Inzell Lectures on Orthogonal Polynomials
W. zu Castell, F. Filbir, B. Forster (eds.)
Advances in the Theory of Special
Functions and Orthogonal Polynomials
Nova Science Publishers
Volume 2, 2004, Pages 135–188

Lecture notes on orthogonal polynomials of several variables

Yuan Xu

Department of Mathematics, University of Oregon
Eugene, Oregon 97403-1222, U.S.A.
yuan@math.uoregon.edu

Summary: These lecture notes provide an introduction to orthogonal polynomials of several variables. It will cover the basic theory but deal mostly with examples, paying special attention to those orthogonal polynomials associated with classical type weight functions supported on the standard domains, for which fairly explicit formulae exist. There is little prerequisites for these lecture notes, a working knowledge of classical orthogonal polynomials of one variable satisfies.

Contents

1. Introduction

1.1. **Definition: one variable vs several variables.** Let μ be a positive Borel measure on $\mathbb{R}$ with finite moments. For $n \in \mathbb{N}_0$, a nonnegative integer, the number $\mu_n = \int_{\mathbb{R}} t^n d\mu$ is the n-th moment of $d\mu$. The standard Gram-Schmidt process applied to the sequence $\{\mu_n\}$ with respect to the inner product

$$\langle f, g\rangle = \int_{\mathbb{R}} f(t)g(t)d\mu(t)$$

of $L^2(d\mu)$ gives a sequence of orthogonal polynomials $\{p_n\}_{n=0}^{\infty}$, which satisfies $\langle p_n, p_m\rangle = 0$, if $n \neq m$, and p_n is a polynomial of degree exactly n. The orthogonal polynomials are unique up to a constant multiple. They are called orthonormal if, in addition, $\langle p_n, p_n\rangle = 1$, and we assume that the measure is normalized by $\int_{\mathbb{R}} d\mu = 1$ when dealing with orthonormality. If $d\mu = w(t)dt$, we say that p_n are associated with the weight function w. The orthogonal polynomials enjoy many properties, which make them a useful tool in various applications and a rich source of research problems. A starting point of orthogonal polynomials of several variables is to extend those properties from one to several variables.

To deal with polynomials in several variables we use the standard multi-index notation. A multi-index is denoted by $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$. For $\alpha \in \mathbb{N}_0^d$ and $x \in \mathbb{R}^d$ a monomial in variables $x_1, \ldots, x_d$ of index α is defined by

$$x^\alpha = x_1^{\alpha_1} \ldots x_d^{\alpha_d}.$$

The number $|\alpha| = \alpha_1 + \cdots + \alpha_d$ is called the total degree of x^α. We denote by $\mathcal{P}_n^d := \operatorname{span}\{x^\alpha : |\alpha| = n, \alpha \in \mathbb{N}_0^d\}$ the space of homogeneous polynomials of degree n, by $\Pi_n^d := \operatorname{span}\{x^\alpha : |\alpha| \leq n, \alpha \in \mathbb{N}_0^d\}$ the space of polynomials of (total) degree at most n, and we write $\Pi^d = \mathbb{R}[x_1, \ldots, x_d]$ for the space of all polynomials of d variables. It is well known that

$$r_n^d := \dim \mathbb{P}_n^d = \binom{n+d-1}{n} \qquad \text{and} \qquad \dim \Pi_n^d = \binom{n+d}{n}.$$

Let μ be a positive Borel measure on $\mathbb{R}^d$ with finite moments. For $\alpha \in \mathbb{N}_0^d$, denote by $\mu_\alpha = \int_{\mathbb{R}^d} x^\alpha d\mu(x)$ the moments of μ. We can apply the Gram-Schmidt process to the monomials with respect to the inner product

$$\langle f, g \rangle_\mu = \int_{\mathbb{R}^d} f(x) g(x) d\mu(x)$$

of $L^2(d\mu)$ to produce a sequence of orthogonal polynomials of several variables. One problem, however, appears immediately: *orthogonal polynomials of several variables are not unique*. In order to apply the Gram-Schmidt process, we need to give a linear order to the moments μ_α which means an order amongst the multi-indices of $\mathbb{N}_0^d$. There are many choices of well-defined total order (for example, the lexicographic order or the graded lexicographic order); but there is no natural choice and different orders will give different sequences of orthogonal polynomials. Instead of fixing a total order, we shall say that $P \in \Pi_n^d$ is an orthogonal polynomial of degree n with respect to $d\mu$ if

$$\langle P, Q \rangle = 0, \qquad \forall Q \in \Pi^d \quad \text{with} \quad \deg Q < \deg P.$$

This means that P is orthogonal to all polynomials of lower degrees, but it may not be orthogonal to other orthogonal polynomials of the same degree. We denote by $\mathcal{V}_n^d$ the space of orthogonal polynomials of degree exactly n; that is,

$$\mathcal{V}_n^d = \{P \in \Pi_n^d : \langle P, Q \rangle = 0, \quad \forall Q \in \Pi_{n-1}^d\}. \tag{1.1}$$

If μ is supported on a set Ω that has nonempty interior, then the dimension of $\mathcal{V}_n^d$ is the same as that of $\mathcal{P}_n^d$. Hence, it is natural to use a multi-index to index the elements of an orthogonal basis of $\mathcal{V}_n^d$. A sequence of orthogonal polynomials $P_\alpha \in \mathcal{V}_n^d$ are called orthonormal, if $\langle P_\alpha, P_\beta \rangle = \delta_{\alpha,\beta}$. The space $\mathcal{V}_n^d$ can have many different bases and the bases do not have to be orthonormal. This non-uniqueness is at the root of the difficulties that we encounter in several variables.

Since the orthogonality is defined with respect to polynomials of different degrees, certain results can be stated in terms of $\mathcal{V}_0^d, \mathcal{V}_1^d, \ldots, \mathcal{V}_n^d, \ldots$ rather than in terms of a particular basis in each $\mathcal{V}_n^d$. For such results, a degree of uniqueness is restored.

For example, this allows us to derive a proper analogy of the three-term relation for orthogonal polynomials in several variables and proves a Favard's theorem. We adopt this point of view and discuss results of this nature in Section 2.

