Growth in tensor powers

Victor Ostrik
University of Oregon
vostrik@uoregon.edu

July 7

arxiv: 2107.02372, 2301.00885, 2301.09804
(jt with Kevin Coulembier, Pavel Etingof, Daniel Tubbenhauer)
Tensor powers

Setup

$F$ – any field; $\Gamma$ – any group (or affine group scheme)
$V$ – any finite dimensional representation of $\Gamma$ over $F$

$V \otimes^n = V \otimes V \otimes \ldots \otimes V$ ($n$ times)

$V \otimes^n = \bigoplus_{i=1}^{b_n(V)} W_i$ where $W_i$ are indecomposable $\Gamma$–modules

Question: What can we say about sequence $b_n(V)$? e.g. its growth?

Theorem (K. Coulembier, V. O., D. Tubbenhauer)

$$\lim_{n \to \infty} \sqrt[n]{b_n(V)} = \dim(V)$$

Example

$\Gamma$ finite, $F = \mathbb{C}$, $M = \max\{\chi(1) | \chi - \text{irreducible complex character}\}$

$$\frac{1}{M} \dim(V)^n \leq b_n(V) \leq \dim(V \otimes^n) = \dim(V)^n$$
Example

Γ finite, $F$ of characteristic $p > 0$
Then Γ typically has indecomposable representations of arbitrarily large dimension
However Γ has finitely many *projective indecomposable modules* (PIMs)

**Theorem.** (R. Bryant - L. Kovacs) Assume $V$ is faithful. Then $V \otimes^n$ contain a projective summand for $n \gg 0$.

**Corollary** Almost all summands of $V \otimes^n$ are projective over the image of Γ in $GL(V)$ (i.e. dimension of all non-projective summands in $V \otimes^n$ is less than $Kr^n$ where $r < \dim(V)$ and $K > 0$).

\[ M = \max \{\dim(P) \mid P - \text{PIM for image of } \Gamma \text{ in } GL(V)\} \]

\[ \frac{1}{M}(\dim(V)^n - Kr^n) \leq b_n(V) \leq \dim(V)^n \]
\( \text{SL}(2): \text{characteristic zero} \)

**Example**

\( F = \mathbb{C}, \Gamma = \text{SL}(2), \, V - \text{tautological 2-dimensional representation} \)

\[
\text{ch}(V) = q + q^{-1}, \quad \text{ch}(V \otimes^n) = (q + q^{-1})^n
\]

\[
b_n(V) = \left( \begin{array}{c} n \\ \left\lfloor \frac{n}{2} \right\rfloor \end{array} \right) \sim \frac{2^n}{\sqrt{\pi n/2}}
\]

**Example**

\( F = \mathbb{C}, \Gamma = \text{SL}(2), \, V_2 - \text{irreducible 3-dimensional representation} \)

\[
\text{ch}(V_2) = q^2 + 1 + q^{-2}, \quad \text{ch}(V_2 \otimes^n) = (q^2 + 1 + q^{-2})^n
\]

\[
b_n(V) = \text{free term of } (q^2 + 1 + q^{-2})^n \sim K \frac{3^n}{\sqrt{n}} \quad \text{(CLT)}
\]

**Generalization (P. Biane (1993) et al):**

\( \Gamma \text{ reductive over } F = \mathbb{C}: \ b_n(V) \sim K \frac{\dim(V)^n}{n^{b/2}} \text{ where } b = |R_+| \text{ integer} \)
Growth knows about $\Gamma$

**Theorem (K. Coulembier, V. O., D. Tubbenhauer)**

Assume $\text{char } F = 0$ and there is $K > 0$ such that $b_n(V) \geq K \dim(V)^n$. Then Zariski closure of the image of $\Gamma$ in $GL(V)$ is a finite group extended by torus. Equivalently, $\Gamma \supset \Gamma_0$ such that $[\Gamma : \Gamma_0] < \infty$ and the image of $\Gamma_0$ consists of simultaneously diagonalizable matrices.

**Question:** What about $\text{char } F > 0$?

**Remark:** Even for $F = \mathbb{C}$, $\Gamma$ finite the limit

$$\lim_{n \to \infty} \frac{b_n(V)}{\dim(V)^n}$$

might fail to exist.

