Conformal field theories and tensor categories
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Tensor categories in Conformal Field Theory

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Correlation functions and conformal blocks

Conformal Field Theory is determined by *correlation functions*

Variables: complex numbers or (better) points on some complex curve

$$\psi(z_1, \ldots, z_n) = \sum_p F_p(z_1, \ldots, z_n)\overline{G_p(z_1, \ldots, z_n)}$$

where F_p and G_p are *holomorphic* multivaluable functions with poles at the diagonals $z_i = z_j$
more precisely: F_p and G_p are (flat) sections of bundles of conformal blocks

Monodromy is described by representations of various braid groups

Example: Pure braid group $PB_n = \pi_1(\mathbb{C}^n \setminus \cup_{i \neq j}\{z_i = z_j\})$

Representations of vertex algebras

Fact: Conformal blocks are controlled by vertex algebras and their representations

Representations of a vertex algebra V

Notation: $\mathrm{Rep}(V)$ – representations of V
$\mathrm{Rep}(V)$ is a *category*: we have morphisms of representations with associative composition
$\mathrm{Rep}(V)$ is $\mathbb{C}-$*linear* category: $\mathrm{Hom}(M, N)$ is $\mathbb{C}-$vector space and composition is bilinear
$\mathrm{Rep}(V)$ is *abelian* category: we can talk about kernels and cokernels of morphisms

Rationality

Rational vertex algebra: any $M \in \mathrm{Rep}(V)$ is a direct sum of irreducibles; there are just finitely many of irreducibles

Tensor categories

Theorem (Huang)

Let V be a good rational vertex algebra. Then $Rep(V)$ has a natural structure of a Modular Tensor Category (MTC).

Definition

Tensor category: sextuple $(\mathcal{C}, \otimes, a_{\bullet\bullet\bullet}, \mathbf{1}, l_\bullet, r_\bullet)$
$\mathcal{C}$ – category
$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ – *(bi)functor*
$a_{XYZ} : (X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$ (functorial) *associativity constraint*
$\mathbf{1}$ – *unit object*
$l_X : \mathbf{1} \otimes X \simeq X$
$r_X : X \otimes \mathbf{1} \simeq X$
subject to axioms

Axioms

Pentagon axiom:

$$
\begin{array}{ccccc}
 & & (X \otimes Y) \otimes (Z \otimes T) & & \\
 & \nearrow^{a_{X\otimes Y,Z,T}} & & \searrow^{a_{X,Y,Z\otimes T}} & \\
((X \otimes Y) \otimes Z) \otimes T & & & & X \otimes (Y \otimes (Z \otimes T)) \\
\downarrow{\scriptstyle a_{X,Y,Z}\otimes \mathrm{id}_T} & & & & \uparrow{\scriptstyle \mathrm{id}_X \otimes a_{Y,Z,T}} \\
(X \otimes (Y \otimes Z)) \otimes T & & \xrightarrow{a_{X,Y\otimes Z,T}} & & X \otimes ((Y \otimes Z) \otimes T)
\end{array}
$$

and *Triangle axiom*

$$
\begin{array}{ccc}
(X \otimes \mathbf{1}) \otimes Y & \xrightarrow{a_{X,\mathbf{1},Y}} & X \otimes (\mathbf{1} \otimes Y) \\
 & {\scriptstyle r_X \otimes \mathrm{id}_Y}\searrow \quad \swarrow{\scriptstyle \mathrm{id}_X \otimes l_Y} & \\
 & X \otimes Y &
\end{array}
$$

Rigidity

$X \in \mathcal{C}$: *right dual* $X^* \in \mathcal{C}$ and *left dual* ${}^*X \in \mathcal{C}$

$$\mathbf{1} \to X \otimes X^* \quad \textit{coevaluation} \quad \mathbf{1} \to {}^*X \otimes X$$
$$X^* \otimes X \to \mathbf{1} \quad \textit{evaluation} \quad X \otimes {}^*X \to \mathbf{1}$$

Axiom: the maps below are identities:

$$X = \mathbf{1} \otimes X \to X \otimes X^* \otimes X \to X \otimes \mathbf{1} = X$$

$$X^* = X^* \otimes \mathbf{1} \to X^* \otimes X \otimes X^* \to \mathbf{1} \otimes X^* = X^*$$

Definition

A tensor category $\mathcal{C}$ is rigid if any object has both left and right dual

We will consider only $\mathbb{C}-$linear abelian tensor categories (with bilinear tensor product)

Useful fact (Deligne, Milne): In a rigid abelian tensor category tensor product is *exact*

Finiteness conditions

Finite category: abelian category equivalent to $\text{Rep}^{fd}(A)$ where A is a finite dimensional algebra.
Equivalently: f.d. Hom's, finitely many irreducible objects, any object has finite length, and enough projective objects.
Finite multi-tensor category: rigid tensor category which is finite.
Finite tensor category: finite multi-tensor category with $\text{End}(\mathbf{1}) = \mathbb{C}$.
Fusion category: finite tensor category which is semisimple (that is each object is a direct sum of irreducible ones).
Multi-fusion category: finite multi-tensor category which is semisimple.

