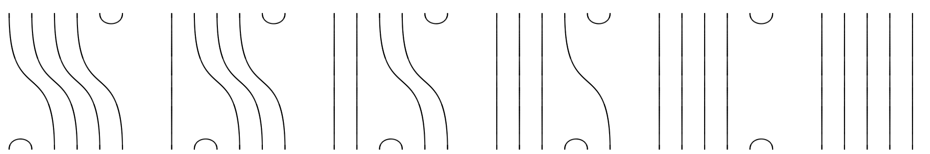


This is a well known result. The original paper is [Wen87]. Various proofs can be found in any of [CFS95, Kau01, KL94, Lic93].

3. Simplifications of the Wenzl recurrence formula

We will now consider the last term, $\frac{[n]}{[n+1]} f^{(n)} e_n f^{(n)}$, in the Wenzl recurrence formula. By expanding this appropriately, we will see that many of the terms do not contribute.

Let P denote the leftmost $n-2$ points along the top edge of an n strand diagram. Define $\mathcal{J}_n \subset TL_n$ as the linear span of those diagrams in which any two points of P are connected together by a strand. This is a left ideal; multiplying by any diagram on the right does not change this condition. Further we can write $TL_n = \mathcal{J}_n \bigoplus \mathcal{K}_n$, where $\mathcal{K}_n$ is spanned by the diagrams in which the points of P are all connected to points on the bottom edge of the diagram. This collection of diagrams consists of those diagrams with a single cup at the top right, and a single cap at some position along the bottom edge, along with the identity diagram. We denote these diagrams by $g_{n,i}$, with $i = 1, \ldots, n-1$, with the subscript i indicating the position of the cap. Further, for convenience we write $g_{n,n} = \mathbf{1}$. This is illustrated for $n = 6$ in Figure 3. From this, we see $\mathcal{J}_n$ has codimension n.



$g_{6,1}$ $g_{6,2}$ $g_{6,3}$ $g_{6,4}$ $g_{6,5}$ $g_{6,6} = \mathbf{1}$

Figure 4. The diagrams spanning $\mathcal{K}_6$.

Lemma 3.1. *The left ideal $\mathcal{J}_n$ is contained in the kernel of the map $TL_n \subset TL_{n+1} \to TL_{n+1}$ given by $h \mapsto f^{(n)} e_n h$.*

Proof. If h is a diagram in $\mathcal{J}_n$, then we can write $h = e_i h'$ for some $1 \leq i \leq n-2$, and $h' \in TL_n$. Then $f^{(n)} e_n e_i = f^{(n)} e_i e_n = 0$. □

This immediately allows us to simplify the Wenzl recurrence relation. Write $f^{(n)} = f^{(n)}_{\mathcal{J}} + f^{(n)}_{\mathcal{K}}$, with $f^{(n)}_{\mathcal{J}} \in \mathcal{J}_n$ and $f^{(n)}_{\mathcal{K}} \in \mathcal{K}_n$. Then we have

$$\begin{aligned} f^{(n)} e_n f^{(n)} &= f^{(n)} e_n (f^{(n)}_{\mathcal{J}} + f^{(n)}_{\mathcal{K}}) \\ &= f^{(n)} e_n f^{(n)}_{\mathcal{K}}. \end{aligned}$$

Now $\mathcal{K}_n$ is spanned by the diagrams $g_{n,i}$ for $i = 1, \ldots, n$, so we can write

$$f^{(n)}_{\mathcal{K}} = \sum_{i=1}^{n} \underset{\in f^{(n)}}{\operatorname{coeff}} (g_{n,i})\, g_{n,i}.$$

From this we easily obtain

Proposition 3.2 (Simplified recurrence formula)**.** *The Jones-Wenzl idempotents satisfy*

$$f^{(n+1)} = f^{(n)} \left(\sum_{i=1}^{n} \frac{[n]}{[n+1]} \underset{\in f^{(n)}}{\operatorname{coeff}} (g_{n,i})\, g_{n+1,i} + g_{n+1,n+1} \right). \tag{3.1}$$