**Example**

$F = \mathbb{C}$, $\Gamma = D_8$, $V$ - 2-dimensional irreducible

$$\frac{b_n(V)}{\dim(V)^n} = 1 \text{ or } \frac{1}{2} \text{ depending on parity of } n$$
Example

\[ \text{char } F = p > 0, \Gamma = SL(2), \ V - \text{tautological 2-dimensional representation} \]
\[ V \otimes^n - \text{direct sum of tilting } SL(2) - \text{modules} \]

H. H. Andersen/S. Donkin: combinatorial description of \( b_n(V) \)

Numerically: \( K' \frac{2^n}{n^{\alpha_p}} \leq b_n(V) \leq K'' \frac{2^n}{n^{\alpha_p}} \) for some \( \alpha_p \) and \( K', K'' > 0 \)

where \( \alpha_2 \approx 0.7075, \alpha_3 \approx 0.6845 \)

Conjecture (P. Etingof): \( K' \frac{2^n}{n^{\alpha_p}} \leq b_n(V) \leq K'' \frac{2^n}{n^{\alpha_p}} \) for some \( K', K'' > 0 \)

\[
\begin{align*}
\alpha_2 &= \frac{1}{2} \log_2 \frac{8}{3} \\
\alpha_3 &= \frac{1}{2} \log_3 \frac{9}{2} \\
\alpha_p &= \frac{1}{2} \log_p \frac{2p^2}{p + 1}
\end{align*}
\]

perhaps \( b_n(V) \sim K \frac{2^n}{n^{\alpha_p}} \) for \( p \geq 3 \)

Question: What about other representations of \( SL(2) \)?

Is the exponent \( \alpha_p \) universal?

Question What about other groups? e.g. \( SL(3) \)?
Few words about proof

**Theorem (K. Coulembier, V. O., D. Tubbenhauer)**

For any group $\Gamma$, field $F$, representation $V$ we have

$$\lim_{n \to \infty} \sqrt[n]{b_n(V)} = \dim(V)$$

**Step 1:** Clearly $b_n(V) \leq \dim(V)^n$ so we need a lower bound for $b_n(V)$

Hence we can assume $\Gamma = GL(V)$ (done if char $F = 0$!)

**Step 2:** $GL(V)$–module $V \otimes^n$ is a direct sum of tilting modules, so it is determined by its character

**Difficulty:** characters of indecomposable tilting modules are not known for $\dim(V) \geq 3$ (conjecture by Lusztig-Williamson for $\dim(V) = 3$)

Use partial information (block of Steinberg module)...

**Remark:** $\Gamma$ can be *Lie algebra, semigroup, super group* or *super Lie algebra, quantum group at root of 1*

Also $V$ can be an object of a *Tannakian category*

**Warning:** counterexamples for comodules over Hopf algebras
Other counts: non-projective summands

D. Benson, P. Symonds: $\Gamma$ finite, char $F = p > 0$

$c_n(V) =$ total dimension of non-projective summands in $V \otimes^n$

$$\gamma(V) := \lim_{n \to \infty} n^{\sqrt{c_n(V)}}$$

- The limit exists! but difficult to compute...
- $\gamma(V)$ is not necessarily an integer
- $0 \leq \gamma(V) \leq \dim(V)$, $\gamma(V) = 0 \iff V$ is projective
- $\gamma(V) > 0 \implies \gamma(V) \geq 1$, $\gamma(V) > 1 \implies \gamma(V) \geq \sqrt{2}$
- **Conjecture:** $\gamma(V)$ is an algebraic integer
- $\gamma(V \oplus W) \neq \gamma(V) + \gamma(W)$ in general
- $\gamma(V \otimes W) \neq \gamma(V)\gamma(W)$ in general

Consider $c'_n(V) =$ number of non-projective summands in $V \otimes^n$
and define $\gamma'(V) = \lim_{n \to \infty} n^{\sqrt{c'_n(V)}}$

- Open True/False question: is $\gamma(V) = \gamma'(V)$ for all $V$?
Example

$\Gamma = \mathbb{Z}/5\mathbb{Z}$, $p = 5$, representation: $1 \mapsto A$, $A^5 = \text{Id} \iff (A - \text{Id})^5 = 0$

Indecomposable representations: Jordan cells $J_1, J_2, J_3, J_4, J_5$

$J_3 : 1 \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$J_1$ is trivial and the only simple

$J_5$ is the only projective

Tensor products: $J_1 \otimes J_i = J_i$ $J_3 \otimes J_3 = J_1 + J_3 + J_5$ $J_3 \otimes J_5 = 3J_5$

Take $V = J_3$ and let $V^\otimes n = A_nJ_1 + B_nJ_3 + C_nJ_5$

Then $A_{n+1} = B_n$ (so $A_n = B_{n-1}$) $B_{n+1} = A_n + B_n$ $C_{n+1} = B_n + 3C_n$

Hence $B_{n+1} = B_{n-1} + B_n = F_n = c'_n(V)$ (Fibonacci number) and $c_n(V) = A_n + 3B_n = B_{n+2} + B_n$ (Lucas number) $\implies \gamma(V) = \frac{1 + \sqrt{5}}{2}$

Exercise. Compute $\gamma(J_2)$ and $\gamma(J_4)$ (of course $\gamma(J_1) = 1$ and $\gamma(J_5) = 0$)
Other counts: non-negligible summands