Examples

- *Vec* – finite dimensional vector spaces over $\mathbb{C}$
- $\text{Rep}(G)$ (G – finite group) – f.d. representations of G over $\mathbb{C}$
- $\text{Rep}^{fd}(H)$ (H – f.d. (weak/quasi) Hopf algebra)

Pointed Example (Hoang Sinh)

Example

G is a (semi)group;
simple objects $g \in G$; $g \otimes h = gh$;
$a_{g,h,k} \in \mathbb{C}^{\times}$;
pentagon axiom $\Leftrightarrow a_{gh,k,l}a_{g,h,kl} = a_{g,h,k}a_{g,hk,l}a_{h,k,l}$
$\Leftrightarrow \partial a = 1$, that is a is a 3-cocycle;
triangle axiom $\Leftrightarrow$ 3-cocycle a is normalized;
rigidity $\Leftrightarrow$ G is a group
Fact: the category above depends only on the class $\omega = [a] \in H^3(G, \mathbb{C}^{\times})$
Notation: Vec_G^{ω}

Ocneanu rigidity

For a finite group G the group $H^3(G, \mathbb{C}^{\times})$ is finite.
Generalization (Ocneanu; Etingof, Nikshych, O): there are just countably many fusion categories.

Pivotal and spherical structures

Pivotal structure: choice of an isomorphism of *tensor* functors Id $\to$ **
(that is functorial isomorphism $X \simeq X^{**}$ compatible with tensor product)
Allows to define *traces* and *dimensions*

$$\mathrm{Tr}(f):\ \mathbf{1} \to X \otimes X^* \xrightarrow{f \otimes \mathrm{id}} X \otimes X^* \to X^{**} \otimes X^* \to \mathbf{1}$$

$$\dim(X) = \mathrm{Tr}(\mathrm{id}_X)$$

Spherical structure: pivotal structure with $\dim(X) = \dim(X^*)$ for all X

Question: Is it true that any fusion category has a spherical structure?

Theorem (Etingof, Nikshych, O)

For any fusion category there is a distinguished isomorphism of tensor functors Id $\to$ ****

Braided structure

Braiding: functorial isomorphism $c_{XY} : X \otimes Y \simeq Y \otimes X$ satisfying *hexagon* axioms for c and $c^{rev}_{XY} := c^{-1}_{YX}$

$$(X \otimes Y) \otimes Z \xrightarrow{a_{X,Y,Z}} X \otimes (Y \otimes Z) \xrightarrow{c_{X,Y \otimes Z}} (Y \otimes Z) \otimes X \xrightarrow{a_{Y,Z,X}} Y \otimes (Z \otimes X)$$

$$(X \otimes Y) \otimes Z \xrightarrow{c_{X,Y} \otimes \mathrm{id}_Z} (Y \otimes X) \otimes Z \xrightarrow{a_{Y,X,Z}} Y \otimes (X \otimes Z) \xrightarrow{\mathrm{id}_Y \otimes c_{X,Z}} Y \otimes (Z \otimes X)$$

If $\mathcal{C}$ is a braided tensor category then the pure braid group PB_n acts on $X_1 \otimes \ldots \otimes X_n$ and the braid group B_n acts on $X^{\otimes n}$.

Remark: c is a braiding $\iff$ c^{rev} is a braiding

Notation: $\mathcal{C}^{rev} = \mathcal{C}$ as a tensor category but c is replaced by c^{rev}

Pointed Example II

Example (Joyal-Street)

What are possible braided structures on Vec_G^ω?
$G = A$ should be abelian (since $a \otimes b = ab$ and $b \otimes a = ba$)
For any $a \in A$ the braiding $c_{aa} : a \otimes a \to a \otimes a$ is just a scalar $q(a) \in \mathbb{C}^\times$
Claim: $q : A \to \mathbb{C}^\times$ is a *quadratic form*:
$B(a,b) := \frac{q(ab)}{q(a)q(b)}$ is bilinear and
$q(a^{-1}) = q(a)$
Fact: Braided tensor category above is uniquely determined by (A, q)
In particular ω is determined by q. For example:
ω is trivial $\Leftrightarrow q(a) = \tilde{B}(a,a)$ for some bilinear (possibly non-symmetric) form $\tilde{B} : A \times A \to \mathbb{C}^\times$
Notation: $\mathcal{C}(A, q)$

Number of braidings on Vec_G^ω is finite.
Generalization (Ocneanu): Number of braidings on a fusion category is finite.

