Sumy–Eugene

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Growth in tensor powers

Victor Ostrik

University of Oregon

vostrik@uoregon.edu

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(jt with Kevin Coulembier, Pavel Etingof, Daniel Tubbenhauer)

Tensor powers

Setup

F – any field; Γ – any group (or affine group scheme)
V – any finite dimensional representation of Γ over F
$V^{\otimes n} = V \otimes V \otimes \ldots \otimes V$ (n times)
$V^{\otimes n} = \bigoplus_{i=1}^{b_n(V)} W_i$ where W_i are indecomposable $\Gamma-$modules

Question: What can we say about sequence $b_n(V)$? e.g. its growth?

Theorem (K. Coulembier, V. O., D. Tubbenhauer)

$$\lim_{n\to\infty} \sqrt[n]{b_n(V)} = \dim(V)$$

Example

Γ finite, $F = \mathbb{C}$, $M = \max\{\chi(1) \mid \chi -$ irreducible complex character$\}$

$$\frac{1}{M}\dim(V)^n \le b_n(V) \le \dim(V^{\otimes n}) = \dim(V)^n$$

Finite groups: modular case

Example

Γ finite, F of characteristic $p > 0$
Then Γ typically has indecomposable representations of arbitrarily large dimension
However Γ has finitely many *projective indecomposable modules* (PIMs)

Theorem. (R. Bryant - L. Kovacs) Assume V is faithful. Then $V^{\otimes n}$ contain a projective sumand for $n \gg 0$.

Corollary Almost all summands of $V^{\otimes n}$ are projective over the image of Γ in $GL(V)$ (i.e. dimension of all non-projective summands in $V^{\otimes n}$ is less than Kr^n where $r < \dim(V)$ and $K > 0$).

$M = \max\{\dim(P) \mid P - \text{PIM for image of } \Gamma \text{ in } GL(V)\}$

$$\frac{1}{M}(\dim(V)^n - Kr^n) \leq b_n(V) \leq \dim(V)^n$$

$SL(2)$: characteristic zero

Example

$F = \mathbb{C}$, $\Gamma = SL(2)$, V – tautological 2-dimensional representation
$\mathrm{ch}(V) = q + q^{-1}$, $\mathrm{ch}(V^{\otimes n}) = (q + q^{-1})^n$

$$b_n(V) = \binom{n}{[\frac{n}{2}]} \sim \frac{2^n}{\sqrt{\pi n/2}}$$

Example

$F = \mathbb{C}$, $\Gamma = SL(2)$, V_2 – irreducible 3-dimensional representation
$\mathrm{ch}(V_2) = q^2 + 1 + q^{-2}$, $\mathrm{ch}(V_2^{\otimes n}) = (q^2 + 1 + q^{-2})^n$

$$b_n(V) = \text{free term of } (q^2 + 1 + q^{-2})^n \sim K \frac{3^n}{\sqrt{n}} \text{ (CLT)}$$

Generalization (P. Biane (1993) et al):

Γ reductive over $F = \mathbb{C}$: $b_n(V) \sim K \frac{\dim(V)^n}{n^{b/2}}$ where $b = |R_+|$ integer

Growth knows about Γ

Theorem (K. Coulembier, V. O., D. Tubbenhauer)

Assume char $F = 0$ and there is $K > 0$ such that $b_n(V) \geq K \dim(V)^n$. Then Zariski closure of the image of Γ in $GL(V)$ is a finite group extended by torus. Equivalently, $\Gamma \supset \Gamma_0$ such that $[\Gamma : \Gamma_0] < \infty$ and the image of Γ_0 consists of simultaneously diagonalizable matrices.

Question: What about char $F > 0$?

Remark: Even for $F = \mathbb{C}$, Γ finite the limit

$$\lim_{n\to\infty} \frac{b_n(V)}{\dim(V)^n}$$

might fail to exist.