1.2. **Example: orthogonal polynomials on the unit disc.** Before we go on with the general theory, let us consider an example of orthogonal polynomials with respect to the weight function

$$W_\mu(x,y) = \frac{2\mu+1}{2\pi}(1-x^2-y^2)^{\mu-1/2}, \qquad \mu > -1/2, \qquad (x,y) \in B^2,$$

on the unit disc $B^2 = \{(x,y) : x^2+y^2 \le 1\}$. The weight function is normalized so that its integral over B^2 is 1. Among all possible choices of orthogonal bases for $\mathcal{V}_n^d$, we are interested in those for which fairly explicit formulae exist. Several families of such bases are given below.

For polynomials of two variables, the monomials of degree n can be ordered by $\{x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n\}$. Instead of using the notation P_α, $|\alpha| = |\alpha_1+\alpha_2| = n$, to denote a basis for $\mathcal{V}_n^2$, we sometimes use the notation P_k^n with $k = 0, 1, \ldots, n$.

The orthonormal bases given below are in terms of the classical Jacobi and Gegenbauer polynomials. The Jacobi polynomials are denoted by $P_n^{(a,b)}$, which are orthogonal polynomials with respect to $(1-x)^a(1+x)^b$ on $[-1,1]$ and normalized by $P_n^{(a,b)}(1) = \binom{n+a}{n}$, and the Gegenbauer polynomials are denoted by C_n^λ, which are orthogonal with respect to $(1-x^2)^{\lambda-1/2}$ on $[-1,1]$, and

$$C_n^\lambda(x) = ((2\lambda)_n/(\lambda+1/2)_n)P_n^{(\lambda-1/2,\lambda-1/2)}(x),$$

where $(c)_n = c(c+1)\ldots(c+n-1)$ is the Pochhammer symbol.

1.2.1. *First orthonormal basis.* Consider the family

$$P_k^n(x,y) = h_{k,n}C_{n-k}^{k+\mu+\frac{1}{2}}(x)(1-x^2)^{\frac{k}{2}}C_k^\mu\Big(\frac{y}{\sqrt{1-x^2}}\Big), \qquad 0 \le k \le n,$$

where $h_{k,n}$ are the normalization constants.

Since $C_k^\lambda(x)$ is an even function if k is even and is an odd function if k is odd, P_k^n are indeed polynomials in Π_n^2. The orthogonality of these polynomials can be verified using the formula

$$\begin{aligned}\int_{B^2} f(x,y)dxdy &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)dxdy \\ &= \int_{-1}^1 \int_{-1}^1 f(x,\sqrt{1-x^2}t)\sqrt{1-x^2}dxdt.\end{aligned}$$

1.2.2. *Second orthonormal basis.* Using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, we define

$$h^n_{j,1} P_j^{(\mu-\frac{1}{2}, n-2j+\frac{d-2}{2})}(2r^2-1) r^{n-2j} \cos(n-2j)\theta, \qquad 0 \le 2j \le n,$$
$$h^n_{j,2} P_j^{(\mu-\frac{1}{2}, n-2j+\frac{d-2}{2})}(2r^2-1) r^{n-2j} \sin(n-2j)\theta, \qquad 0 \le 2j \le n-1,$$

where $h^n_{j,i}$ are the normalization constants.

For each n these give exactly $n+1$ polynomials. That they are indeed polynomials in (x, y) of degree n can be verified using the relations $r = \|x\|$,

$$\cos m\theta = T_m(x/\|x\|), \quad \text{and} \quad \sin m\theta/\sin\theta = U_{m-1}(x/\|x\|),$$

where T_m and U_m are the Chebyshev polynomials of the first and the second kind. The orthogonality of these polynomials can be verified using the formula

$$\int_{B^2} f(x,y)dxdy = \int_0^1 \int_0^{2\pi} f(r\cos\theta, r\sin\theta) d\theta\, r\, dr.$$

1.2.3. *An orthogonal basis.* A third set is given by

$$P^n_k(x,y) = C_n^{\mu+1/2}\Big(x\cos\frac{k\pi}{n+1} + y\sin\frac{k\pi}{n+1}\Big), \qquad 0 \le k \le n.$$

In particular, if $\mu = 1/2$, then the polynomials

$$P^n_k(x,y) = \frac{1}{\sqrt{\pi}} U_n\Big(x\cos\frac{k\pi}{n+1} + y\sin\frac{k\pi}{n+1}\Big), \qquad 0 \le k \le n,$$

form an orthonormal basis with respect to the Lebesgue measure on B^2. The case $\mu = 1/2$ first appeared in [22] in connection with a problem in computer tomography.

1.2.4. *Appell's monomial and biorthogonal bases.* The polynomials in these bases are denoted by V^n_k and U^n_k for $0 \le k \le n$ (cf. [2]). The polynomials V^n_k are defined by the generating function

$$(1 - 2(b_1 x + b_2 y) + \|b\|^2)^{-\mu-1/2} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} b_1^k b_2^{n-k} V^n_k(x,y), \quad b = (b_1, b_2),$$

and they are called the monomial orthogonal polynomials since

$$V^n_k(x,y) = x^k y^{n-k} + q(x,y),$$

where $q \in \Pi^2_{n-1}$. The polynomials U^n_k are defined by

$$U^n_k(x,y) = (1 - x^2 - y^2)^{-\mu+\frac{1}{2}} \frac{\partial^k}{\partial x^k} \frac{\partial^{n-k}}{\partial y^{n-k}} (1 - x^2 - y^2)^{n+\mu-1/2}.$$

Both V_k^n and U_k^n belong to $\mathcal{V}_n^2$, and they are biorthogonal in the sense that

$$\int_{B^2} V_k^n(x,y) U_j^n(x,y) W_\mu(x,y) = 0, \qquad k \neq j.$$

The orthogonality follows from a straightforward computation of integration by parts.

1.3. **Orthogonal polynomials for classical type weight functions.** In the ideal situation, one would like to have fairly explicit formulae for orthogonal polynomials and their various structural constants (such as L^2-norm). The classical orthogonal polynomials of one variable are good examples. These polynomials include the Hermite polynomials $H_n(t)$ associated with the weight function e^{-t^2} on $\mathbb{R}$, the Laguerre polynomials $L_n^a(t)$ associated with $t^a e^{-t}$ on $\mathbb{R}_+ = [0,\infty)$, and the Jacobi polynomials $P_n^{(a,b)}(t)$ associated with $(1-t)^a(1+t)^b$ on $[-1,1]$. Up to an affine linear transformation, they are the only families of orthogonal polynomials (with respect to a positive measure) that are eigenfunctions of a second order differential operator.