Symmetric tensor categories

Definition

A braided tensor category is *symmetric* if $c_{YX} \circ c_{XY} = \mathrm{Id}$ (equivalently, $c^{rev} = c$).

Examples

- *Vec*
- $\mathcal{C}(A, q)$ is symmetric $\Leftrightarrow B \equiv 1$ $(q(ab) = q(a)q(b))$
- $\mathrm{Rep}(G)$ with $c_{XY}(x \otimes y) = y \otimes x$
- Modify $\mathrm{Rep}(G)$: pick a central involution $z \in G$ and set

$$c'_{XY}(x \otimes y) = (-1)^{mn} y \otimes x$$

$$\text{if } zx = (-1)^m x, zy = (-1)^n y$$

Notation: $\mathrm{Rep}(G, z)$

- Super vector spaces: $sVec = \mathrm{Rep}(\mathbb{Z}/2\mathbb{Z}, z)$ (z nontrivial)

Classification of symmetric tensor categories

Theorem (Grothendieck, Saavedra Rivano, Doplicher, Roberts, Deligne)

A rigid symmetric tensor category satisfying some finiteness assumptions is of the form $Rep(G, z)$ *where* G *is a (super) group.*

Remark

Any finite tensor category satisfies the assumptions of the Theorem. However there are reasonable examples for which Theorem fails.

Drinfeld center

This is a construction of braided tensor category $\mathcal{Z}(\mathcal{C})$ starting with any tensor category $\mathcal{C}$
Objects of $\mathcal{Z}(\mathcal{C})$: $(X, c_\bullet)$ with $X \in \mathcal{C}$, $c_Y : X \otimes Y \simeq Y \otimes X$ satisfying one hexagon axiom
Tensor product: $(X, c_\bullet) \otimes (Y, d_\bullet) = (X \otimes Y, \widetilde{cd}_\bullet)$
Braiding: use c_Y to identify $X \otimes Y$ and $Y \otimes X$
Remark: there is no reason for the braiding to be symmetric

Remarks

- There is a *forgetful functor* $\mathcal{Z}(\mathcal{C}) \to \mathcal{C}$, $(X, c_\bullet) \mapsto X$
- If $\mathcal{C}$ is braided, then there are obvious tensor functors $\mathcal{C} \to \mathcal{Z}(\mathcal{C})$ and $\mathcal{C}^{rev} \to \mathcal{Z}(\mathcal{C})$; moreover we can combine them

$$\mathcal{C} \boxtimes \mathcal{C}^{rev} \to \mathcal{Z}(\mathcal{C})$$

Drinfeld center II

Theorem (Müger; Etingof, Nikshych, O)

Assume $\mathcal{C}$ is fusion category. Then $\mathcal{Z}(\mathcal{C})$ is also a fusion category.

Theorem (Etingof, O)

Assume $\mathcal{C}$ is finite tensor category. Then $\mathcal{Z}(\mathcal{C})$ is also a finite tensor category.

Example

$\mathcal{Z}(Vec_G^\omega)$ – twisted Drinfeld double of G
If $\omega = 0$ then

$$Irr(\mathcal{Z}(Vec_G^\omega)) = \{(x,\rho) | x \in G, \rho \in Irr(C_G(x))\}/G$$

If $\omega \neq 0$ use projective representations of $C_G(x)$

Non-degeneracy

Non-degenerate braided tensor category: "opposite" of symmetric

3 equivalent definitions of non-degenerate braided fusion category

1) (Turaev) $S-$matrix is non-degenerate: $S_{ij} = \mathrm{Tr}(c_{X_iX_j} \circ c_{X_jX_i})$
2) (Bruguières, Müger) No transparent objects: if $c_{XY} \circ c_{YX} = \mathrm{id}_{Y \otimes X}$ for all $Y \in \mathcal{C}$ then X is a multiple of $\mathbf{1}$
3) $\mathcal{C}$ is factorizable: the functor $\mathcal{C} \boxtimes \mathcal{C}^{rev} \to \mathcal{Z}(\mathcal{C})$ is an equivalence

Example

$\mathcal{C}(A, q)$ is non-degenerate $\Leftrightarrow$ $B(a, b) = \frac{q(ab)}{q(a)q(b)}$ is non-degenerate

What about Logarithmic CFT?