Proof. We use the fact that

$$e_n g_{n,i} = g_{n+1,i}, \tag{3.2}$$

as illustrated in Figure 3, and calculate as follows:

$$\begin{aligned}
f^{(n+1)} &= f^{(n)} + \frac{[n]}{[n+1]} f^{(n)} e_n \sum_{i=1}^{n} \underset{\in f^{(n)}}{\mathrm{coeff}} \left(g_{n,i}\right) g_{n,i} \\
&= f^{(n)} + \frac{[n]}{[n+1]} f^{(n)} \sum_{i=1}^{n} \underset{\in f^{(n)}}{\mathrm{coeff}} \left(g_{n,i}\right) e_n g_{n,i} \\
&= f^{(n)} g_{n+1,n+1} + \frac{[n]}{[n+1]} f^{(n)} \sum_{i=1}^{n} \underset{\in f^{(n)}}{\mathrm{coeff}} \left(g_{n,i}\right) g_{n+1,i} \\
&= f^{(n)} \left(\sum_{i=1}^{n} \frac{[n]}{[n+1]} \underset{\in f^{(n)}}{\mathrm{coeff}} \left(g_{n,i}\right) g_{n+1,i} + g_{n+1,n+1} \right).
\end{aligned}$$

□

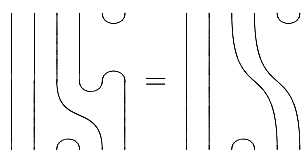


FIGURE 5. A sample calculation, $e_5 g_{5,3} = g_{6,3}$, illustrating Equation 3.2.

This simplification of the Wenzl recurrence relation is not in itself particularly useful. It is still 'quadratic' in the sense that when expanded, each term contains two unknown coefficients. However, we can now use it to make a direct calculation of the quantities $\mathrm{coeff}(g_{n,i})$, which will enable us to further simplify the recurrence relation to a 'linear' form.

Proposition 3.3 (Further simplified recurrence formula)**.** *The coefficients of the diagrams with 'a single right cup' are given by*

$$\underset{\in f^{(n)}}{\mathrm{coeff}} \left(g_{n,i}\right) = \frac{[i]}{[n]}, \tag{3.3}$$

and the recurrence formula thus becomes

$$f^{(n+1)} = \frac{f^{(n)}}{[n+1]} \left(\sum_{i=1}^{n+1} [i] g_{n+1,i} \right). \tag{3.4}$$

Proof. At $n = 1$, there is only one such diagram, $\mathbf{1} = g_{1,1}$, with coefficient 1, as required. Now assume Equation 3.3 holds for some value of n. Equation 3.4 follows immediately from Equation

3.1, by the following calculation:

$$\begin{aligned} f^{(n+1)} &= f^{(n)} \left(\sum_{i=1}^{n} \frac{[n]}{[n+1]} \frac{[i]}{[n]} g_{n+1,i} + g_{n+1,n+1} \right) \\ &= f^{(n)} \left(\sum_{i=1}^{n} \frac{[i]}{[n+1]} g_{n+1,i} + \frac{[n+1]}{[n+1]} g_{n+1,n+1} \right) \\ &= \frac{f^{(n)}}{[n+1]} \left(\sum_{i=1}^{n+1} [i] g_{n+1,i} \right). \end{aligned}$$

We will now use this to calculate the coefficient of $g_{n+1,i}$ in $f^{(n+1)}$. Suppose h is a diagram in TL_n, and consider the term $\frac{[i]}{[n+1]} \operatorname{coeff}(h) h g_{n+1,i}$ on the right hand side of Equation 3.4. We will determine the diagrams h and values of i for which this term contributes to the $g_{n+1,j}$ term in $f^{(n+1)}$. There are several cases to consider.

(1) The diagram h contains a cap connecting two of the leftmost $n-1$ points at the bottom of the diagram. In this case $hg_{n+1,i}$ has $n-4$ or fewer through strands, and so can not contribute to the $g_{n+1,j}$ term in $f^{(n+1)}$. An example of this appears in Figure 2.

