THE MEAN CURVATURE FLOW

1. Evolution equation

Given a $n$ dimensional manifold $\Sigma$ and an immersion $F_0 : \Sigma \rightarrow \mathbb{R}^m$, consider a one parameter family of immersions $F(\cdot, t) : \Sigma \rightarrow \mathbb{R}^m$ evolved by the mean curvature flow

$$\frac{d}{dt}F(x, t) = H(x, t), \ F(x, 0) = F_0.$$ 

$\Sigma(t) = F(t) \circ \Sigma$ denotes the image of the immersion and $H(x, t)$ is the mean curvature vector of $\Sigma(t)$ at $F(x, t)$. For any $p \in \Sigma$, choose a local coordinates of $(x_1, \cdots, x_n)$, the induced metric on $\Sigma(t)$ is given by

$$g_{ij} = \langle \partial_i F, \partial_j F \rangle, \ \partial_i = \frac{\partial}{\partial x_i}.$$ 

Around $F(p, t) \in \Sigma(t) \subset \mathbb{R}^m$, we can choose a local frame consisting of $\{\partial_i, v_{\alpha}\}$, where $v_{\alpha}$ is an orthonormal basis of normal bundle along $\Sigma(t)$ for each $\alpha = n + 1, \cdots, m$ and we identify $\partial_i$ with $F_*(t)\partial_i$ along $\Sigma(t)$. We can always assume that at one point, $g_{ij} = \delta_{ij}$, $\partial_k g_{ij} = 0$. The second fundamental form is given by

$$A_{ij}^{\alpha} = A_{ij}^{\alpha} v_{\alpha},$$ 

where

$$A^\alpha = A^\alpha_{ij} dx_i \otimes dx_j$$

is a symmetric two tensor and

$$A^\alpha_{ij} = \langle v_{\alpha}, \partial_i \partial_j F \rangle.$$ 

The mean curvature vector is defined by

$$H^\alpha v_{\alpha} = g^{ij} A^\alpha_{ij} v_{\alpha}.$$ 

The Riemannian curvature tensor, the Ricci tensor and the scalar curvature are given by

$$R_{ijkl} = A^\alpha_{ik} A^\alpha_{jl} - A^\alpha_{il} A^\alpha_{jk},$$

$$R_{ik} = H^\alpha A^\alpha_{ik} - A^\alpha_{il} A^\alpha_{kl},$$

$$R = H^\alpha H^\alpha - A^\alpha_{ij} A^\alpha_{ij}.$$ 

Note we use the convention

$$R(\partial_i, \partial_j)\partial_k = \nabla_i \nabla_j \partial_k - \nabla_j \nabla_i \partial_k = R^{l}_{ijk} \partial_l,$$

$$R_{ijkl} = \langle R(\partial_i, \partial_j)\partial_k, \partial_l \rangle = R_{ijkl} g_{mk},$$

$$\langle \nabla_i \nabla_j - \nabla_j \nabla_i \rangle X^k = R^k_{ijkl} X^l,$$

$$\langle \nabla_i \nabla_j - \nabla_j \nabla_i \rangle Y_k = R_{ijkl} Y_l.$$
We calculate the evolution equations along the mean curvature flow. First the induced
metric is evolved by

\[ \frac{\partial}{\partial t} g_{ij} = -2H^\alpha A^\alpha_{ij}. \]  

To see this, we have

\[ \frac{\partial}{\partial t} g_{ij} = \frac{\partial}{\partial t} \langle \partial_i F, \partial_j F \rangle \]
\[ = 2\langle \partial_i (H^\alpha v_\alpha), \partial_j F \rangle \]
\[ = 2H^\alpha \langle \partial_i v_\alpha, \partial_j F \rangle \]
\[ = -2H^\alpha \langle v_\alpha, \partial_i \partial_j F \rangle \]
\[ = -2H^\alpha A^\alpha_{ij}. \]

Using the Gauss-Weingarten equation, we can write

\[ \partial_k v_\alpha = -A^\alpha_{ik} \partial_k F + B^\alpha_{i\beta} v_\beta. \]

It is clear that

\[ B^\beta_{i\alpha} + B^\alpha_{i\beta} = 0. \]

We are ready to derive the evolution equation of the second fundamental form. Denote

\[ \left( \frac{\partial v_\alpha}{\partial t} \right) \downarrow = b^\alpha_{\beta} v_\beta. \]

It is clear that

\[ b^\beta_{\alpha} + b^\alpha_{\beta} = 0. \]