Guess: factorizable categories

Modular Tensor Categories

Definition (Turaev)

MTC is a non-degenerate braided fusion category with a choice of spherical structure.

Examples

- $\mathcal{Z}(\mathcal{A})$ where $\mathcal{A}$ is a spherical fusion category (e.g. $\mathcal{A} = Vec_G^\omega$) is MTC.
- Wess-Zumino-Witten model: let $\mathfrak{g}$ be a simple finite dimensional Lie algebra and $k \in \mathbb{Z}_{>0}$. Then $\mathcal{C}(\mathfrak{g}, k) =$ integrable $\hat{\mathfrak{g}}-$modules of level k has a structure of MTC.
- Dijkgraaf-Witten: given a compact group G and $\omega \in H^4(BG, \mathbb{Z})$ (satisfying some non-degeneracy condition) we should have MTC
  - G is simple and simply connected: $H^4(BG, \mathbb{Z}) = \mathbb{Z}$: WZW model
  - G is finite: $H^4(BG, \mathbb{Z}) = H^3(G, \mathbb{C}^\times)$: $\mathcal{Z}(Vec_G^\omega)$
  - G is torus: pointed category $\mathcal{C}(A, q)$
  - general G: not known

Module categories

Definition

Let $\mathcal{C}$ be a tensor category. Module category over $\mathcal{C}$ is a quadruple $(\mathcal{M}, \otimes, a_{\bullet\bullet\bullet}, l_\bullet)$
$\mathcal{M}$ is an abelian $\mathbb{C}-$linear category
$\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ is an exact bifunctor
$a_{XYM} : (X \otimes Y) \otimes M \simeq X \otimes (Y \otimes M)$
$l_M : \mathbf{1} \otimes M \simeq M$
satisfying the pentagon and triangle axioms

Example

Let $\mathcal{C} = Vec$. The module categories over $\mathcal{C}$ are all abelian $\mathbb{C}-$linear categories. Thus it is a bad idea to study all module categories over given $\mathcal{C}$.

Reasonable class of module categories for a fusion category $\mathcal{C}$: finite semisimple ones

Module categories II

Examples

- $\mathcal{M} = \mathcal{C}$ is module category over $\mathcal{C}$
- If $F : \mathcal{C} \to \mathcal{D}$ is a tensor functor then $\mathcal{D}$ is a module category over $\mathcal{C}$
- Let H be a Hopf algebra and let $\mathcal{C} = \mathrm{Rep}^{fd}(H)$. Then there is a forgetful tensor functor $\mathcal{C} \to Vec$. Thus Vec is a module category over $\mathrm{Rep}^{fd}(H)$
- In general: $\mathcal{M}$ is module category over $\mathcal{C}$ $\Leftrightarrow$ there is a tensor functor $\mathcal{C} \to Fun(\mathcal{M}, \mathcal{M})$ (category of exact functors $\mathcal{M} \to \mathcal{M}$)

Direct sums

There is an easy operation of direct sum $\mathcal{M}_1 \oplus \mathcal{M}_2$. Each module category as above is a direct sum of indecomposable ones in a unique way. Thus for a compete classification it is enough to describe *indecomposable* module categories.

Module categories and correlation functions

Theorem (Fjelstad, Fuchs, Runkel, Schweigert)

Let V be a rational vertex algebra and let $\mathcal{M}$ be an indecomposable (finite semisimple) module category over Rep(V) satisfying some condition. Then there is a way to combine conformal blocks of V into a consistent system of correlation functions.

Full RCFT: good rational vertex algebra V and an indecomposable module category over Rep(V).

Physical interpretation of objects of $\mathcal{M}$: *boundary conditions*

Guess for LCFT: *exact module categories*

Let $\mathcal{C}$ be a finite tensor category. A module category over $\mathcal{C}$ is exact if $P \otimes M$ is projective whenever $P \in \mathcal{C}$ is (notice that $X \otimes M$ is automatically projective for a projective $M \in \mathcal{M}$).

Classifications of module categories

Theorem (Etingof, Nikshych, O)

For a given fusion category $\mathcal{C}$ there are just finitely many indecomposable module categories.