(2) There is no such cap in h, but there is a cap connected the rightmost two points at the bottom of the diagram. In this case the diagram $hg_{n+1,i}$ has a vertical strand on the right hand side, and so again can not contribute. An example appears in Figure 2.

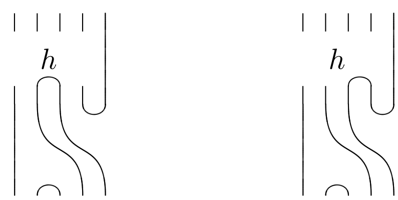


FIGURE 6. Examples illustrating the first two cases in Proposition 3.3.

(3) There are no such caps, and h is the identity diagram. In this case

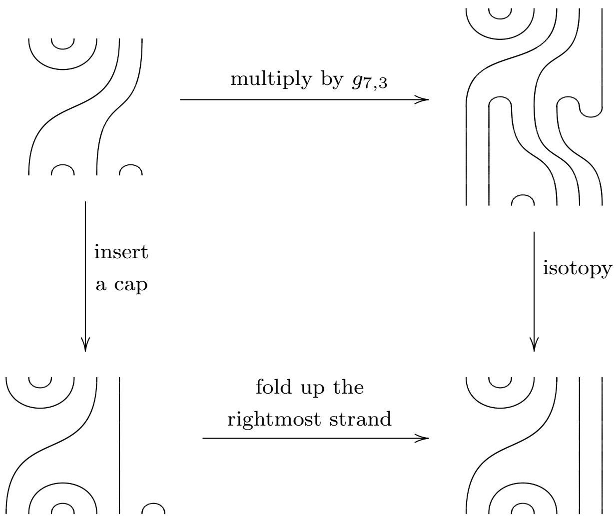
$$\frac{[i]}{[n+1]} \operatorname*{coeff}_{\in f^{(n)}}(h)\, h g_{n+1,i} = \frac{[i]}{[n+1]} g_{n+1,i}.$$

These cases are exhaustive, and so it is easily seen that there is exactly one contribution to the $g_{n+1,j}$ term in $f^{(n+1)}$, coming from the identity term in $f^{(n)}$ and the $g_{n+1,j}$ term of the summation, and so the coefficient of $g_{n+1,j}$ in $f^{(n+1)}$ is exactly $\frac{[j]}{[n+1]}$. Thus by induction the claimed result holds for all values of n. □

Remark. An analogue of this 'linear' recurrence relation for idempotents in the $\mathfrak{sl}_3$ spider (c.f. [Kup96]) appears in Dongseok's work [Kim07, Kim03], where it is called a 'single clasp expansion'.

4. Unfolding the recurrence formula

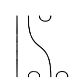
Let's now think about the map (diagram) $\mapsto$ (diagram) $\sum_{i=1}^{n+1} \frac{[i]}{[n+1]} g_{n+1,i}$. Multiplying an n strand diagram by $g_{n+1,i}$ can be thought of as 'inserting a cap at the i-th position, and folding up the right strand':

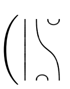
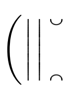
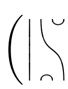


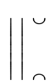
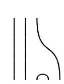
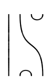
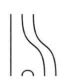
Each diagrammatic term in $f^{(n+1)}$ thus arises from a sum of contributions generated in this way. Choose some diagram D in TL_{n+1}. To determine which terms in $f^{(n)}$ contribute to the coefficient of D in $f^{(n+1)}$, we should take D, and 'fold down the right strand, then select and remove an innermost cap'. It is only the terms in $f^{(n)}$ involving these diagrams which matter in calculating the coefficient of D in $f^{(n+1)}$. Suppose we chose to remove an innermost cap at position i. The resulting diagram, when multiplied by the $g_{n+1,i}$, gives the original diagram D.