We compute

\[ \frac{\partial v_\alpha}{\partial t} = \left\langle \frac{\partial v_\alpha}{\partial t}, \partial_i F \right\rangle \partial_i F + b^\alpha_{\beta} v_\beta \]
\[ = -\langle v_\alpha, \partial_i (H^\beta v_\beta) \rangle \partial_i F + b^\alpha_{\beta} v_\beta \]
\[ = -\left\langle v_\alpha, \partial_i (H^\beta v_\beta) \right\rangle \partial_i F + b^\alpha_{\beta} v_\beta \]
\[ = -\nabla H^\alpha + H^\beta B^\alpha_{i\beta} \partial_i F + b^\alpha_{\beta} v_\beta. \]

It is also clear that

\[ \partial_i \partial_j F - \Gamma^k_{ij} \partial_k F = A^\alpha_{ij} v_\alpha, \]
where $\Gamma_{ij}^k$ is the Levi-Civita connection of the induced metric on $\Sigma$. It follows that

$$\frac{\partial}{\partial t} A_{ij}^\alpha = \frac{\partial}{\partial t} \langle v_\alpha, \partial_i \partial_j F \rangle = \left( \frac{\partial}{\partial t} \langle v_\alpha, \partial_i \partial_j F \rangle \right) + \langle v_\alpha, \partial_i \partial_j \partial_t F \rangle = A_{ij}^\beta b_\beta^\alpha + \langle v_\alpha, \partial_i \partial_j \left( H^\beta v_\beta \right) \rangle = \nabla_i \nabla_j H^\alpha - \nabla_i H^\beta B_{ij}^\alpha - \nabla_j H^\beta B_{ij}^\alpha + A_{ij}^\beta b_\beta^\alpha + H^\beta \langle v_\alpha, \partial_i \partial_j v_\beta \rangle = \nabla_i \nabla_j H^\alpha - \nabla_i H^\beta B_{ij}^\alpha - \nabla_j H^\beta B_{ij}^\alpha + A_{ij}^\beta b_\beta^\alpha - H^\beta A_{ik}^\alpha A_{jk}^\beta - H^\beta \nabla_i B_{ij}^\alpha + H^\beta B_{ij}^\gamma B_{ij}^\gamma.

It follows that

$$\nabla_k A_{ij}^\alpha = \nabla_i A_{kj}^\alpha + A_{ij}^\beta B_{kj}^\alpha - A_{kj}^\alpha B_{ij}^\beta.$$ 

We compute

$$\nabla_i \nabla_j H^\alpha = \nabla_i \nabla_j A_{ij}^\alpha + \nabla_i \left( H^\beta B_{ij}^\alpha \right) - \nabla_i \left( A_{kj}^\beta B_{ij}^\alpha \right) = \nabla_k \nabla_i A_{ij}^\alpha + R_{ikkl} A_{ij}^\alpha + R_{ikjl} A_{kl}^\alpha + \nabla_i \left( H^\beta B_{ij}^\alpha \right) - \nabla_i \left( A_{kj}^\beta B_{ij}^\alpha \right) = \Delta A_{ij}^\alpha + \nabla_k \left( A_{kj}^\beta B_{ij}^\alpha - A_{ij}^\beta B_{kj}^\alpha \right) + R_{ikkl} A_{ij}^\alpha + R_{ikjl} A_{kl}^\alpha + \nabla_i \left( H^\beta B_{ij}^\alpha \right) - \nabla_i \left( A_{kj}^\beta B_{ij}^\alpha \right) = \Delta A_{ij}^\alpha + A_{ik}^\beta A_{kj}^\alpha + A_{ij}^\beta A_{kl}^\alpha - H^\beta A_{ik}^\alpha A_{ij}^\alpha - A_{ij}^\beta A_{kl}^\alpha + H^\beta B_{ij}^\gamma B_{ij}^\gamma - \nabla_k \left( A_{kj}^\beta B_{ij}^\alpha \right) - \nabla_i \left( H^\beta B_{ij}^\alpha \right) - \nabla_i \left( A_{kj}^\beta B_{ij}^\alpha \right).$$