Examples

- Rep(G): Rep$^{\psi}(H)$ – representations of twisted group algebra $\mathbb{C}[H]_{\psi}$ where $H \subset G$, $\psi \in H^2(H, \mathbb{C}^{\times})$ (Bezrukavnikov, O)
- Vec_G^{ω}: (H, ψ) where $H \subset G$, $\partial\psi = \omega|_H$ (O)
- $\mathcal{Z}(Vec_G^{\omega})$: (H, ψ), $H \subset G \times G$, $\partial\psi = \tilde{\omega}|_H$ (O)
- $\mathcal{C}(sl_2, k)$: ADE classification (Cappelli, Itzykson, Zuber et al)
- $\mathcal{C}(sl_n, k)$: classification is known for $n = 3, 4$ (Ocneanu)
- Haagerup subfactor (Grossman, Snyder)

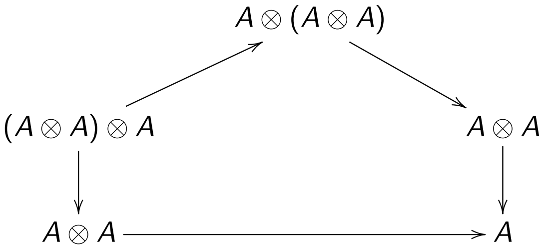
Problem: Classify module categories over $\mathcal{C}(\mathfrak{g}, k)$.

Algebras

Definition

An associative algebra with unit $A \in \mathcal{C}$ is a triple (A, m, i) where
$A \in \mathcal{C}$ is an object
$m : A \otimes A \to A$ *multiplication morphism*
$i : \mathbf{1} \to A$ *unit morphism*

Associativity axiom:



Algebras II

Unit axiom:

$A = \mathbf{1} \otimes A \to A \otimes A \to A$ is id_A

$A = A \otimes \mathbf{1} \to A \otimes A \to A$ is id_A

Examples

- If $X \in \mathcal{C}$ then $A = X \otimes X^*$ is an algebra:

$$i : \mathbf{1} \xrightarrow{coev} X \otimes X^*, \; m : X \otimes X^* \otimes X \otimes X^* \xrightarrow{\mathrm{id} \otimes ev \otimes \mathrm{id}} X \otimes X^*$$

- For $H \subset G$, $\mathbb{C}[H]_\psi$ is an algebra in Vec_G
- For $H \subset G$, $\psi \in Z^2(H, \mathbb{C}^\times)$ with $\partial\psi = \omega|_H$, $\mathbb{C}[H]_\psi$ is an algebra in Vec_G^ω

Commutative algebras

If $\mathcal{C}$ is braided we say that an algebra $A \in \mathcal{C}$ is *commutative* if $A \otimes A \xrightarrow{c_{AA}} A \otimes A \xrightarrow{m} A$ equals $m : A \otimes A \to A$

Modules

Definition

Let $A \in \mathcal{C}$ be an algebra. Right $A-$module is a pair (M, μ), $M \in \mathcal{C}$, $\mu : M \otimes A \to M$ such that
$(M \otimes A) \otimes A \xrightarrow{\mu \otimes \mathrm{id}_A} M \otimes A \xrightarrow{\mu} M$ coincides with
$(M \otimes A) \otimes A \xrightarrow{\alpha_{MAA}} M \otimes (A \otimes A) \xrightarrow{\mathrm{id}_M \otimes m} M \otimes A \xrightarrow{\mu} M$
and $M = M \otimes \mathbf{1} \to M \otimes A \to M$ is id_M.

Category of $A-$modules

Right $A-$modules form an abelian category $\mathcal{C}_A$: morphism from (M, μ) to (N, ν) is $f : M \to N$ such that

$$\begin{array}{ccc} M \otimes A & \xrightarrow{f \otimes \mathrm{id}} & N \otimes A \\ \downarrow \mu & & \downarrow \nu \\ M & \xrightarrow{f} & N \end{array}$$

commutes.

Observation: $\mathcal{C}_A$ has an obvious structure of module category over $\mathcal{C}$: $X \otimes M \in \mathcal{C}_A$ for $X \in \mathcal{C}$, $M \in \mathcal{C}_A$

Modules II

Definition

Assume $\mathcal{C}$ is fusion category. $A \in \mathcal{C}$ is *separable* if $\mathcal{C}_A$ is semisimple.

Theorem (O)

For a fusion category $\mathcal{C}$ any (semisimple) module category over $\mathcal{C}$ is of the form $\mathcal{C}_A$ for some separable algebra A.

Morita equivalence

Algebra A in the Theorem above is not unique!
Module categories over $\mathcal{C} \leftrightarrow$ separable algebras in $\mathcal{C}$ up to *Morita equivalence*

Example

Algebra $A = X \otimes X^*$ is Morita equivalent to algebra $\mathbf{1}$.

Bimodules and dual categories

For any algebra A we consider category ${}_A\mathcal{C}_A$ of $A-$bimodules.
${}_A\mathcal{C}_A$ is tensor category with tensor product $\otimes_A$ and unit object A.