One obvious extension to several variables is using tensor product. For $1 \le j \le d$ let w_j be the weight function on the interval $I_j \subset \mathbb{R}$ and denote by $p_{n,j}$ orthogonal polynomials of degree n with respect to w_j. Then for the product weight function

$$W(x) = w_1(x_1)\ldots w_d(x_d), \qquad x \in I_1 \times \ldots \times I_d,$$

the product polynomials $P_\alpha(x) = \prod_{j=1}^d p_{\alpha_j,j}(x_j)$ are orthogonal with respect to W. Hence, as extensions of the classical orthogonal polynomials, we can have product Hermite polynomials associated with

$$W^H(x) = e^{-\|x\|^2}, \qquad x \in \mathbb{R}^d,$$

product Laguerre polynomials associated with

$$W_\kappa^L(x) = x^\kappa e^{-|x|}, \qquad x \in \mathbb{R}_+^d, \quad \kappa_i > -1,$$

the product Jacobi polynomials associated with

$$W_{a,b}(x) = \prod_{i=1}^d (1-x_i)^{a_i}(1+x_i)^{b_i}, \qquad x \in [-1,1]^d, \qquad a_i, b_i > -1,$$

as well as the mixed product of these polynomials. Throughout this lecture, the notation $\|x\|$ stands for the Euclidean norm and $|x|$ stands for the ℓ^1-norm of $x \in \mathbb{R}^d$. The product basis in this case is also the monomial basis. There are other interesting orthogonal bases; for example, for the product Hermite weight function, another orthonormal basis can be given in polar coordinates.

More interesting, however, are extensions that are not of product type. The geometry of $\mathbb{R}^d$ is rich, i.e., there are other attractive regular domains; the unit ball $B^d = \{x : \|x\| \le 1\}$ and the simplex $T^d = \{x \in \mathbb{R}^d : x_1 \ge 0, \ldots, x_d \ge$

$0, 1 - |x| \geq 0\}$ are two examples. There are orthogonal polynomials on B^d and on T^d for which explicit formulae exist, and their study goes back at least as far as Hermite (cf. [2] and [11, Vol. 2, Chapt. 12]). The weight functions are

$$W_\mu^B(x) = (1 - \|x\|^2)^{\mu - 1/2}, \qquad x \in B^d, \qquad \mu > -1/2,$$

and

$$W_\kappa^T(x) = \prod_{i=1}^{d} |x_i|^{\kappa_i - 1/2}(1 - |x|)^{\kappa_{d+1} - 1/2}, \qquad x \in T^d, \qquad \kappa_i > -1/2.$$

In both cases, there are explicit orthonormal bases that can be given in terms of Jacobi polynomials. In Section 4 and 5 we discuss these two cases and their extensions.

There is no general agreement on what should be called classical orthogonal polynomials of several variables. For $d = 2$ Krall and Sheffer [18] gave a classification of orthogonal polynomials that are eigenfunctions of a second order partial differential operator, which shows that only five such families are orthogonal with respect to a positive measure: product Hermite, product Laguerre, product Hermite-Laguerre, orthogonal polynomials with respect to W_μ^B on the disc and with respect to W_κ^T on the triangle T^2. Clearly these families should be called classical, but perhaps equally entitled are product Jacobi polynomials and a score of others.

1.4. **Harmonic and h-harmonic polynomials.** Another classical example of orthogonal polynomials of several variables is the spherical harmonics. The Laplace operator Δ on $\mathbb{R}^d$ is defined by

$$\Delta = \partial_1^2 + \cdots + \partial_d^2,$$

where $\partial_i = \partial/\partial x_i$. Harmonic polynomials are polynomials that satisfy $\Delta P = 0$, and spherical harmonics are the restriction of homogeneous harmonic polynomials on the sphere S^{d-1}. Let $\mathcal{H}_n^d$ be the set of homogeneous harmonic polynomials of degree n; $\mathcal{H}_n^d = \mathcal{P}_n^d \cap \ker \Delta$. It is known that

$$P \in \mathcal{H}_n^d \qquad \text{if and only if} \qquad \int_{S^{d-1}} PQ d\omega = 0, \quad \forall Q \in \Pi^d, \quad \deg Q < n,$$

where $d\omega$ is the surface measure of S^{d-1}. An orthonormal basis for spherical harmonics can also be given in terms of Jacobi polynomials. The fact that the Lebesgue measure $d\omega$ is invariant under the orthogonal group $O(d)$ plays an important role in the theory of spherical harmonics.

An important extension of harmonic polynomials are the h-harmonics introduced by Dunkl [7], in which the role of the rotation group is replaced by a reflection group. The h-harmonics are homogeneous polynomials that satisfy $\Delta_h P = 0$, where Δ_h is a second order differential-difference operator. This h-Laplacian can be decomposed as

$$\Delta_h = \mathcal{D}_1^2 + \ldots + \mathcal{D}_d^2,$$

where $\mathcal{D}_i$ are the first order differential-difference operators (Dunkl's operators) which commute, that is, $\mathcal{D}_i\mathcal{D}_j = \mathcal{D}_j\mathcal{D}_i$. The h-harmonics are orthogonal with respect to $h^2 d\omega$ where h is a weight function invariant under the underlying reflection group. Examples include $h(x) = \prod_{i=1}^d |x_i|^{\kappa_i}$ invariant under $\mathbb{Z}_2^d$ and $h(x) = \prod_{i<j} |x_i - x_j|^{\kappa}$ invariant under the symmetric group.

It turns out that there is a close relation between orthogonal polynomials on the sphere and those on the simplex and the ball. The classical examples of orthogonal polynomials on these two domains can be derived from the corresponding results for h-harmonics associated to the product weight function $\prod_{i=1}^d |x_i|^{\kappa_i}$. We discuss h-harmonics in Section 3, giving special emphasis to the case of product weight function since it can be developed without prerequisites of reflection groups.