It follows that

$$\frac{\partial}{\partial t} A_{ij}^\alpha = \Delta A_{ij}^\alpha - H^\beta \left( A_{ik}^\alpha A_{kj}^\beta + A_{ik}^\beta A_{kj}^\alpha \right) + A_{ij}^\beta A_{kl}^\alpha A_{kl}^\alpha + A_{ik}^\beta A_{kj}^\alpha A_{kj}^\beta - A_{ij}^\beta B_{ij}^\alpha + A_{ij}^\beta B_{ij}^\gamma B_{ij}^\gamma - \nabla_k \left( A_{kj}^\beta B_{ij}^\alpha \right) - \nabla_i \left( H^\beta B_{ij}^\alpha \right) - \nabla_i \left( A_{kj}^\beta B_{ij}^\alpha \right).$$

\begin{equation}
(1.2)
\end{equation} 

In the codimension one case, $B = b = 0$, and $\alpha = \beta$, it follows that

$$\frac{\partial}{\partial t} A_{ij} = \Delta A_{ij} - 2 HA_{ik} A_{kj} + |A|^2 A_{ij}.$$
We compute

\[ \frac{\partial}{\partial t} (A_{ij}^\alpha A_{kl}^\beta) = A_{ij}^\alpha \frac{\partial}{\partial t} A_{kl}^\beta + \frac{\partial}{\partial t} A_{ij}^\alpha A_{kl}^\beta \]

\[ = A_{ij}^\alpha \Delta A_{kl}^\beta + A_{kl}^\beta \Delta A_{ij}^\alpha + A_{ij}^\beta A_{st}^\alpha A_{kl}^\beta + A_{kl}^\beta A_{st}^\alpha A_{ij}^\alpha \]

\[ - A_{ik}^\alpha \Delta A_{kl}^\beta (A_{ks}^\alpha A_{sj}^\alpha + A_{is}^\beta A_{sj}^\alpha) - A_{ik}^\beta \Delta A_{kl}^\alpha (A_{ks}^\beta A_{sj}^\alpha + A_{is}^\alpha A_{sj}^\beta) \]

\[ + A_{ik}^\alpha A_{is}^\beta A_{jk}^\alpha A_{tl}^\beta + A_{ij}^\alpha A_{ks}^\beta A_{tl}^\alpha - A_{ik}^\beta A_{is}^\alpha A_{jk}^\alpha A_{tl}^\beta - A_{ij}^\alpha A_{ks}^\beta A_{tl}^\alpha \]

\[ + A_{ik}^\alpha A_{ij}^\beta B^\alpha_{kl} + A_{ij}^\alpha A_{ij}^\beta B^\alpha_{kl} + A_{kl}^\beta H^\beta B^\gamma_{ij} B^\gamma_{kl} + A_{ij}^\alpha H^\beta B^\alpha_{kl} B^\gamma_{ij} \]

\[ - A_{ik}^\alpha \nabla j H^\beta B^\gamma_{ij} - A_{ij}^\alpha \nabla l H^\beta B^\gamma_{ij} + A_{ik}^\alpha \nabla s \left( A^\beta_{js} B^\alpha_{ij} \right) \]

\[ + A_{ij}^\alpha \nabla s \left( A^\beta_{kl} B^\alpha_{sj} \right) - A_{ik}^\alpha \nabla s \left( A^\beta_{kl} B^\alpha_{sj} \right) - A_{ij}^\alpha \nabla s \left( A^\beta_{kl} B^\alpha_{sj} \right) \]

2. Minimal Surface

In this section we derive a comparison geometric property of minimal surface. Suppose \( \Sigma^n \to \mathbb{R}^n \) is an immersed minimal surface \((n = 2)\). For any \( p \in \Sigma \), denote \( d(x, p) \) to be the distance of \( p \) and \( x \). Consider the functional

\[ V(p, \tau) = \int_{\Sigma} \tau^{-1} \exp \left( -\frac{d^2(x, p)}{4\tau} \right) \, dx. \]

**Lemma 2.1.** \( V(p, \tau) \) is monotone non-decreasing on \( \tau \) for any \( p \).

**Proof.** This is actually the comparison geometry for manifolds with negative curvature. However, the comparison geometry works only for negative sectional curvature. Our method works only for minimal surface, not for general minimal sub-manifold. We have

\[ \frac{\partial}{\partial \tau} V(p, \tau) = \int_{\Sigma} \tau^{-1} \exp \left( -\frac{d^2(x, p)}{4\tau} \right) \left( -\frac{1}{\tau} + \frac{d^2(x, p)}{4\tau^2} \right) \, dx. \]