Example

$F = \mathbb{C}$, $\Gamma = D_8$, V - 2-dimensional irreducible
$\frac{b_n(V)}{\dim(V)^n} = 1$ or $\frac{1}{2}$ depending on parity of n

SL(2): modular case

Example

char $F = p > 0$, $\Gamma = SL(2)$, V – tautological 2-dimensional representation
$V^{\otimes n}$ - direct sum of tilting $SL(2)-$modules
H. H. Andersen/S. Donkin: combinatorial description of $b_n(V)$
Numerically: $K' \frac{2^n}{n^{\alpha_p}} \leq b_n(V) \leq K'' \frac{2^n}{n^{\alpha_p}}$ for some α_p and $K', K'' > 0$
where $\alpha_2 \approx 0.7075$, $\alpha_3 \approx 0.6845$

Conjecture (P. Etingof): $K' \frac{2^n}{n^{\alpha_p}} \leq b_n(V) \leq K'' \frac{2^n}{n^{\alpha_p}}$ for some $K', K'' > 0$

$$\alpha_2 = \frac{1}{2} \log_2 \frac{8}{3} \qquad \alpha_3 = \frac{1}{2} \log_3 \frac{9}{2} \qquad \alpha_p = \frac{1}{2} \log_p \frac{2p^2}{p+1}$$

perhaps $b_n(V) \sim K \frac{2^n}{n^{\alpha_p}}$ for $p \geq 3$

Question: What about other representations of $SL(2)$?
Is the exponent α_p universal?
Question What about other groups? e.g. $SL(3)$?

Few words about proof

Theorem (K. Coulembier, V. O., D. Tubbenhauer)

For any group Γ, field F, representation V we have

$$\lim_{n\to\infty} \sqrt[n]{b_n(V)} = \dim(V)$$

Step 1: Clearly $b_n(V) \leq \dim(V)^n$ so we need a lower bound for $b_n(V)$
Hence we can assume $\Gamma = GL(V)$ (done if char $F = 0$!)
Step 2: $GL(V)-$module $V^{\otimes n}$ is a direct sum of tilting modules,
so it is determined by its character
Difficulty: characters of indecomposable tilting modules are not known for $\dim(V) \geq 3$ (conjecture by Lusztig-Williamson for $\dim(V) = 3$)
Use partial information (block of Steinberg module)...
Remark: Γ can be *Lie algebra*, *semigroup*, *super group* or *super Lie algebra*, *quantum group at root of 1*
Also V can be an object of a *Tannakian category*
Warning: counterexamples for comodules over Hopf algebras

Other counts: non-projective summands

D. Benson, P. Symonds: Γ finite, char $F = p > 0$

$c_n(V)$ = total dimension of non-projective summands in $V^{\otimes n}$

$$\gamma(V) := \lim_{n\to\infty} \sqrt[n]{c_n(V)}$$

- The limit exists! but difficult to compute...
- $\gamma(V)$ is not necessarily an integer
- $0 \le \gamma(V) \le \dim(V)$, $\gamma(V) = 0 \Leftrightarrow V$ is projective
- $\gamma(V) > 0 \Rightarrow \gamma(V) \ge 1$, $\gamma(V) > 1 \Rightarrow \gamma(V) \ge \sqrt{2}$
- **Conjecture:** $\gamma(V)$ is an algebraic integer
- $\gamma(V \oplus W) \neq \gamma(V) + \gamma(W)$ in general
- $\gamma(V \otimes W) \neq \gamma(V)\gamma(W)$ in general

Consider $c_n'(V)$ = number of non-projective summands in $V^{\otimes n}$
and define $\gamma'(V) = \lim_{n\to\infty} \sqrt[n]{c_n'(V)}$
• Open True/False question: is $\gamma(V) = \gamma'(V)$ for all V?

Example

$\Gamma = \mathbb{Z}/5\mathbb{Z}$, $p = 5$, representation: $1 \mapsto A$, $A^5 = \mathrm{Id} \Leftrightarrow (A - \mathrm{Id})^5 = 0$

Indecomposable representations: Jordan cells J_1, J_2, J_3, J_4, J_5

$$J_3 : 1 \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

J_1 is trivial and the only simple

J_5 is the only projective

Tensor products: $\boxed{J_1 \otimes J_i = J_i}$ $\boxed{J_3 \otimes J_3 = J_1 + J_3 + J_5}$ $\boxed{J_3 \otimes J_5 = 3J_5}$

Take $V = J_3$ and let $V^{\otimes n} = A_n J_1 + B_n J_3 + C_n J_5$

Then $\boxed{A_{n+1} = B_n}$ (so $A_n = B_{n-1}$) $\boxed{B_{n+1} = A_n + B_n}$ $\boxed{C_{n+1} = B_n + 3C_n}$

Hence $B_{n+1} = B_{n-1} + B_n = F_n = c'_n(V)$ (Fibonacci number) and $c_n(V) = A_n + 3B_n = B_{n+2} + B_n$ (Lucas number)$\Rightarrow \gamma(V) = \frac{1+\sqrt{5}}{2}$