1.5. **Fourier orthogonal expansion.** Let μ be a positive measure with finite moments such that the space of polynomials is dense in $L^2(d\mu)$. Let P_α be a sequence of orthonormal polynomials with respect to $d\mu$. Then the standard Hilbert space theory shows that every $f \in L^2(d\mu)$ can be expanded in terms of P_α as

$$f(x) = \sum_{\alpha\in\mathbb{N}_0^d} a_\alpha(f) P_\alpha(x) \qquad \text{with} \qquad a_\alpha(f) = \int_{\mathbb{R}^d} f(x) P_\alpha(x) d\mu(x). \tag{1.2}$$

This is the Fourier orthogonal expansion. Just as the case of the classical Fourier series, the expansion does not hold pointwisely in general if f is merely a continuous function. We define the n-th partial sum of the expansion by

$$S_n(f;x) := \sum_{k=0}^n \sum_{|\alpha|=k} a_\alpha(f) P_\alpha(x) = \int_{\mathbb{R}^d} f(y)\mathbf{K}_n(x,y) d\mu(y),$$

where the first equation is the definition and the second equation follows from the formula for $a_\alpha(f)$, where

$$\mathbf{K}_n(x,y) = \sum_{k=0}^n \mathbf{P}_k(x,y) \qquad \text{with} \qquad \mathbf{P}_k(x,y) = \sum_{|\alpha|=k} P_\alpha(x) P_\alpha(y);$$

the function $\mathbf{K}_n(x,y)$ is the reproducing kernel of the space Π_n^d in the sense that $\int \mathbf{K}_n(x,y)P(y)d\mu(y) = P(x)$ for all $P \in \Pi_n^d$, and the function $\mathbf{P}_n(x,y)$ is the reproducing kernel of the space $\mathcal{V}_n^d$. In particular, the definition of the kernels and thus $S_n(f)$ are independent of the choices of particular orthonormal bases. As an application, we discuss the convergence of Fourier orthogonal expansions for the classical type weight functions in Section 7.

1.6. **Literature.** The main early references of orthogonal polynomials of several variables are Appell and de Fériet [2], and Chapter 12 of Erdélyi et. al. [11], as well as the influential survey of Koornwinder on orthogonal polynomials of two variables [15]. There is also a more recent book of Suetin [30] on orthogonal polynomials of two variables, which is in the spirit of the above references. We follow the presentation in the recent book of Dunkl and Xu [10]. However, our main development for orthogonal polynomials with respect to the classical type weight functions follows a line that does not require background in reflection groups, and we also include some more recent results. We will not give references to every single result in the main body of the text; the main references and the historical notes will appear at the end of the lecture notes.

2. General properties

By general properties we mean those properties that hold for orthogonal polynomials associated with weight functions that satisfy some mild conditions but are not any more specific.

2.1. **Three-term relations.** For orthogonal polynomials of one variable, one important property is the three-term relation, which states that every system of orthogonal polynomials $\{p_n\}_{n=0}^{\infty}$ with respect to a positive measure satisfies a three-term relation

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x), \qquad n \ge 0, \tag{2.1}$$

where $p_{-1}(x) = 0$ by definition, $a_n, b_n, c_n \in \mathbb{R}$ and $a_n c_{n+1} > 0$; if p_n are orthonormal polynomials, then $c_n = a_{n-1}$. Furthermore, Favard's theorem states that every sequence of polynomials that satisfies such a relation must be orthogonal.

Let $\{P_\alpha : \alpha \in \mathbb{N}_0^d\}$ be a sequence of orthogonal polynomials in d variables and assume that $\{P_\alpha : |\alpha| = n\}$ is a basis of $\mathcal{V}_n^d$. The orthogonality clearly implies that $x_i P_\alpha(x)$ is orthogonal to all polynomials of degree at most $n-2$ and at least $n+2$, so that it can be written as a linear combination of orthogonal polynomials of degree $n-1, n, n+1$, although there are many terms for each of these three degrees. Clearly this can be viewed as a three-term relation in terms of $\mathcal{V}_{n-1}^d, \mathcal{V}_n^d, \mathcal{V}_{n+1}^d$. This suggests to introduce the following vector notation:

$$\mathbb{P}_n(x) = (P_\alpha^n(x))_{|\alpha|=n} = (P_{\alpha^{(1)}}(x), \ldots, P_{\alpha^{(r_n^d)}}(x))^T = G_n \mathbf{x}^n + \ldots,$$

where $\alpha^{(1)}, \ldots, \alpha^{(r_n)}$ is the arrangement of elements in $\{\alpha \in \mathbb{N}_0^d : |\alpha| = n\}$ according to the lexicographical order (or any other fixed order), and $\mathbf{x}^n = (x^{\alpha^{(1)}}, \ldots, x^{\alpha^{(r_n^d)}})^T$ is the vector of the monomials of degree n; the matrix G_n of size $r_n^d \times r_n^d$ is called the leading coefficient of $\mathbb{P}_n$, and it is invertible. We note that if S is a nonsingular matrix of size $|\alpha|$, then the components of $S\mathbb{P}_n$ are also a basis for $\mathcal{V}_n^d$. In terms of $\mathbb{P}_n$, we have the three-term relation:

Theorem 2.1. *Let P_α be orthogonal polynomials. For $n \ge 0$, there exist unique matrices $A_{n,i} : r_n^d \times r_{n+1}^d$, $B_{n,i} : r_n^d \times r_n^d$, and $C_{n,i}^T : r_n^d \times r_{n-1}^d$, such that*

$$x_i \mathbb{P}_n = A_{n,i}\mathbb{P}_{n+1} + B_{n,i}\mathbb{P}_n + C_{n,i}\mathbb{P}_{n-1}, \qquad 1 \le i \le d, \tag{2.2}$$

where we define $\mathbb{P}_{-1} = 0$ and $C_{-1,i} = 0$. If P_α are orthonormal polynomials, then $C_{n,i} = A_{n-1,i}^T$.

Proof. Looking at the components, the three-term relation is evident. The coefficient matrices satisfy $A_{n,i}H_{n+1} = \int x_i \mathbb{P}_n \mathbb{P}_{n+1}^T d\mu$, $B_{n,i}H_n = \int x_i \mathbb{P}_n \mathbb{P}_n^T d\mu$, and $A_{n,i}H_{n+1} = H_n C_{n+1,i}^T$, where $H_n = \int \mathbb{P}_n \mathbb{P}_n^T d\mu$ is an invertible matrix. Hence the coefficient matrices are unique. If P_α are orthonormal, then H_n is an identity matrix and $C_{n,i} = A_{n-1,i}^T$. ∎

For orthonormal polynomials, $A_{n,i} = \int x_i \mathbb{P}_n \mathbb{P}_{n+1}^T d\mu$ and $B_{n,i} = \int x_i \mathbb{P}_n \mathbb{P}_n^T d\mu$, which can be used to compute the coefficient matrices.