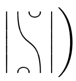
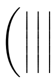
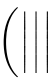
Proposition 4.1. *Suppose D is a diagram in TL_{n+1}. Let $\hat{D} \in TL_{n,n+2}$ be the diagram obtained by folding down the top right end point of D. Let $\{i\}$ be the set of positions of innermost caps in $\hat{D}$, and $D_i \in TL_n$ be the diagram obtained by removing that innermost cap. Then*

$$\operatorname*{coeff}_{\in f^{(n+1)}}(D) = \sum_{\{i\}} \frac{[i]}{[n+1]} \operatorname*{coeff}_{\in f^{(n)}}(D_i). \tag{4.1}$$

Example. Consider the diagram  $\in TL_5$. Folding down the rightmost strand gives . There are now two innermost caps we can remove, at positions 2 and 5. Thus

$\operatorname*{coeff}_{\in f^{(5)}}\Big($  $\Big) = \frac{[2]}{[5]} \operatorname*{coeff}_{\in f^{(4)}}\Big($  $\Big) + \frac{[5]}{[5]} \operatorname*{coeff}_{\in f^{(4)}}\Big($  $\Big)$.

We can continue in this way. The diagram  folds down to give , with only one cap to remove, and similarly  folds down to . Thus

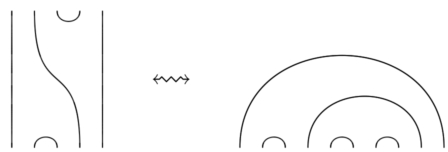
$\operatorname*{coeff}_{\in f^{(5)}}\Big($  $\Big) = \frac{[2][3]}{[5][4]} \operatorname*{coeff}_{\in f^{(3)}}\Big($  $\Big) + \frac{[5][2]}{[5][4]} \operatorname*{coeff}_{\in f^{(3)}}\Big($  $\Big) = \frac{[2][3]+[5][2]}{[5][4]}$.

Thus the coefficient of a diagram is a certain sum over sequences of choices of arcs to remove. Iterating the calculation in Equation 4.1 allows us to find the coefficient of any diagram. Although this calculation is based on a recursive step, it is very different from Wenzl's formula in Equation 2.2. In particular, we never need to perform any multiplications in the Temperley-Lieb algebra, and we can find the coefficient of a diagram without calculating the entire projection, by performing simple combinatorial operations on the diagrams.

5. An explicit formula

It is possible to write down an explicit formula giving the result of this calculation, but it is made somewhat awkward by the fact that the numbering of the strands changes as we successively remove innermost caps.

A good way to think about the diagrams is as a 'capform' [Kau01], produced by 'folding the diagram down to the right'.



Now, for a diagram with n strands, let

$$S = \left\{ (s_1, \ldots, s_n) \in \mathbb{N}^n \;\middle|\; \begin{array}{l} \text{the } s_i \text{ are all distinct, } 1 \leq s_i \leq n+i-1, \\ s_i \text{ is the position of the left end of a cap} \\ \text{for each } i\text{, and if } \tilde{s_i} \text{ denotes the position of} \\ \text{the corresponding right end, then if } i < j, \\ \text{and } s_i < s_j\text{, then } \tilde{s_i} < s_j \text{ also} \end{array} \right\}.$$

The sequences in S specify choices of orders in which to remove strands. The restriction $1 \leq s_i \leq n+i-1$ ensures that we only remove a strand when its initial point is in the left half of the capform, and the second restriction ensures that we remove only innermost caps.

This set S is not quite what is needed, because although it describes the orders in which we can remove strands, the factors appearing in Equation 4.1 depend on the position of the cap at the moment we remove it.

This position is given by the map $\tau : s \mapsto s - \kappa(s)$, where

$$\kappa(s)_i = \# \{ 1 \leq j \leq i-1 \mid s_j < s_i \}.$$

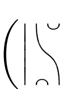
Thus for example $\tau(s)_2 = \begin{cases} s_2 & \text{if } s_1 > s_2 \\ s_2 - 2 & \text{if } s_1 < s_2 \end{cases}$.