Exercise. Compute $\gamma(J_2)$ and $\gamma(J_4)$ (of course $\gamma(J_1) = 1$ and $\gamma(J_5) = 0$)

Other counts: non-negligible summands

Assume F is algebraically closed

W – indecomposable representation of a group Γ (or super group scheme)

Definition

W is *negligible* if $\dim(W) = 0 \in F$ (take $\mathrm{sdim}(W)$ for super groups)

W is non-negligible if $\dim(W) \neq 0 \in F$

Remark: More generally, (possibly decomposable) W is negligible if every indecomposable summand is negligible

Negligible representations form tensor ideal

$d_n(V) =$ total number of non-negligible summands in $V^{\otimes n}$

$$\delta(V) := \lim_{n\to\infty} \sqrt[n]{d_n(V)}$$

Observation: $d_{n+m}(V) \geq d_n(V) d_m(V)$ and $d_n(V) \leq \dim(V)^n$

Fekete's Lemma implies $\delta(V) := \lim_{n\to\infty} \sqrt[n]{d_n(V)}$ exists

Properties of δ

Obvious properties:

- $\delta(V \oplus W) \geq \delta(V) + \delta(W)$
- $\delta(V \otimes W) \geq \delta(V)\delta(W)$
- $\delta(V) = 0 \Leftrightarrow V$ is negligible
- $\delta(V) > 0 \Rightarrow 1 \leq \delta(V) \leq \dim(V)$

Theorem (K. Coulembier, P. Etingof, V. O.)

1. $\delta(V \oplus W) = \delta(V) + \delta(W)$ and $\delta(V \otimes W) = \delta(V)\delta(W)$.
2. Let $q = q_p = e^{\frac{\pi i}{p}}$ and $[m]_q := \frac{q^m - q^{-m}}{q - q^{-1}} = q^{m-1} + \ldots + q^{1-m}$ for $m \in \mathbb{N}$. Then $\delta(V) =$ linear combination of $[m]_q, 1 \leq m \leq \frac{p}{2}$ with nonnegative integer coefficients.

Example

For $p = 2$ or $p = 3$ we say that $\delta(V) \in \mathbb{Z}_{\geq 0}$
For $p = 5$, $\delta(V) = a + b\frac{1+\sqrt{5}}{2}$ where $a, b \in \mathbb{Z}_{\geq 0}$ (since $[2]_{q_5} = \frac{1+\sqrt{5}}{2}$)

Example

Γ	p	V	$\dim(V)$	$\gamma(V)$	$\delta(V)$	$d_n(V)$	note
$\mathbb{Z}/5\mathbb{Z}$	5	J_3	3	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	F_n	$= c'_n(V)$
$\mathbb{Z}/8\mathbb{Z}$	2	J_5	5	3	1	1	
$\mathbb{Z}/9\mathbb{Z}$	3	J_5	5	3	2	$\frac{1}{3}(2^{n+1} + (-1)^n)$	$= d_n(W_{S_3})$

W_{S_3} - 2-dimensional representation of S_3 over $\mathbb{C}$

Example

Assume $p = 2$ and $\dim(V) = 3$ or $p = 3$ and $\dim(V) = 2$

Then exactly one of the following is true:

(a) all summands of $V^{\otimes n}$ are non-negligible for all n

(b) exactly one summand of each $V^{\otimes n}$ is non-negligible for all n

Define $d'_n(V) =$ total dimension of non-negligible summands in $V^{\otimes n}$ and $\delta'(V) := \lim_{n\to\infty} \sqrt[n]{d'_n(V)}$

Question: is $\delta(V) = \delta'(V)$ for any V?

More

Theorem (K. Coulembier, P. Etingof, V. O.)

There are constants $K', K'' > 0$ such that

$$K'\delta(V)^n \leq d_n(V) \leq K''\delta(V)^n$$

In fact we can take $K'' = 1$ (elementary) and we prove that for $p > 0$

$$c(V) = \liminf_{n\to\infty} \frac{d_n(V)}{\delta(V)^n} > 0$$

Conjecture: $c(V) \geq e^{-a_p\delta(V)}$ for some $a_p \in \mathbb{R}_{>0}$.
This is true for $p = 2$ and $p = 3$ with

$$a_2 = \frac{4\ln(3)}{3} \approx 1.464, \quad a_3 = 24$$

For $p \geq 5$ we have $c(V) \geq \exp(-a_p\delta(V) - \frac{\pi\ln(2)}{2}(p-2)\delta(V)^2)$

More knowledge about tensor categories is required!

Thanks for listening!