Example. For the first orthonormal basis on the unit disc in the previous section, we have

$$B_{n,1} = B_{n,2} = 0, \qquad A_{n,1} = \begin{bmatrix} a_{0,n} & & & \bigcirc & 0 \\ & a_{1,n} & & & 0 \\ & & \ddots & & \vdots \\ \bigcirc & & & a_{n,n} & 0 \end{bmatrix},$$

and

$$A_{n,2} = \begin{bmatrix} e_{0,n} & d_{0,n} & & & \bigcirc & 0 \\ c_{1,n} & e_{1,n} & d_{1,n} & & & \vdots \\ & \ddots & \ddots & \ddots & & \vdots \\ & & c_{n-1,n} & & d_{n-1,n} & 0 \\ \bigcirc & & & c_{n,n} & e_{n,n} & d_{n,n} \end{bmatrix},$$

where the coefficients can all be computed explicitly.

For $d = 1$ the relation reduces to the classical three-term relation. Moreover, let $A_n = (A_{n,1}^T | \dots | A_{n,d}^T)^T$ denote the joint matrix of $A_{n,1}, \dots, A_{n,d}$, then the following is an analog of the condition $a_n c_{n+1} > 0$:

Theorem 2.2. *For $n \ge 0$ and $1 \le i \le d$,* $\operatorname{rank} A_{n,i} = \operatorname{rank} C_{n+1,i} = r_n^d$. *Moreover, for the joint matrix A_n of $A_{n,i}$ and the joint matrix C_n^T of $C_{n,i}^T$,*

$$\operatorname{rank} A_n = r_{n+1}^d \quad \textit{and} \quad \operatorname{rank} C_{n+1}^T = r_{n+1}^d.$$

Proof. Comparing the leading coefficients of the both sides of (2.2) shows that $A_{n,i}G_{n+1} = G_n L_{n,i}$, where $L_{n,i}$ is the transformation matrix defined by $L_{n,i}\mathbf{x}^{n+1} = x_i \mathbf{x}^n$, which implies that $\operatorname{rank} L_{n,i} = r_n^d$. Hence, $\operatorname{rank} A_{n,i} = r_n^d$ as G_n is invertible. Furthermore, let L_n be the joint matrix of $L_{n,1}, \dots, L_{n,d}$. Then the components of $L_n\mathbf{x}^{n+1}$ contain every x^α, $|\alpha| = n+1$. Hence L_n has full rank, $\operatorname{rank} L_n = r^d_{n+1}$. Furthermore, $A_n G_{n+1} = \operatorname{diag}\{G_n, \ldots, G_n\} L_n$ from which follows that $\operatorname{rank} A_n = r^d_{n+1}$. The statement on $C^T_{n,i}$ and C^T_n follows from the relation $A_{n,i}H_{n+1} = H_n C^T_{n+1,i}$. ∎

Just as in the one variable case, the three-term relation and the rank conditions of the coefficients characterize the orthogonality. A linear functional $\mathcal{L}$ is said to be positive definite if $\mathcal{L}(p^2) > 0$ for all nonzero polynomials $p \in \Pi^d_n$. The following is the analog of Favard's theorem, which we only state for the case of $C_{n,i} = A^T_{n-1,i}$ and P_α orthonormal.

Theorem 2.3. *Let $\{\mathbb{P}_n\}_{n=0}^\infty = \{P^n_\alpha : |\alpha| = n, n \in \mathbb{N}_0\}$, $\mathbb{P}_0 = 1$, be an arbitrary sequence in Π^d. Then the following statements are equivalent.*

(1). There exists a linear function $\mathcal{L}$ which defines a positive definite linear functional on Π^d and which makes $\{\mathbb{P}_n\}_{n=0}^\infty$ an orthogonal basis in Π^d.

(2). For $n \geq 0$, $1 \leq i \leq d$, there exist matrices $A_{n,i}$ and $B_{n,i}$ such that

(a) the polynomials $\mathbb{P}_n$ satisfy the three-term relation (2.2) with $C_{n,i} = A^T_{n-1,i}$,

(b) the matrices in the relation satisfy the rank condition in Theorem 2.2.

The proof follows roughly the line that one uses to prove Favard's theorem of one variable. The orthogonality in the theorem is given with respect to a positive definite linear functional. Further conditions are needed in order to show that the linear functional is given by a nonnegative Borel measure with finite moments. For example, if $\mathcal{L}f = \int f d\mu$ for a measure μ with compact support in (1) of Theorem 2.3, then the theorem holds with one more condition

$$\sup_{k\geq 0} \|A_{k,i}\|_2 < \infty \quad \text{and} \quad \sup_{k \geq 0} \|B_{k,i}\|_2 < \infty, \quad 1 \leq i \leq d$$

in (2). The known proof of such refined results uses the spectral theory of self-adjoint operators.

Although Favard's theorem shows that the three-term relation characterizes orthogonality, it should be pointed out that the relation is not as strong as in the case of one variable. In one variable, the coefficients of the three-term relation (2.1) can be any real numbers satisfying $a_n > 0$ (in the case of orthonormal polynomials $c_n = a_{n-1}$). In several variables, however, the coefficients of the three-term relations have to satisfy additional conditions.

Theorem 2.4. *The coefficients of the three-term relation of a sequence of orthonormal polynomials satisfy*

$$\begin{aligned} A_{k,i}A_{k+1,j} &= A_{k,j}A_{k+1,i}, \\ A_{k,i}B_{k+1,j} + B_{k,i}A_{k,j} &= B_{k,j}A_{k,i} + A_{k,j}B_{k+1,i}, \\ A^T_{k-1,i}A_{k-1,j} + B_{k,i}B_{k,j} + A_{k,i}A^T_{k,j} &= A^T_{k-1,j}A_{k-1,i} + B_{k,j}B_{k,i} + A_{k,j}A^T_{k,i}, \end{aligned}$$

for $i \neq j$, $1 \leq i, j \leq d$, and $k \geq 0$, where $A_{-1,i} = 0$.