Assume $F$ is algebraically closed
$W$ – indecomposable representation of a group $\Gamma$ (or super group scheme)

**Definition**

$W$ is **negligible** if $\dim(W) = 0 \in F$ (take $\text{sdim}(W)$ for super groups)
$W$ is **non-negligible** if $\dim(W) \neq 0 \in F$

**Remark:** More generally, (possibly decomposable) $W$ is negligible if every indecomposable summand is negligible
Negligible representations form tensor ideal

$$d_n(V) = \text{total number of non-negligible summands in } V \otimes^n$$

$$\delta(V) := \lim_{n \to \infty} n \sqrt[n]{d_n(V)}$$

**Observation:** $d_{n+m}(V) \geq d_n(V)d_m(V)$ and $d_n(V) \leq \dim(V)^n$

**Fekete’s Lemma** implies $\delta(V) := \lim_{n \to \infty} \sqrt[n]{d_n(V)}$ exists
Properties of $\delta$

Obvious properties:

- $\delta(V \oplus W) \geq \delta(V) + \delta(W)$
- $\delta(V \otimes W) \geq \delta(V)\delta(W)$
- $\delta(V) = 0 \iff V$ is negligible
- $\delta(V) > 0 \Rightarrow 1 \leq \delta(V) \leq \dim(V)$

**Theorem (K. Coulembier, P. Etingof, V. O.)**

1. $\delta(V \oplus W) = \delta(V) + \delta(W)$ and $\delta(V \otimes W) = \delta(V)\delta(W)$.
2. Let $q = q_p = e^{\frac{\pi i}{p}}$ and $[m]_q := \frac{q^m - q^{-m}}{q - q^{-1}} = q^{m-1} + \ldots + q^{1-m}$ for $m \in \mathbb{N}$. Then $\delta(V) =$ linear combination of $[m]_q$, $1 \leq m \leq \frac{p}{2}$ with nonnegative integer coefficients.

**Example**

For $p = 2$ or $p = 3$ we say that $\delta(V) \in \mathbb{Z}_{\geq 0}$

For $p = 5$, $\delta(V) = a + b\frac{1+\sqrt{5}}{2}$ where $a, b \in \mathbb{Z}_{\geq 0}$ (since $[2]_{q_5} = \frac{1+\sqrt{5}}{2}$)
Example

\[ \Gamma \quad p \quad V \quad \dim(V) \quad \gamma(V) \quad \delta(V) \quad d_n(V) \quad \text{note} \]

| \( \mathbb{Z}/5\mathbb{Z} \) | 5 | \( J_3 \) | 3 | \( \frac{1+\sqrt{5}}{2} \) | \( \frac{1+\sqrt{5}}{2} \) | \( F_n \) | \( = c'_n(V) \) |
| \( \mathbb{Z}/8\mathbb{Z} \) | 2 | \( J_5 \) | 5 | 3 | 1 | 1 | |
| \( \mathbb{Z}/9\mathbb{Z} \) | 3 | \( J_5 \) | 5 | 3 | 2 | \( \frac{1}{3}(2^{n+1} + (-1)^n) \) | \( = d_n(W_{S_3}) \) |

\( W_{S_3} - 2 \)-dimensional representation of \( S_3 \) over \( \mathbb{C} \)

Example

Assume \( p = 2 \) and \( \dim(V) = 3 \) or \( p = 3 \) and \( \dim(V) = 2 \)

Then exactly one of the following is true:

(a) all summands of \( V \otimes^n \) are non-negligible for all \( n \)
(b) exactly one summand of each \( V \otimes^n \) is non-negligible for all \( n \)

Define \( d'_n(V) = \text{total dimension of non-negligible summands in } V \otimes^n \)
and \( \delta'(V) := \lim_{n \to \infty} \sqrt[n]{d'_n(V)} \)

**Question:** is \( \delta(V) = \delta'(V) \) for any \( V \)?
Theorem (K. Coulembier, P. Etingof, V. O.)

There are constants $K', K'' > 0$ such that

$$K' \delta(V)^n \leq d_n(V) \leq K'' \delta(V)^n$$

In fact we can take $K'' = 1$ (elementary) and we prove that for $p > 0$

$$c(V) = \liminf_{n \to \infty} \frac{d_n(V)}{\delta(V)^n} > 0$$

**Conjecture:** $c(V) \geq e^{-a_p \delta(V)}$ for some $a_p \in \mathbb{R}_{>0}$.

This is **true** for $p = 2$ and $p = 3$ with

$$a_2 = \frac{4 \ln(3)}{3} \approx 1.464, \quad a_3 = 24$$

For $p \geq 5$ we have

$$c(V) \geq \exp(-a_p \delta(V) - \frac{\pi \ln(2)}{2}(p - 2)\delta(V)^2)$$

More knowledge about tensor categories is required!
Thanks for listening!