Theorem (Etingof, Nikshych, O)

$\mathcal{C}$ is fusion category and $A \in \mathcal{C}$ is separable $\Rightarrow$ ${}_A\mathcal{C}_A$ is a fusion category.

Fact: ${}_A\mathcal{C}_A$ depends only on Morita equivalence class of A.
Notation: dual category $\mathcal{C}^*_{\mathcal{M}} := {}_A\mathcal{C}_A$ where $\mathcal{M} = \mathcal{C}_A$.
Fact (Müger): $\mathcal{C} \sim \mathcal{C}^*_{\mathcal{M}}$ is an equivalence relation.
This is weak Morita equivalence, or 2-Morita equivalence.
Example: $\mathrm{Rep}^{fd}(H)$ is 2-Morita equivalent to $\mathrm{Rep}^{fd}(H^*)$.

Theorem (Drinfeld; Kitaev; Etingof, Nikshych, O)

$\mathcal{C}$ and $\mathcal{D}$ are 2-Morita equivalent $\Leftrightarrow$ $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{Z}(\mathcal{D})$

Physical interpretation of objects of $\mathcal{C}^*_{\mathcal{M}}$: labels for defect lines

Étale algebras

Observation

Assume that $A \in \mathcal{C}$ is commutative. Then $\mathcal{C}_A$ is tensor category (with $\otimes_A$ as a tensor product)

Definition

An étale algebra in a braided fusion category $\mathcal{C}$ is algebra which is both commutative and separable.

An étale algebra $A \in \mathcal{C}$ is *connected* if $\mathrm{Hom}(\mathbf{1}, A) = \mathbb{C}$
Any étale algebra decomposes uniquely into a direct sum of connected ones

Lemma

Assume that $A \in \mathcal{C}$ is connected étale. Then $\mathcal{C}_A$ is a fusion category (usually not braided). Moreover, we have a surjective *tensor functor $\mathcal{C} \to \mathcal{C}_A$, $X \mapsto X \otimes A$*

Roles of étale algebras

1. Extensions of vertex algebras
2. Kernels of central functors
3. Kernels of tensor functors
4. Quantum Manin pairs
5. Modular invariants
6. Left/right centers

Extensions of vertex algebras

Let V be a vertex algebra.
Question: What are possible extensions $W \supset V$?

Theorem (Kirillov Jr.,O; Huang,Kirillov Jr.,Lepowsky)

Assume that V is good rational, so $Rep(V)$ is MTC.
Vertex algebra extensions $\leftrightarrow$ (some) étale algebras in $Rep(V)$.

This produces many interesting examples for categories $\mathcal{C}(\mathfrak{g}, k)$ via the theory of *conformal embeddings*

Dyslexia (Pareigis)

What is $\mathrm{Rep}(W)$ in the categorical terms?
Answer: *dyslectic* (or *local*) modules
$\mathcal{C}_A^0 = \{M \in \mathcal{C}_A | \mu \circ c_{AM} \circ c_{MA} = \mu\} \subset \mathcal{C}_A$

Kernels of central functors

Central functors

Let $\mathcal{C}$ be a braided category and $\mathcal{D}$ be a tensor category. A *central functor* $F : \mathcal{C} \to \mathcal{D}$ is a tensor functor together with isomorphisms $F(X) \otimes Y \simeq Y \otimes F(X)$ satisfying some axioms. Equivalently, this is a factorization $\mathcal{C} \to \mathcal{Z}(\mathcal{D}) \to \mathcal{D}$ where functor $\mathcal{C} \to \mathcal{Z}(\mathcal{D})$ is braided.

Observation: The functor $\mathcal{C} \to \mathcal{C}_A$ has a natural structure of central functor.

Theorem (Davydov, Müger, Nikshych, O)

Conversely, let $F : \mathcal{C} \to \mathcal{D}$ be a central functor between fusion categories. Let $I : \mathcal{D} \to \mathcal{C}$ be the right adjoint functor *of F. Then $A = I(\mathbf{1}) \in \mathcal{C}$ has a natural structure of (connected) étale algebra; moreover the central functor $\mathcal{C} \to F(\mathcal{C}) \subset \mathcal{D}$ is isomorphic to $\mathcal{C} \to \mathcal{C}_A$*

Kernels of tensor functors

Let $\mathcal{C}$ be a tensor category and let $A \in \mathcal{Z}(\mathcal{C})$ be a commutative algebra.
Observation (Schauenburg): $\mathcal{C}_A$ is a tensor category and there is a tensor functor $\mathcal{C} \to \mathcal{C}_A$, $X \mapsto X \otimes A$.