Consider

\[ \Delta \exp \left( -\frac{d^2(x, p)}{4\tau} \right) = - \exp \left( -\frac{d^2(x, p)}{4\tau} \right) \left( \frac{\Delta d^2}{4\tau} - \frac{\nabla d^2 \nabla d^2}{16\tau^2} \right). \]

If the metric is flat, it follows that

\[ \Delta (d^2) = 4, |\nabla d^2|^2 = 4d^2. \]

It implies that

\[ \frac{\partial}{\partial \tau} V(p, \tau) = \int_{\Sigma} \Delta \left( \tau^{-1} \exp \left( -\frac{d^2(x, p)}{4\tau} \right) \right) \, dx = 0. \]

In general, note that

\[ |\nabla d| \equiv 1. \]

To conclude monotonicity, we need to show that

\[ \Delta (d^2) \geq 4. \]

\[ \square \]
3. Mean curvature flow

Suppose \( F_0 : \Sigma^m \to \mathbb{R}^m \) is an immersed surface. If \( F(t) : \Sigma \to \mathbb{R}^m, t \in [0, T) \) is a mean curvature flow solution
\[
\frac{\partial}{\partial t} F = \nabla \cdot (\kappa n)
\]
such that \( F(0) = F_0 \). For any fixed point \( p \in \Sigma \), consider a curve \( \gamma(s) : s \in [0, \tau] \) such that \( \gamma(0) = p, \gamma(\tau) = q \). We associate two distance functionals of \( \gamma(s) \) along the mean curvature flow by
\[
L_1(\gamma) = \int_0^\tau \left( |\dot{\gamma}|^2_{g(s)} + |\gamma(s)|^2 \right) ds
\]
and
\[
L_2(\gamma) = \int_0^\tau \sqrt{s} \left( |\dot{\gamma}|^2_{g(s)} + |\gamma(s)|^2 \right) ds.
\]
For \((p, 0)\) fixed, we can also define two distance functional along the mean curvature flow by
\[
L_1(q, \tau) = \inf_{\gamma} \int_0^{\tau} \left( |\dot{\gamma}|^2_{g(s)} + |\gamma(s)|^2 \right) ds
\]
and
\[
L_2(q, \tau) = \inf_{\gamma} \int_0^{\tau} \sqrt{s} \left( |\dot{\gamma}|^2_{g(s)} + |\gamma(s)|^2 \right) ds.
\]
where \( \gamma(s) \) such that \( \gamma(0) = p, \gamma(\tau) = q \) is a path connecting \( p, q \). Also we define
\[
l_1(q, \tau) = \frac{L_1(q, \tau)}{4}
\]
and
\[
l_2(q, \tau) = \frac{L_2(q, \tau)}{2\sqrt{\tau}}.
\]
We can also define two volume functionals as follows
\[
V_1(p, \tau) = \int_{\Sigma} \tau^{-1} \exp(-l_1(q, \tau)) dq
\]
and
\[
V_2(p, \tau) = \int_{\Sigma} \tau^{-1} \exp(-l_2(q, \tau)) dq,
\]
where we use \( dq \) to denote the volume form of induced metric \( g(s) = F^* ds^2 \). We are aiming to show that \( V_1(p, \tau) \) and \( V_2(p, \tau) \) are monotone non-decreasing for any \( p \). The idea is that the monotonicity formula for minimal surface should be generalized to the mean curvature flow. One can view minimal surface as a solution of a elliptic system then the mean curvature flow as a solution of a parabolic system. The key point is that as Perelman, Ricci-flat manifold is fixed point of the Ricci flow. One can deduce comparison geometry for Ricci flat manifold. Then Perelman’s seminal monotonicity formula is the comparison geometry for Ricci flow. A minimal surface is the fixed point of the mean curvature flow. But the story is not
exactly the same since minimal surface has negative Ricci curvature (non-positive). The comparison geometry holds only for non-positive sectional curvature. Hence our method holds only for surface case \((n = 2)\), not for general dimensional sub-manifold.