Then we have

Proposition 5.1. *The coefficient in $f^{(n)}$ of a diagram D with index set S, as given above, is*

$$\underset{\in f^{(n)}}{\mathrm{coeff}}(D) = \frac{1}{[n]!} \sum_{s \in S} [\tau(s)], \tag{5.1}$$

using the convenient notations $[n]! = [n][n-1] \cdots [1]$ and $[(t_1, \ldots, t_n)] = [t_1] \cdots [t_n]$.

Example. We redo the calculation of $\mathrm{coeff}_{\in f^{(5)}}$ (). The index set has two elements, $S = \{(2,5,7,4,1), (5,2,7,4,1\}$. Then $\tau(S) = \{(2,3,3,2,1), (5,2,3,2,1)\}$, and so

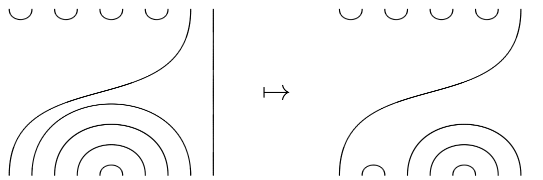
$\underset{\in f^{(5)}}{\mathrm{coeff}}$ () $= \dfrac{[2][3][3][2][1] + [5][2][3][2][1]}{[5]!} = \dfrac{[2][3] + [5][2]}{[5][4]}$,

as we calculated before.

6. k-MOVES

We'll next apply this algorithm for computing coefficients to prove 'k-move invariance'. A k-move acts on the capform of a diagram transforming a collection of k nested caps with centre strictly in the left half of the capform into $k-1$ nested caps to the right of a single cap, while leaving the rest of the diagram unchanged. We apply k-moves to rectangular Temperley-Lieb diagrams by converting to a capform, applying the move as described, and converting back.

Thus, a valid 4-move is illustrated below.



The condition that the centre of the capform must lie in the left half of the diagram requires that the move does not decrease the number of through strands in the original diagram.

The following theorem relating the coefficients of diagrams obtained by k-moves allows very efficient calculations in many situations.

Proposition 6.1. *If D' is obtained from a diagram $D \in TL_n$ by a k-move then*

$$[k] \underset{\in f^{(n)}}{\operatorname{coeff}}(D) = \underset{\in f^{(n)}}{\operatorname{coeff}}(D').$$

The proof is a somewhat complicated combinatorial argument, based on the algorithm above, and manipulation of relations amongst the quantum integers.

We use the notation of Proposition 5.1. First we describe the structure of the index set S' for the diagram D', in terms of the index set S for D.

Each $s \in S$ describes an order in which to successively remove strands. In particular, it tells us the (increasing) times at which we remove each of the k nested caps. Associated to this ordering we have several possible orderings for the diagram D'. Instead of removing the k caps in order, we can now remove the additional single cap at any point instead. Thus we obtain k different elements of S', which remove strands in the rest of the diagram at exactly the same times as s. At some point (different for each of the k elements) instead of removing the current innermost cap of the k nested caps, we remove the new single cap. It is not too hard to see that we obtain all valid sequences in S' this way, and each exactly once. This is formalised in the next paragraph.

Suppose the leftmost arc of the k nested caps in D is the a-th strand. For each $s \in S$, define $s^{(1)}, \ldots, s^{(k)} \in S'$ as follows. Let $j_1 < \cdots < j_k$ be the positions in s of the numbers $a+k-1, \ldots, a$, and call these positions 'marked'. Because of the nested structure, we have $s_{j_i} = a+k-i$. In the following we'll often need to describe the elements of a sequence of the marked positions, so we'll introduce the following notation:

$$((s)) = (s_{j_i})_{i=1}^k = (a+k-1, a+k-2, \ldots, a).$$

Now let $s^{(i)}$ be the same as s in the unmarked positions, and

$$\left(\left(s^{(i)}\right)\right) = (a+k, a+k-1, \ldots, a+k+2-i, \underbrace{a}_{i\text{-th position}}, a+k-i+1, \ldots, a+3, a+2).$$

That is, $\left(\left(s^{(i)}\right)\right) = ((s)) + (1,1,1,\ldots,1,i-k,2,\ldots,2,2)$.