Proof. The relations are obtained from computing the matrices $\mathcal{L}(x_i x_j \mathbb{P}_k \mathbb{P}^T_{k+2})$, $\mathcal{L}(x_i x_j \mathbb{P}_k \mathbb{P}^T_k)$, and $\mathcal{L}(x_i x_j \mathbb{P}_k \mathbb{P}^T_{k+1})$ in two different ways, using the three-term relation (2.2) to replace $x_i \mathbb{P}_n$ and $x_j \mathbb{P}_n$, respectively. ■

These equations are called the commuting conditions. Since they are necessary for polynomials to be orthogonal, we cannot choose arbitrary matrices to generate a family of polynomials satisfying the three-term relation and hope to get orthogonal polynomials.

As an application, let us mention that the three-term relation implies a Christoffel-Darboux formula. Recall that reproducing kernel $\mathbf{K}_n(x, y)$ is defined in Section 1.5.

Theorem 2.5. *(The Christoffel-Darboux formula) For $n \geq 0$,*

$$\mathbf{K}_n(x, y) = \frac{\left[A_{n,i}\mathbb{P}_{n+1}(x)\right]^T \mathbb{P}_n(y) - \mathbb{P}^T_n(x) A_{n,i} \mathbb{P}_{n+1}(y)}{x_i - y_i}, \quad 1 \leq i \leq d,$$

for $x \neq y$ and

$$\mathbf{K}_n(x, x) = \mathbb{P}^T_n(x) A_{n,i} \partial_i \mathbb{P}_{n+1}(x) - [A_{n,i}\mathbb{P}_{n+1}(x)]^T \mathbb{P}_n(x).$$

The proof follows just as in the case of one variable. Note, that the right hand side depends on i, but the left hand side is independent of i.

2.2. **Common zeros of orthogonal polynomials.** If $\{p_n\}$ is a sequence of orthogonal polynomials of one variable, then all zeros of p_n are real and distinct, and these zeros are the eigenvalues of the truncated Jacobi matrix J_n.

$$J_n = \begin{bmatrix} b_0 & a_0 & & & \bigcirc \\ a_0 & b_1 & a_1 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-2} & b_{n-1} & a_{n-1} \\ \bigcirc & & & a_{n-1} & b_n \end{bmatrix},$$

where a_n and b_n are coefficients of the three-term relation satisfied by the orthonormal polynomials. This fact has important applications in a number of problems.

The zero set for a polynomial in several variables can be a point, a curve, and an algebraic variety in general – a difficult object to study. However, using the three-term relation, it is possible to study the common zeros of $\mathbb{P}_n(x)$; that is, the common zeros of all P_α, $|\alpha| = n$. Note, that this means the common zeros of all polynomials in $\mathcal{V}_n^d$, which are independent of the choice of the bases. Throughout this subsection we assume that $\{P_\alpha\}$ is a sequence of orthonormal polynomials with respect to a positive measure μ.

Using the coefficient matrices of the three-term relation (2.2), we define the truncated block Jacobi matrices $J_{n,i}$ as follows:

$$J_{n,i} = \begin{bmatrix} B_{0,i} & A_{0,i} & & & \bigcirc \\ A_{0,i}^T & B_{1,i} & A_{1,i} & & \\ & \ddots & \ddots & \ddots & \\ & & A_{n-3,i}^T & B_{n-2,i} & A_{n-2,i} \\ \bigcirc & & & A_{n-2,i}^T & B_{n-1,i} \end{bmatrix}, \qquad 1 \le i \le d.$$

Then $J_{n,i}$ is a square matrix of size $N \times N$ with $N = \dim \Pi_{n-1}^d$. We say that $\Lambda = (\lambda_1, \ldots, \lambda_d)^T \in \mathbb{R}^d$ is a joint eigenvalue of $J_{n,1}, \ldots, J_{n,d}$, if there is a $\xi \neq 0$, $\xi \in \mathbb{R}^N$, such that $J_{n,i}\xi = \lambda_i \xi$ for $i = 1, \ldots, d$; the vector ξ is called a joint eigenvector associated to Λ.

Theorem 2.6. *A point $\Lambda = (\lambda_1, \ldots, \lambda_d)^T \in \mathbb{R}^d$ is a common zero of $\mathbb{P}_n$ if and only if it is a joint eigenvalue of $J_{n,1}, \ldots, J_{n,d}$; moreover, a joint eigenvector of Λ is $(\mathbb{P}_0^T(\Lambda), \ldots, \mathbb{P}_{n-1}^T(\Lambda))^T$.*

Proof. If $\mathbb{P}_n(\Lambda) = 0$, then the three-term relation for $\mathbb{P}_k$, $0 \le k \le n-1$, is the same as $J_{n,i}\xi = \lambda_i \xi$ with $\xi = (\mathbb{P}_0^T(\Lambda), \ldots, \mathbb{P}_{n-1}^T(\Lambda))^T$. On the other hand, suppose $\Lambda = (\lambda_1, \ldots, \lambda_d)$ is an eigenvalue of $J_{n,1}, \ldots, J_{n,d}$ with a joint eigenvector $\xi = (\mathbf{x}_0^T, \ldots, \mathbf{x}_{n-1}^T)^T$, $\mathbf{x}_j \in \mathbb{R}^{r_j^d}$. Let us define $\mathbf{x}_n = 0$. Then $J_{n,i}\xi = \lambda_i\xi$ implies that $\{\mathbf{x}_k\}_{k=0}^n$ satisfies the same (first $n-1$ equations of the) three-term relation as $\{\mathbb{P}_k(\Lambda)\}_{k=0}^n$ does. The rank condition on $A_{n,i}$ shows inductively that $\mathbf{x}_0 \neq 0$ unless ξ is zero. But $\xi \neq 0$ as an eigenvector and we can assume that $\mathbf{x}_0 = 1 = \mathbb{P}_0$. Then $\{\mathbf{y}_k\}_{k=0}^n$ with $\mathbf{y}_k = \mathbf{x}_k - \mathbb{P}_k$ satisfies the same three-term relation. But $\mathbf{y}_0 = 0$, it follows from the rank condition that $\mathbf{y}_k = 0$ for all $1 \le k \le n$. In particular, $\mathbf{y}_n = \mathbb{P}_n(\Lambda) = 0$. ∎