To show the monotonicity formula, we need to understand the behavior of the distance functional. Denote \(X(s) = \dot{\gamma}(s)\) and \(Y(s)\) is a vector field along \(\gamma(s)\). Denote 
\[
S = ||^2, M = .
\]
Along the mean curvature flow, let \(\tau = -t\), so 
\[
\frac{\partial}{\partial \tau}g = 2M.
\]
We compute the first variation of the \(L\) distance.
\[
\begin{align*}
\delta Y_{L1} &= \int_0^\tau (2\langle \nabla_X^2X, Y \rangle + \langle \nabla S, Y \rangle) ds, \\
\delta Y_{L2} &= \int_0^\tau \sqrt{s}(2\langle \nabla_X^2X, Y \rangle + \langle \nabla S, Y \rangle) ds.
\end{align*}
\]
Note that 
\[
\frac{d}{ds}(X, Y) = \langle \nabla_X^2X, Y \rangle + \langle X, \nabla_XY \rangle + 2M(X, Y).
\]
It follows that
\[
\begin{align*}
\delta Y_{L1} &= \int_0^\tau (2\langle \nabla_X^2X, Y \rangle + \langle \nabla S, Y \rangle) ds \\
&= 2\langle X, Y \rangle - \int_0^\tau (2\langle \nabla_X^2X, Y \rangle + 4M(X, Y) - \langle \nabla S, Y \rangle) ds,
\end{align*}
\]
and
\[
\begin{align*}
\delta Y_{L2} &= \int_0^\tau \sqrt{s}(2\langle \nabla_X^2X, Y \rangle + \langle \nabla S, Y \rangle) ds \\
&= 2\sqrt{\tau}\langle X, Y \rangle - \int_0^\tau \sqrt{s}(2\langle \nabla_X^2X, Y \rangle + 4M(X, Y) + s^{-1}\langle X, Y \rangle - \langle \nabla S, Y \rangle) ds.
\end{align*}
\]
It follows that the \(L_1\) and \(L_2\) geodesic equations are
\[
\begin{align*}
2\nabla_X^2X + 4M(X, \cdot) - \nabla S &= 0 \\
2\nabla_X^2X + 4M(X, \cdot) + s^{-1}X - \nabla S &= 0.
\end{align*}
\]
Also we get 
\[
\nabla L_1 = 2X, \text{ and } \nabla L_2 = 2\sqrt{\tau}X.
\]
It is clear that 
\[
L_{1,\tau} = |X|^2 + S - \langle \nabla L_1, X \rangle = S - |X|^2
\]
and
\[
L_{2,\tau} = \sqrt{\tau}(|X|^2 + S) - \langle \nabla L_2, X \rangle = \sqrt{\tau}(S - |X|^2).
\]
We compute the second variation of $L_1, L_2$.

\[
\delta^2_Y L_1 = \int_0^\tau \left( 2\langle \nabla_Y \nabla_Y X, X \rangle + 2|\nabla_Y X|^2 + Y \cdot Y \cdot S \right) ds
\]

\[
= \int_0^\tau \left( 2\langle \nabla_X \nabla_Y Y, X \rangle + 2\langle R(Y, X)Y, X \rangle + 2|\nabla_Y Y|^2 + Y \cdot Y \cdot S \right) ds,
\]

and

\[
\delta^2_Y L_2 = \int_0^\tau \sqrt{s} \left( 2\langle \nabla_Y \nabla_Y X, X \rangle + 2|\nabla_Y X|^2 + Y \cdot Y \cdot S \right) ds
\]

\[
= \int_0^\tau \sqrt{s} \left( 2\langle \nabla_X \nabla_Y Y, X \rangle + 2\langle R(Y, X)Y, X \rangle + 2|\nabla_Y Y|^2 + Y \cdot Y \cdot S \right) ds.
\]

Note that

\[
\frac{d}{ds} \langle \nabla_Y Y, X \rangle = \langle \nabla_X \nabla_Y Y, X \rangle + \langle \nabla_Y Y, \nabla_X X \rangle + 2\mathcal{M}(\nabla_Y Y, X) + \langle \nabla_Y Y, Y \rangle.
\]

It is also clear that

\[
\langle \nabla_Y Y, X \rangle = 2(\nabla_Y \mathcal{M})(X, Y) - (\nabla_X \mathcal{M})(Y, Y).
\]

It follows that

\[
\delta^2_Y L_1 = 2\langle \nabla_Y Y, X \rangle + \int_0^\tau \left( 2\langle R(Y, X)Y, X \rangle + 2|\nabla_Y Y|^2 + Y \cdot Y \cdot S \right) ds
\]

\[
- \int_0^\tau \left( 2\langle \nabla_Y Y, \nabla_X X \rangle + 4\mathcal{M}(\nabla_Y Y, X) \right) ds
\]

\[
- \int_0^\tau \left( 4(\nabla_Y \mathcal{M})(X, Y) - 2(\nabla_X \mathcal{M})(Y, Y) \right) ds,
\]