Theorem (Schauenburg)

$\mathcal{Z}(\mathcal{C}_A) = \mathcal{Z}(\mathcal{C})_A^0$

Theorem (Kitaev; Bruguières, Natale; Davydov, Müger, Nikshych, O)

Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor between (multi-)fusion categories. Let $I : \mathcal{D} \to \mathcal{C}$ be the right adjoint functor *of F. Then $A = I(\mathbf{1}) \in \mathcal{C}$ has a natural lift to $\mathcal{Z}(\mathcal{C})$; in addition $I(\mathbf{1}) \in \mathcal{Z}(\mathcal{C})$ has a natural structure of étale algebra. Moreover the tensor functor $\mathcal{C} \to F(\mathcal{C}) \subset \mathcal{D}$ is isomorphic to $\mathcal{C} \to \mathcal{C}_A$*

Quantum Manin pairs

Let $\mathcal{A}$ be a fusion category. The forgetful functor $\mathcal{Z}(\mathcal{A}) \to \mathcal{A}$ is central and surjective. Let $A = I(\mathbf{1}) \in \mathcal{Z}(\mathcal{A})$. Then $\mathcal{A} = \mathcal{Z}(\mathcal{A})_A$.

Theorem (Kitaev; Davydov, Müger, Nikshych, O)

Let $\mathcal{C}$ be a non-degenerate braided fusion category and $A \in \mathcal{C}$ be an étale algebra. The functor $\mathcal{C} \to \mathcal{C}_A$ is isomorphic to the forgetful functor $\mathcal{Z}(\mathcal{A}) \to \mathcal{A}$ if and only if $\mathcal{C}_A^0 = Vec$.

Definition

Lagrangian algebra: connected étale algebra A in a non-degenerate braided fusion category $\mathcal{C}$ such that $\mathcal{C}_A^0 = Vec$.
Quantum Manin pair: $(\mathcal{C}, A)$ where $A \in \mathcal{C}$ is Lagrangian.

Example (non-degenerate pointed category $\mathcal{C}(A, q)$)

étale algebras in $\mathcal{C}(A, q) \leftrightarrow$ isotropic subgroups $(H \subset A, q|_H = 1)$
Lagrangian algebras in $\mathcal{C}(A, q) \leftrightarrow$ Lagrangian subgroups $(H = H^\perp)$

Quantum Manin pairs II

Example

There is a conformal embedding $so(5)_{12} \subset (E_8)_1$. Since $\mathcal{C}(E_8, 1) = Vec$ we see that $\mathcal{C}(so(5), 12) = \mathcal{Z}(\mathcal{A})$ for some $\mathcal{A}$.

Module category and Lagrangian algebras

Assume that $\mathcal{M}$ is a module category over $\mathcal{A}$. Then there is a functor $\mathcal{A} \to Fun(\mathcal{M}, \mathcal{M})$ described by a connected étale algebra $B \in \mathcal{Z}(\mathcal{A})$.

Theorem (Kong, Runkel; Etingof, Nikshych, O; Davydov, Müger, Nikshych, O)

Algebra $B \in \mathcal{Z}(\mathcal{A})$ is Lagrangian. Moreove, the assignment $\mathcal{M} \mapsto B$ is a bijection: indecomposable module categories over $\mathcal{A} \leftrightarrow$ Lagrangian algebras $B \in \mathcal{Z}(\mathcal{A})$

Aside: lattice of subcategories of $\mathcal{A}$ is anti-isomorphic to lattice of étale subalgebras of $I(\mathbf{1}) \in \mathcal{Z}(\mathcal{A})$

Modular invariants (after Rehren)

Reminder: full RCFT $\Leftrightarrow$ vertex algebra V and module category $\mathcal{M}$ over $\mathcal{C} = \text{Rep}(V) \Leftrightarrow$ vertex algebra V and Lagrangian algebra $\mathcal{L} \in \mathcal{Z}(\mathcal{C})$.
$\mathcal{C}$ is MTC, so $\mathcal{Z}(\mathcal{C}) = \mathcal{C} \boxtimes \mathcal{C}^{rev}$; $\mathcal{L} \in \mathcal{C} \boxtimes \mathcal{C}^{rev}$ is bulk algebra
The class $[\mathcal{L}] \in K(\mathcal{Z}(\mathcal{C}))$ can be written as $\sum_{i,j} Z_{ij}[X_i \boxtimes X_j]$ where $Z_{ij} \in \mathbb{Z}_{\geq 0}$, $Z_{00} = 1$ (since $\mathcal{L}$ is connected).