Lemma 6.2.

$$S' = \left\{ s^{(i)} \mid s \in S, i \in 1, \ldots, k \right\}$$

and so

$$\operatorname*{coeff}_{\in f^{(n)}}\left(D'\right) = \frac{1}{[n]!}\sum_{s\in S}\sum_{i=1}^{k}[\tau(s^{(i)})].$$

Proof of Proposition 6.1. We calculate $\tau(s^{(i)})$, then prove that $\sum_{i=1}^{k}[\tau(s^{(i)})] = [k][\tau(s)]$.

Firstly, suppose $((\kappa(s))) = (\kappa_1, \ldots, \kappa_k)$, so $\tau(s)_{j_i} = a + k - i - 2\kappa_i$. For brevity we'll define $b_i = a + k - i - 2\kappa_i$. Outside the marked positions, $\kappa(s^{(i)})$ agrees with $\kappa(s)$, and $\left(\left(\kappa(s^{(i)})\right)\right) = (\kappa_1, \kappa_2, \ldots, \kappa_{i-1}, \kappa_i, \kappa_{i+1} + 1, \ldots, \kappa_k + 1)$. Thus

$$\begin{aligned}\left(\left(\tau(s^{(i)})\right)\right) &= (a + k - 2\kappa_1, a + k - 1 - 2\kappa_2, \ldots, a + k - (i - 2) - 2\kappa_{i-1}, a - 2\kappa_i, a + k - (i + 1) - 2\kappa_{i+1}, \ldots, a - 2\kappa_k)\\ &= (b_1 + 1, b_2 + 1, \ldots, b_{i-1} + 1, b_i - k + i, b_{i+1}, \ldots, b_k).\end{aligned}$$

We want to prove that $\sum_{i=1}^{k}[\left(\left(\tau(s^{(i)})\right)\right)] = [k][(b_1, \ldots, b_k)]$. To this end, define the partial sum $T_l = \sum_{i=1}^{l}[\left(\left(\tau(s^{(i)})\right)\right)]$. We will show that

$$T_k = T_l + \prod_{j=1}^{l}[b_j + 1]\cdot[k - l]\cdot\prod_{j=l+1}^{k}[b_j] \tag{6.1}$$

for each l, and so, evaluating at $l = 0$, $T_k = [k]\prod_{j=1}^{k}[b_j]$, as required.

Certainly Equation 6.1 holds for $l = k$, since $[0] = 0$. Suppose it holds for some value l. We can pull out the final term of the summation, and obtain

$$\begin{aligned}T_k &= T_l + \prod_{j=1}^{l}[b_j + 1]\cdot[k - l]\cdot\prod_{j=l+1}^{k}[b_j]\\ &= T_{l-1} + \prod_{j=1}^{l-1}[b_j + 1]\cdot([b_l - k + l] + [b_l + 1][k - l])\prod_{j=l+1}^{k}[b_j]\end{aligned}$$

and by Lemma 1.1, this is

$$= T_{l-1} + \prod_{j=1}^{l-1}[b_j + 1]\cdot[k - l + 1][b_l]\cdot\prod_{j=l+1}^{k}[b_j].$$

Thus Equation 6.1 also holds for $l - 1$, establishing the result. □

7. Results of Ocneanu and of Reznikoff

A similar formula has previously been published for these coefficients, by Ocneanu [Ocn02], although a proof of that formula was not given. His formula uses the alternative convention that closed loops have value [2].

Subsequently, a proof of special cases of this formula was been provided by Reznikoff [Rez02, Rez07]. The proof confirms Ocneanu's formula for diagrams in TL_n with $n - 2$ or $n - 4$ through strings, and uses very different methods (via the Brauer representation of the Temperley-Lieb algebra) from those employed here.

The method presented here readily reproduces Reznikoff's results. Some examples of this are given below. In doing so, this proves that Ocneanu's formula and the formula here are equivalent for diagrams with $n - 2$ or $n - 4$ through strings. However, I have been unable to obtain a direct proof that the formulas agree for all diagrams.