and

\[
\delta^2_Y L_2 = 2\sqrt{s} \langle \nabla_Y Y, X \rangle + \int_0^\tau \sqrt{s} \left( 2\langle R(Y, X)Y, X \rangle + 2|\nabla_Y Y|^2 + Y \cdot Y \cdot S \right) ds
\]

\[
- \int_0^\tau \sqrt{s} \left( s^{-1} \langle \nabla_Y Y, X \rangle + 2\langle \nabla_Y Y, \nabla_X X \rangle + 4\mathcal{M}(\nabla_Y Y, X) \right) ds
\]

\[
- \int_0^\tau \sqrt{s} \left( 4(\nabla_Y \mathcal{M})(X, Y) - 2(\nabla_X \mathcal{M})(Y, Y) \right) ds.
\]

We compute the Hessian of $L_1$ and $L_2$

\[
\mathcal{H}_{L_1}(Y, Y) = \delta^2_{\nabla_Y} L_1 - \delta_{\nabla_Y} L_1
\]

\[
= \int_0^\tau \left( 2\langle R(Y, X)Y, X \rangle + 2|\nabla_Y Y|^2 + \mathcal{H}(Y, Y) \right) ds
\]

\[
- \int_0^\tau \left( 4(\nabla_Y \mathcal{M})(X, Y) - 2(\nabla_X \mathcal{M})(Y, Y) \right) ds,
\]
and
\[ \mathcal{H}L_2(Y, Y) = \delta^2 Y - \delta_{XY} Y \]
\[ = \int_0^\tau \sqrt{s} \left( 2\langle R(Y, X)Y, X \rangle + 2\nabla_X Y \right)^2 + \mathcal{HS}(Y, Y) \right) ds \]

and
\[ - \int_0^\tau \sqrt{s} \left( 4\nabla_Y M(X, Y) - 2\nabla_X M(Y, Y) \right) ds, \]
where \( \mathcal{HS} \) is the Hessian of \( S \). We also use the quadratic form \( Q \) to represent the Hessian of \( L_i \).

Now we try to derive the Jacobian equation along the \( L \) geodesic. By definition, the Jacobian field is the derivative of a variation of \( \gamma \) along \( L \) geodesics. Suppose \( \gamma_1(s, \eta), \gamma_2(s, \eta) \) are two-parameter family of \( L \) geodesics respectively, then it follows that
\[ \nabla_{X_1} X_1 = \frac{1}{2} \nabla S - 2M(X_1, \cdot) \]
and
\[ \nabla_{X_2} X_2 = \frac{1}{2} \nabla S - 2M(X_2, \cdot) - \frac{1}{2s} X_2. \]
Denote \( Y_i(s) = \frac{\partial}{\partial \eta} \gamma_i \) at \( \eta = 0 \). Differentiating the geodesic equations in the \( \eta \) direction along the curve \( \eta = 0 \) yields
\[ \nabla_{Y_1} \nabla_{X_1} X_1 = \frac{1}{2} \nabla_{Y_1} \nabla S - 2\nabla_{Y_1} (M(X_1, \cdot)) \]
and
\[ \nabla_{Y_2} \nabla_{X_2} X_2 = \frac{1}{2} \nabla_{Y_2} \nabla S - 2\nabla_{Y_2} (M(X_2, \cdot)) - \frac{1}{2s} \nabla_{Y_2} X_2. \]
Note
\[ \nabla_{Y} (M(X, \cdot)) = (\nabla_{Y} M)(X, \cdot) + M(\nabla_{Y} X, \cdot). \]
We derive the \( L_i \) Jacobian equation as follows
\[ \nabla_{X_1} \nabla_{X_1} Y_1 + R(Y_1, X_1)X_1 - \frac{1}{2} \nabla_{Y_1} (\nabla S) + 2(\nabla_{Y_1} M)(X_1, \cdot) + 2M(\nabla_{X_1} Y_1, \cdot) = 0 \]
and
\[ \nabla_{X_2} \nabla_{X_2} Y_2 + R(Y_2, X_2)X_2 - \frac{1}{2} \nabla_{Y_2} (\nabla S) + \frac{1}{2s} \nabla_{X_2} Y_2 \]
\[ + 2(\nabla_{Y_2} M)(X_2, \cdot) + 2M(\nabla_{X_2} Y_2, \cdot) = 0. \]
If \( Y_i \) is \( L_i \) Jacobian vector field, it is clear that
\[ \mathcal{H}L_1(Y_1, Y_1) = 2\langle \nabla_{X_1} Y_1(\tau), Y_1(\tau) \rangle \]
and
\[ \mathcal{H}L_2(Y_2, Y_2) = 2\sqrt{\tau} \langle \nabla_{X_2} Y_2(\tau), Y_2(\tau) \rangle. \]
The Hessian of the function \( L_i \) can be computed as follows. Assume \( q \in \Sigma \) and let \( \gamma_i(s) : [0, \tau] \rightarrow \Sigma \) be the minimizing \( L_i \) geodesics such that \( \gamma_i(0) = p, \gamma_i(\tau) = q \). Given \( w \in T_q M \), take a short geodesic \( c : (-\xi, \xi) \rightarrow \Sigma \) with \( c(0) = q, c'(0) = w \). Let \( \gamma_i(s, \eta) \) be the family of \( L_i \)
geodesics with $\gamma_i(0, \eta) = p, \gamma_i(\tau, \eta) = c(\eta)$. Then $Y_i(s) = \partial_\eta \gamma_i(s, \eta)|_\eta = 0$ is an $L_i$ Jacobian vector filed along $\gamma_i$ with $Y_i(0) = 0, Y_i(\tau) = w$. We have

