Homework 6

1. Section 24

24.9 Consider \( f_n(x) = nx^n(1-x) \) for \( x \in [0,1] \).

(a) Find \( f(x) = \lim f_n(x) \). (b) Does \( f_n \to f \) uniformly on \([0,1]\)? Justify.

24.10 (a) Prove that if \( f_n \to f \) uniformly, on a set \( S \), and if \( g_n \to g \) uniformly on \( S \), then \( f_n + g_n \to f + g \)

(b) Do you believe the analogue of (a) holds for products? If so, see the next exercise.

24.13 Prove that if \( (f_n) \) is a sequence of uniformly continuous functions on an interval \( (a,b) \), and if \( f_n \to f \) uniformly on \( (a,b) \), then \( f \) is also uniformly continuous on \( (a,b) \). \( \text{Hint:} \) Try an \( \epsilon/3 \) argument as in the proof of Theorem 24.3.

24.14 Let \( f_n(x) = \frac{nx^n}{1+n^2x^2} \) and \( f(x) = 0 \) for \( x \in \mathbb{R} \).

(a) Show \( f_n \to f \) pointwise on \( \mathbb{R} \).

(b) Does \( f_n \to f \) uniformly on \([0,1]\)? Justify.

(c) Does \( f_n \to f \) uniformly on \([1,\infty)\)? Justify.

2. Section 25

25.2 Let \( f_n(x) = \frac{x^n}{n} \). Show \( (f_n) \) is uniformly convergent on \([-1,1]\) and specify the limit function.

25.4 Let \( (f_n) \) be a sequence of functions on a set \( S \subset \mathbb{R} \), and suppose \( f_n \to f \) uniformly on \( S \). Prove \( (f_n) \) is uniformly Cauchy on \( S \). \( \text{Hint:} \) Use the proof of Lemma 10.9 on page 63 as a model, but be careful.

25.5 Let \( (f_n) \) be a sequence of bounded functions on a set \( S \), and suppose \( f_n \to f \) uniformly on \( S \). Prove \( f \) is a bounded function on \( S \).

25.6 (a) Show that if \( \sum |a_k| < \infty \), then \( \sum a_kx^k \) converges uniformly on \([-1,1]\) to a continuous function.

(b) Does \( \sum_{n=1}^{\infty} \frac{x^n}{n} \) represent a continuous function on \([-1,1]\)?

25.10 (a) Show \( \sum \frac{x^n}{1+x^n} \) converges for \( x \in [0,1) \).

(b) Show that the series converges uniformly on \([0,a] \) for each \( a, 0 < a < 1 \).

(c) Does the series converge uniformly on \([0,1)\)? Explain.