Homework 3

1. Section 8

8.2 Determine the limits of the following sequences, and then prove your claims.

(a) $\lim_{n\to\infty} \frac{n}{n^2+1}$

(c) $\lim_{n\to\infty} \frac{4n+1}{7n-5}$

8.4 Let (t_n) be a bounded sequence, *i.e.*, there exists M such that $|t_n| \le M$ for all n, and let (s_n) be a sequence such that $\lim s_n = 0$. Prove $\lim(s_n t_n) = 0$.

8.5 (a) Consider three sequences (a_n), (b_n) and (s_n) such that $a_n \le s_n \le b_n$ for all n and $\lim a_n = \lim b_n = s$. Prove $\lim s_n = s$. This is called the "squeeze lemma."

(b) Suppose (s_n) and (t_n) are sequences such that $|s_n| \le t_n$ for all n and $\lim t_n = 0$. Prove $\lim s_n = 0$.

8.6 Let (s_n) be a sequence in $\mathbb{R}$.

(a) Prove $\lim s_n = 0$ if and only if $\lim |s_n| = 0$.

(b) Observe that if $s_n = (-1)^n$, then $\lim |s_n|$ exists, but $\lim s_n$ does not exist.

8.8 Prove the following [see Exercise 7.5]:

(a) $\lim[\sqrt{n^2+1} - n] = 0$.

(c) $\lim[\sqrt{4n^2+n} - 2n] = \frac{1}{4}$.

8.9 Let (s_n) be a sequence that converges.

(a) Show that if $s_n \ge a$ for all but finitely many n, then $\lim s_n \ge a$.

(b) Show that if $s_n \le b$ for all but finitely many n, then $\lim s_n \le b$.

(c) Conclude that if all but finitely many s_n belong to $[a, b]$, then $\lim s_n$ belongs to $[a, b]$.

2. Section 9

9.2 Suppose $\lim x_n = 3$, $\lim y_n = 7$ and all y_n are nonzero. Determine the following limits:

(a) $\lim(x_n + y_n)$ (b) $\lim \frac{3y_n - x_n}{y_n^2}$

9.4 Let $s_1 = 1$ and for $n \ge 1$ let $s_{n+1} = \sqrt{s_n + 1}$.

(a) List the first four terms of (s_n).

(b) It turns out that (s_n) converges. Assume this fact and prove the limit is $\frac{1}{2}(1+\sqrt{5})$

9.6 Let $x_1 = 1$and $x_{n+1} = 3x_n^2$ for $n \ge 1$.

(a) Show if $a = \lim x_n$, then $a = \frac{1}{3}$ or $a = 0$.

(b) Does $\lim x_n$ exist? Explain.

(c) Discuss the apparent contradiction between parts (a) and (b).

3. Supplement Homework

S1. Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

(a) A sequence with an infinite number of ones that does not converge to one.

(b) A sequence with an infinite number of ones that converges to a limit not equal to one.

(c) A divergent sequence such that for every $n \in \mathbb{N}$ it is possible to find n consecutive ones somewhere in the sequence.

S2. Give an example of each of the following, or state that such a request is impossible by referencing the proper theorem(s):

(a) Sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, which both diverge, but whose sum $(x_n + y_n)_{n\in\mathbb{N}}$ converges.

(b) Sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, such that $(x_n)_{n\in\mathbb{N}}$ converges, $(y_n)_{n\in\mathbb{N}}$ diverges, and $(x_n+y_n)_{n\in\mathbb{N}}$ converges.

(c) A convergent sequence $(b_n)_{n\in\mathbb{N}}$ with $b_n > 0$ for all $n \in \mathbb{N}$, such that $\left(1/b_n\right)_{n\in\mathbb{N}}$ diverges.

(d) An unbounded sequence $(a_n)_{n\in\mathbb{N}}$ and a convergent sequence $(b_n)_{n\in\mathbb{N}}$ with $(a_n - b_n)_{n\in\mathbb{N}}$ bounded.

(e) Two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, such that $(a_n b_n)_{n\in\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}}$ converge but $(b_n)_{n\in\mathbb{N}}$ does not.