$$\mathcal{H}L_1(w, w) = Q(Y_1, Y_1) = 2\langle \nabla_{X_1}Y_1(\tau), Y_1(\tau) \rangle$$

and

$$\mathcal{H}L_2(w, w) = Q(Y_2, Y_2) = 2\sqrt{\tau}\langle \nabla_{X_2}Y_2(\tau), Y_2(\tau) \rangle.$$ 

Now we are in the position to estimate $\triangle L_i$. First we consider $\triangle L_1$. Pick up an orthonormal basis $e_1, e_2 \in T_qM$, it is clear that

$$\triangle L_i = \mathcal{H}L_i(e_1, e_1) + \mathcal{H}L_i(e_2, e_2).$$

It is easy to see that $X_1(s)$ satisfies the Jacobian equation. So we can choose $e_1 = X_1(\tau)/|X_1(\tau)|$ and $e_2 \perp e_1$. It is clear that $X_1(s)$ is a solution of the Jacobian equation. Suppose $e_2(s)$ is the other solution of the Jacobian with $e_2(0) = 0, e_2(\tau) = e_2$. We can get that

$$\mathcal{H}L_1(e_1, e_1) = \langle \nabla_{X_1}e_1(\tau), e_1(\tau) \rangle$$

and

$$\mathcal{H}L_1(e_2, e_2) = \langle \nabla_{X_1}e_2(\tau), e_2(\tau) \rangle.$$ 

We compute

$$\frac{\partial}{\partial \tau}V_1(p, \tau) = \int_\Sigma \tau^{-1} \exp(-l_1(q, \tau)) (-\tau^{-1} - l_{1,\tau} + S) \, dq$$

and

$$\frac{\partial}{\partial \tau}V_2(p, \tau) = \int_\Sigma \tau^{-1} \exp(-l_2(q, \tau)) (-\tau^{-1} - l_{2,\tau} + S) \, dq.$$

Also we have

$$\triangle \exp(-l_1) = \exp(-l_1) (-\triangle l_1 + |\nabla l_1|^2)$$

and

$$\triangle \exp(-l_2) = \exp(-l_2) (-\triangle l_2 + |\nabla l_2|^2).$$

We would like to show that

(3.11) $-\tau^{-1} - l_{1,\tau} + S \geq -\triangle l_1 + |\nabla l_1|^2$

and

(3.12) $-\tau^{-1} - l_{2,\tau} + S \geq -\triangle l_2 + |\nabla l_2|^2.$

We have

$$l_{1,\tau} = \frac{1}{4}L_{1,\tau} = \frac{1}{4}(S - |X|^2),$$

and

$$|\nabla l_1|^2 = \frac{1}{16} |\nabla L_1|^2 = \frac{1}{4} |X|^2.$$ 

To show (3.11), we need to show that

(3.13) $\triangle L_1 + 3S \geq 4\tau^{-1}.$
If $\equiv 0$, (3.13) is reduced to be the ordinary comparison geometry for Riemannian manifold. Note if $\equiv 0$, it follows that

\[ \mathcal{L}_1 = \int_0^\tau |\dot{\gamma}|^2 ds. \]