Theorem (Böckenhauer, Evans, Kawahigashi; Fuchs, Runkel, Schweigert)

Assume that Z_{ij} commutes with $T-$matrix. Then Z_{ij} commutes with $S-$matrix; that is Z_{ij} is a modular invariant.

Remark. If $\dim(X_i) > 0$ then $[Z, T] = 0$ automatically.

Physical modular invariants

Physical modular invariant = modular invariant of the form $[\mathcal{L}]$
Modular invariant can be physical in more than one way.

Modular invariants II

Construction of étale algebras in $\mathcal{C} \boxtimes \mathcal{D}$

Pick étale algebras $A \in \mathcal{C}$, $B \in \mathcal{D}$, tensor subcategories $\mathcal{C}_1 \subset \mathcal{C}_A^0$ and $\mathcal{D}_1 \subset \mathcal{D}_B^0$ and a braided equivalence $\phi : \mathcal{C}_1 \simeq \mathcal{D}_1^{rev}$. Then $\oplus_{M \in Irr(\mathcal{C}_1)} M \boxtimes \phi(M)^*$ has a natural structure of étale algebra.

Theorem (Müger; Davydov, Nikshych, O)

Any connected étale algebra in $\mathcal{C} \boxtimes \mathcal{D}$ is isomorphic to one constructed above.

This applies to $\mathcal{Z}(\mathcal{C}) = \mathcal{C} \boxtimes \mathcal{C}^{rev}$ where $\mathcal{C}$ is non-degenerate (e.g. MTC). Algebra above is Lagrangian $\Leftrightarrow$ $\mathcal{C}_1 = \mathcal{C}_A^0$, $\mathcal{D}_1 = (\mathcal{C}^{rev})_B^0$.

Corollary (Böckenhauer, Evans; Fuchs, Runkel, Schweigert)

Indecomposable module categories over a non-degenerate braided fusion category $\mathcal{C}$ are labeled by triples (A, B, ϕ) where $A, B \in \mathcal{C}$ are connected étale algebras and $\phi : \mathcal{C}_A^0 \to \mathcal{C}_B^0$ is a braided equivalence.

Modular invariants III

Corollary (Etingof, Nikshych, O)

For a non-degenerate $\mathcal{C}$, $Aut^{br}(\mathcal{C}) \leftrightarrow$ invertible module categories $Pic(\mathcal{C})$

Physical interpretation $\sim$ 1989 (Moore, Seiberg; Dijkgraaf, Verlinde)

Algebra $\mathcal{L} \in \mathcal{C} \boxtimes \mathcal{C}^{rev}$ considered as a vector space $\bigoplus_{i,j}(X_i \otimes X_j)^{Z_{ij}}$ – **Hilbert space of states**

$[\mathcal{L}]$ considered as a linear combination of characters $\sum_{i,j} Z_{ij}\chi_i\overline{\chi_j}$ – **partition function** of the theory

type I theory – $A = B$ and $\phi = \mathrm{id}$

type II theory – $A = B$ and $\phi \neq \mathrm{id}$

heterotic theory – $A \neq B$

Example

Example: $\mathcal{C}(G_2, 3)-$modular invariant $|\chi_{00} + \chi_{11}|^2 + 2|\chi_{02}|^2$ has 2 distinct physical realizations.

Left/right centers

Two centers

Let $E \in \mathcal{C}$ be an algebra in a braided category $\mathcal{C}$.

Left center: biggest $C_l(E) \subset E$ such that $C_l(E) \otimes E \xrightarrow{m} E$ equals $C_l(E) \otimes E \xrightarrow{c_{C_l(E)E}} E \otimes C_l(E) \xrightarrow{m} E$.

Right center: biggest $C_r(E) \subset E$ such that $E \otimes C_r(E) \xrightarrow{m} E$ equals $E \otimes C_r(E) \xrightarrow{c_{EC_r(E)}} C_r(E) \otimes E \xrightarrow{m} E$.

Theorem (Fuchs, Schweigert, Runkel)

Let E be a separable algebra in a (non-degenerate) braided fusion category $\mathcal{C}$. Then $C_l(E)$ and $C_r(E)$ are étale. Moreover, there is a braided equivalence $\mathcal{C}^0_{C_l(E)} \simeq \mathcal{C}^0_{C_r(E)}$.

Proof.

Let $\mathcal{L} = \mathcal{L}(A, B, \phi)$ be Lagrangian algebra associated with $\mathcal{M} = \mathcal{C}_E$. Then $C_l(E) = A$ and $C_r(E) = B$. □

Thanks for listening!