Homework 8

1. Section 32

32.1 Find the upper and lower Darboux integrals for \( f(x) = x^3 \) on the interval \([0, b]\). Hint: Exercise 1.3 and Example 1 in §1 will be useful.

32.2 Let \( f(x) = x \) for rational \( x \) and \( f(x) = 0 \) for irrational \( x \).
   (a) Calculate the upper and lower Darboux integrals for \( f \) on the interval \([0, b]\).
   (b) Is \( f \) integrable on \([0, b]\)?

32.6 Let \( f \) be a bounded function on \([a, b]\). Suppose there exist sequences \((U_n)\) and \((L_n)\) of upper and lower Darboux sums for \( f \) such that \( \lim(U_n - L_n) = 0 \). Show \( f \) is integrable and \( \int_a^b f = \lim U_n = \lim L_n \).

32.7 Let \( f \) be integrable on \([a, b]\), and suppose \( g \) is a function on \([a, b]\) such that \( g(x) = f(x) \) except for finitely many \( x \) in \([a, b]\). Show \( g \) is integrable and \( \int_a^b f = \int_a^b g \). Hint: First reduce to the case where \( f \) is the function identically equal to 0.

32.8 Show that if \( f \) is integrable on \([a, b]\), then \( f \) is integrable on every interval \([c, d] \subset [a, b]\).

2. Section 33

33.3 A function \( f \) on \([a, b]\) is called a step function if there exists a partition \( P = \{a = u_0 < u_1 < \cdots < u_m = b\} \) of \([a, b]\) such that \( f \) is constant on each interval \((u_{j-1}, u_j)\), say \( f(x) = c_j \) for \( x \) in \((u_{j-1}, u_j)\).
   (a) Show that a step function \( f \) is integrable and evaluate \( \int_a^b f \).
   (b) Evaluate the integral \( \int_0^4 P(x)dx \) for the postage-stamp function \( P \) in Exercise 17.16.

33.4 Give an example of a function \( f \) on \([0, 1]\) that is not integrable for which \(|f| \) is integrable. Hint: Modify Example 2 in §32.

33.7 Let \( f \) be a bounded function on \([a, b]\), so that there exists \( B > 0 \) such that \(|f(x)| \leq B \) for all \( x \in [a, b] \).
   (a) Show \( \int_a^b U(f^2, P) - L(f^2, P) \leq 2B[\int_a^b U(f, P) - L(f, P)] \)
      for all partitions \( P \) of \([a, b]\). Hint: \( f(x)^2 - f(y)^2 = [f(x) + f(y)] \cdot [f(x) - f(y)] \).
   (b) Show that if \( f \) is integrable on \([a, b]\), then \( f^2 \) also is integrable on \([a, b]\).

33.8 Let \( f \) and \( g \) be integrable functions on \([a, b]\).
   (a) Show \( fg \) is integrable on \([a, b]\). Hint: Use Exercise 33.7 and \( 4fg = (f + g)^2 - (f - g)^2 \).
   (b) Show \( \max(f, g) \) and \( \min(f, g) \) are integrable on \([a, b]\). Hint: Exercise 17.8.

33.11 Let \( f(x) = x \text{sgn}(\sin \frac{1}{x}) \) for \( x \neq 0 \) and \( f(0) = 0 \).
   (a) Show \( f \) is not piecewise continuous on \([-1, 1]\).
   (b) Show \( f \) is not piecewise monotonic on \([-1, 1]\).
   (c) Show \( f \) is integrable on \([-1, 1]\).