The geodesic equation is $\nabla_X X = 0$. The Jacobian equation is $\nabla_X \nabla_X Y + R(Y, X) X = 0$. It is clear that $|X(s)|$ is a nonzero constant. For any $q \in M$, pick up an orthonormal frame $e_1, e_2 \in T_q M$. We can choose $e_1 = X(\tau)/|X(\tau)|$. Let $e_i(s)$ be the solutions of $\nabla_X e_i = 0$ with $e_i(\tau) = e_i$ for $i = 1, 2$. It is clear that $e_1(s) = X(s)/|X(s)|$. It is also clear that $Y_1(s) = sX(s)\tau/|X(s)|$ is a solution of Jacobian equation with $Y_1(0) = 0, Y_1(\tau) = X(\tau)/|X(\tau)| = e_1$. Let $Y_2(s) = f(s)e_2(s)$ be another solution of the Jacobian equation with $f(0) = 0, f(\tau) = 1$. Then

\[ \triangle L_1 = Q(Y_1, Y_1) + Q(Y_2, Y_2). \]

It is clear that

\[ Q(Y_1, Y_1) = 2 \int_0^\tau |\nabla_X Y_1|^2 ds = \frac{2}{\tau}. \]

Let $K$ be the Gaussian curvature, then

\[ K = -(R(e_1, e_2)e_1, e_2). \]

We compute

\[ Q(Y_2, Y_2) = 2 \int_0^\tau \left( (R(Y_2, X)Y_2, X) + |\nabla_X Y_2|^2 \right) ds \]

\[ = 2 \int_0^\tau \left( -f^2 K(s) + |f'|^2 \right) ds. \]

The Jacobian equation gives

\[ f'' + K(s)f = 0. \]

It is clear that

\[ \int_0^\tau |f'|^2 ds \int_0^\tau ds \geq \left( \int_0^\tau f' ds \right)^2 = 1. \]

If $K(s) \leq 0$, \[ Q(Y_2, Y_2) \geq 2 \int_0^\tau |f'|^2 ds \geq \frac{2}{\tau}. \]

It follows that

\[ \triangle L_1 \geq 4\tau^{-1}. \]

For the general case, pick up an orthonormal basis $e_1, e_2 \in T_q M$. Let $e_i(s)$ be the solutions of the ODE $\nabla_X e_i + \mathcal{M}(e_i, \cdot) = 0$ with $e_i(\tau) = e_i$. It is clear that $\{e_1(s), e_2(s)\}$ forms an orthonormal basis along the geodesic $\gamma(s)$. Now suppose $Y_i(s)$ are two solutions of the Jacobian equation with $Y_i(0) = 0, Y_i(\tau) = e_i$. It is clear that

\[ \triangle L_1 = Q(Y_1, Y_1) + Q(Y_2, Y_2). \]
Suppose \( Y(s) = f(s)e_1(s) + g(s)e_2(s) \) is a solution of a Jacobian equation, it is clear that
\[
\nabla_X \nabla_X Y + R(Y, X)X - \frac{1}{2} \nabla_Y (\nabla S) + 2(\nabla_Y \mathcal{M})(X, \cdot) + 2\mathcal{M}(\nabla_X Y, \cdot) = 0.
\]

Denote \( \tilde{Y}_i(s) = \frac{s}{\tau}e_i \), we compute
\[
Q(\tilde{Y}_1, \tilde{Y}_1) + Q(\tilde{Y}_2, \tilde{Y}_2).
\]

We have
\[
Q(\tilde{Y}_1, \tilde{Y}_1) = \int_0^\tau \left( 2\langle R(\tilde{Y}_1, X)\tilde{Y}_1, X \rangle + 2|\nabla_X \tilde{Y}_1|^2 + \mathcal{H}S(\tilde{Y}_1, \tilde{Y}_1) \right) ds
\]
\[
- \int_0^\tau \left( 4(\nabla_{\tilde{Y}_1} \mathcal{M})(X, \tilde{Y}_1) - 2(\nabla_X \mathcal{M})(\tilde{Y}_1, \tilde{Y}_1) \right) ds
\]
\[=\]

Similarly we have
\[
l_{2, \tau} = -\frac{1}{4} \tau^{-3/2}L_2 + \frac{1}{2} (S - |X|^2)
\]
and
\[|
\nabla l_2|^2 = |X|^2.
\]

To show (3.12), we need to show that
\[
\Delta L_2 + \sqrt{\tau} S + \frac{1}{2\tau} L_2 \geq \sqrt{\tau} |X|^2 + 2\tau^{-1/2}.
\]