It is reasonably easy to prove that in limited cases the k-move invariance described in §6 holds for Ocneanu's formula as well. In particular, for two diagrams related by a k-move that involves no through strings at all, the coefficients given by Ocneanu's formula agree with Proposition 6.1.

This suggests a way to prove the equivalence of the formula here and Ocneanu's directly. If we knew the two formulas agreed on some class of simple diagrams, they would also agree on all diagrams obtained from these by a sequence of k-moves and inverse k-moves. However, the equivalence classes of diagrams under these moves are not particularly large; they each contain a single diagram with no nested caps or cups.

8. An application to diagrams with $n-4$ through strings

In this section, we give an explicit calculation of the coefficient of certain diagrams with exactly $n-4$ through strings. Although we only do one case here, all the other cases are no more difficult.

We use a combination of the summation formula of Equation 4.1 and the k-moves of the §6. Hopefully this will illustrate the computational power of these techniques!

A diagram with $n-4$ through strings has exactly 2 caps and 2 cups. We restrict our attention to those diagrams with no nested caps or cups. Consider such a diagram D. Thus we can unambiguously refer to these as the 'left cap', 'right cap', 'left cup', and 'right cup'. Suppose the leftmost points of these arcs occur at positions b_1, b_2, t_1 and t_2. (And of course, $b_2 \geq b_1+2$, $t_2 \geq t_1+2$.)

Because the coefficients of diagrams are preserved when the diagram is reflected in a horizontal line, we may assume that the right cap is no further to the right than the right cup, that is, that $t_2 \geq b_2$.

In this configuration, we can apply an inverse $(n-t_2)$-move, moving to right cup as far to the right as possible, obtaining the diagram D', with $t_2 = n-1$. The coefficients are related by $\operatorname{coeff}_{\in f^{(n)}}(D') = \frac{1}{[n-t_2]} \operatorname{coeff}_{\in f^{(n)}}(D)$, by Proposition 6.1.

We now apply the reduction formula. Folding down the top right point of the diagram turns the right cup into a through strand. Next, we have to choose one of the caps, at positions b_1 and $b-2$, to remove. The resulting diagrams are D'_{b_1}, with a cap at position b_2-2 and a cup at position t_1, and D'_{b_2} with a cap at position b_1 and a cup at position t_1. Then Equation 4.1 then tells us

$$\operatorname*{coeff}_{\in f^{(n)}}\left(D'\right) = \frac{[b_1]}{[n]} \operatorname*{coeff}_{\in f^{(n-1)}}\left(D'_{b_1}\right) + \frac{[b_2]}{[n]} \operatorname*{coeff}_{\in f^{(n-1)}}\left(D'_{b-2}\right). \tag{8.1}$$

The coefficients appearing here depend on the relative ordering of b_2-2 and t_1 (for the first term), and of b_1 and t_1 (for the second term). We'll assume now that $t_1 \geq b_2-2$. (The other two cases, $b_1 \leq t_1 \leq b_2-2$ and $t_1 \leq b_1$, are exactly analogous.) In this case, we can apply an inverse move to each diagram, as above, to move the cup to the far right, and then use Equation 3.3. Thus

$$\begin{aligned}
\operatorname*{coeff}_{\in f^{(n-1)}}\left(D'_{b-1}\right) &= \frac{[b_2-2][n-1-t_1]}{[n-1]} \\
\operatorname*{coeff}_{\in f^{(n-1)}}\left(D'_{b-2}\right) &= \frac{[b_1][n-1-t_1]}{[n-1]}.
\end{aligned}$$

Putting this all together, we obtain

$$\begin{aligned}
\operatorname*{coeff}_{\in f^{(n)}}(D) &= \frac{[b_1]([b_2]+[b_2-2])[n-1-t_1][n-t_2]}{[n][n-1]} \\
&= \frac{[2][b_1][b_2-1][n-1-t_1][n-t_2]}{[n][n-1]}
\end{aligned}$$

This agrees with the formula given as Equation 3 in [Rez07], for 'Style 3' diagrams. (The other two styles of diagrams there correspond exactly to the other two cases described previously.)

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