The main properties of the common zeros of $\mathbb{P}_n$ are as follows:

Corollary 2.1. *All common zeros of $\mathbb{P}_n$ are real distinct points and they are simple. The polynomials in $\mathbb{P}_n$ have at most $N = \dim \Pi_{n-1}^d$ common zeros and $\mathbb{P}_n$ has N zeros if and only if*

$$A_{n-1,i}A_{n-1,j}^T = A_{n-1,j}A_{n-1,i}^T, \quad 1 \le i, j \le d. \tag{2.3}$$

Proof. Since the matrices $J_{n,i}$ are symmetric, the joint eigenvalues are real. If x is a common zero, then the Christoffel-Darboux formula shows that

$$\mathbb{P}_n^T(x)A_{n,i}\partial_i\mathbb{P}_{n+1}(x) = \mathbf{K}_n(x,x) > 0,$$

so that at least one of the partial derivatives of $\mathbb{P}_n$ is not zero at x; that is, the common zero is simple. Since $J_{n,i}$ is an $N\times N$ square matrix, there are at most N eigenvalues, and $\mathbb{P}_n$ has at most N common zeros. Moreover, $\mathbb{P}_n$ has N distinct zeros if and only if $J_{n,1},\ldots,J_{n,d}$ can be simultaneously diagonalized, which holds if and only if $J_{n,1},\ldots,J_{n,d}$ commute,

$$J_{n,i}J_{n,j} = J_{n,j}J_{n,i}, \qquad 1\le i,j\le d.$$

From the definition of $J_{n,i}$ and the commuting conditions in Theorem 2.4, the above equation is equivalent to the condition

$$A_{n-2,i}^T A_{n-2,j} + B_{n-1,i}B_{n-1,j} = A_{n-2,j}^T A_{n-2,i} + B_{n-1,j}B_{n-1,i}.$$

The third equation of the commuting condition leads to the desired result. ■

The zeros of orthogonal polynomials of one variable are nodes of the Gaussian quadrature formula. A similar result can be stated in several variables for the common zeros of $\mathbb{P}_n$. However, it turns out that the condition (2.3) holds rarely; for example, it does not hold for those weight functions that are centrally symmetric (the support set Ω of W is symmetric with respect to the origin and $W(x) = W(-x)$ for all x in Ω). Consequently, $\mathbb{P}_n$ does not have N common zeros in general and the straightforward generalization of Gaussian quadrature usually does not exist. The relation between common zeros of orthogonal polynomials and quadrature formulae in several variables is quite complicated. One needs to study common zeros of subsets of (quasi-)orthogonal polynomials, and the main problem is to characterize or identify subsets that have a large number of common zeros. In the language of polynomial ideals and varieties, the problem essentially comes down to characterize or identify those polynomial ideals generated by (quasi)-orthogonal polynomials whose varieties are large subsets of points, and the size of the variety should equal to the codimension of the ideal. Although some progress has been made in this area, the problem remains open for the most part.

3. h-harmonics and orthogonal polynomials on the sphere

After the first subsection on the relation between orthogonal polynomials on the sphere and those on the ball, we discuss h-harmonics in two steps. The main effort is in a long subsection devoted to the case of the product weight function, which can be developed without any background in reflection groups, and it is this case that offers most explicit formulae. The theory of h-harmonics associated to general reflection groups is summarized in the third subsection.

3.1. **Orthogonal polynomials on the unit ball and on the unit sphere.** If a measure μ is supported on the unit sphere $S^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$ of $\mathbb{R}^d$, then the integrals of the positive polynomials $(1 - \|x\|^2)^{2n}$ over S^{d-1} are zero, so that the general properties of orthogonal polynomials in the previous section no longer hold. There is, however, a close relation between orthogonal structure on the sphere and that on the ball, which can be used to study the general properties for orthogonal polynomials on the sphere.

As a motivating example, recall that in polar coordinates $y_1 = r\cos\theta$ and $y_2 = r\sin\theta$, the spherical harmonics of degree n on S^1 are given by

$$Y_n^{(1)}(y) = r^n \cos n\theta = r^n T_n(y_1/r), \qquad Y_n^{(2)}(y) = r^n \sin n\theta = r^n y_2 U_{n-1}(y_1/r),$$

where T_n and U_n are the Chebyshev polynomials, which are orthogonal with respect to $1/\sqrt{1-x^2}$ and $\sqrt{1-x^2}$ on $B^1 = [-1, 1]$, respectively. This relation can be extended to higher dimension. In the following we work with S^d instead of S^{d-1}.

Let H be a weight function defined on $\mathbb{R}^{d+1}$ and assume that H is nonzero almost everywhere when restricted to S^d, even with respect to y_{d+1}, and centrally symmetric with respect to the variables $y' = (y_1, \ldots, y_d)$; for example, H is even in each of its variables, $H(y) = W(y_1^2, \ldots, y_{d+1}^2)$. Associated with H define a weight function W_H^B on B^d by

$$W_H^B(x) = H(x, \sqrt{1 - \|x\|^2}), \qquad x \in B^d. \tag{3.1}$$

We use the notation $\mathcal{V}_n^d(W)$ to denote the space of orthogonal polynomials of degree n with respect to W. Let $\{P_\alpha\}$ and $\{Q_\alpha\}$ denote systems of orthonormal polynomials with respect to the weight functions

$$W_1^B(x) = 2W_H^B(x)/\sqrt{1 - \|x\|^2} \qquad \text{and} \qquad W_2^B(x) = 2W_H^B(x)\sqrt{1 - \|x\|^2},$$

respectively. We adopt the following notation for polar coordinates: for $y \in \mathbb{R}^{d+1}$ write $y = (y_1, \ldots, y_d, y_{d+1}) = (y', y_{d+1})$ and

$$y = r(x, x_{d+1}), \qquad \text{where} \quad r = \|y\|, \quad (x, x_{d+1}) \in S^d.$$

For $|\alpha| = n$ and $|\beta| = n - 1$ we define the following polynomials

$$Y_\alpha^{(1)}(y) = r^n P_\alpha(x) \qquad \text{and} \qquad Y_\beta^{(2)}(y) = r^n x_{d+1} Q_\beta(x), \tag{3.2}$$

and define $Y_{\beta,0}^{(2)}(y) = 0$. These are in fact homogeneous orthogonal polynomials with respect to $H d\omega$ on S^d:

Theorem 3.1. *Let $H(x) = W(x_1^2, \ldots, x_{d+1}^2)$ be defined as above. Then $Y_\alpha^{(1)}$ and $Y_\alpha^{(2)}$ in* (3.2) *are homogeneous polynomials of degree $|\alpha|$ on $\mathbb{R}^{d+1}$ and they satisfy*

$$\int_{S^d} Y_\alpha^{(i)}(y) Y_\beta^{(j)}(y) H(y) d\omega_d(y) = \delta_{\alpha,\beta}\delta_{i,j}, \qquad i, j = 1, 2.$$

Proof. Since both weight functions W_1^B and W_2^B are even in each of its variables, it follows that P_α and Q_α are sums of monomials of even degree if $|\alpha|$ is even and sums of monomials of odd degree if $|\alpha|$ is odd. This is used to show that $Y_\alpha^{(i)}(y)$ are homogeneous polynomials of degree n in y. Since $Y_\alpha^{(1)}$, when restricted to S^d, is independent of x_{d+1} and $Y_\alpha^{(2)}$ contains a single factor x_{d+1}, it follows that $Y_\alpha^{(1)}$ and $Y_\beta^{(2)}$ are orthogonal with respect to $H\,d\omega_d$ on S^d for any α and β. Since H is even with respect to its last variable, the elementary formula

$$\int_{S^d} f(x)d\omega_d(x) = 2\int_{B^d} f(x, \sqrt{1-\|x\|^2})dx/\sqrt{1-\|x\|^2}$$

shows that the orthogonality of $\{Y_\alpha^{(i)}\}$ follows from that of the polynomials P_α (for $i=1$) or Q_α (for $i=2$), respectively. ■

Let us denote by $\mathcal{H}_n^{d+1}(H)$ the space of homogeneous orthogonal polynomials of degree n with respect to $Hd\omega$ on S^d. Then the relation (3.2) defines a one-to-one correspondence between an orthonormal basis of $\mathcal{H}_n^{d+1}(H)$ and an orthonormal basis of $\mathcal{V}_n^d(W_1^B) \oplus x_{d+1}\mathcal{V}_{n-1}^d(W_2^B)$. Therefore, we can derive certain properties of orthogonal polynomials on the spheres from those on the balls. An immediate consequence is

$$\dim \mathcal{H}_n^{d+1}(H) = \binom{n+d}{d} - \binom{n+d-2}{d} = \dim \mathcal{P}_n^{d+1} - \dim \mathcal{P}_{n-2}^{d+1}.$$

Furthermore, we also have the following orthogonal decomposition:

Theorem 3.2. *For each $n \in \mathbb{N}_0$ and $P \in \mathbb{P}_n^{d+1}$, there is a unique decomposition*

$$P(y) = \sum_{k=0}^{[n/2]} \|y\|^{2k} P_{n-2k}(y), \qquad P_{n-2k} \in \mathcal{H}_{n-2k}^{d+1}(H).$$

The classical example is the Lebesgue measure $H(x) = 1$ on the sphere, which gives the ordinary spherical harmonics. We discuss a family of more general weight functions in detail in the following section.

3.2. **Orthogonal polynomials for the product weight functions.** We consider orthogonal polynomials in $\mathcal{H}_n^{d+1}(h_\kappa^2)$ with h_κ being the product weight function

$$h_\kappa(x) = \prod_{i=1}^{d+1} |x_i|^{\kappa_i}, \qquad \kappa_i > -1, \quad x \in \mathbb{R}^{d+1}.$$

Because of the previous subsection, we consider S^d instead of S^{d-1}. The elements of $\mathcal{H}_n^{d+1}(h_\kappa^2)$ are called the h-harmonics, which is a special case of Dunkl's h-harmonics for reflection groups. The function h_κ is invariant under the group $\mathbb{Z}_2^{d+1}$, which simply means that it is invariant under the sign changes of each variable. We shall work with this case first without referring to general reflection groups. An expository of the theory of h-harmonics is given in the next section.

3.2.1. *An orthonormal basis.* Since the weight function h_κ is of product type, an orthonormal basis with respect to h_κ^2 can be given using the spherical coordinates

$$\begin{aligned} x_1 &= r\cos\theta_d, \\ x_2 &= r\sin\theta_d\cos\theta_{d-1}, \\ &\dots \\ x_d &= r\sin\theta_d\dots\sin\theta_2\cos\theta_1, \\ x_{d+1} &= r\sin\theta_d\dots\sin\theta_2\sin\theta_1, \end{aligned}$$

with $r \ge 0$, $0 \le \theta_1 < 2\pi$, $0 \le \theta_i \le \pi$ for $i \ge 2$. In polar coordinates the surface measure on S^d is

$$d\omega = (\sin\theta_d)^{d-1}(\sin\theta_{d-1})^{d-2}\cdots\sin\theta_2\, d\theta_d\, d\theta_{d-1}\dots d\theta_1.$$

Whenever we speak of orthonormal basis, we mean that the measure is normalized to have integral 1. The normalization constant for h_κ is

$$\sigma_d\int_{S^d} h_\kappa^2(x)d\omega(x) = \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d}{2}}}\frac{\Gamma(\kappa_1+\frac12)\dots\Gamma(\kappa_{d+1}+\frac12)}{\Gamma(|\kappa|+\frac{d+1}{2})},$$

where $\sigma_d^{-1} = \int_{S^d} d\omega = 2\pi^{(d+1)/2}/\Gamma((d+1)/2)$. The orthonormal basis is given in terms of the generalized Gegenbauer polynomials $C_n^{(\lambda,\mu)}$ defined by

$$\begin{aligned} C_{2n}^{(\lambda,\mu)}(x) &= \frac{(\lambda+\mu)_n}{\left(\mu+\frac12\right)_n}P_n^{(\lambda-1/2,\mu-1/2)}(2x^2-1), \\ C_{2n+1}^{(\lambda,\mu)}(x) &= \frac{(\lambda+\mu)_{n+1}}{\left(\mu+\frac12\right)_{n+1}}xP_n^{(\lambda-1/2,\mu+1/2)}(2x^2-1), \end{aligned}$$

which are orthogonal with respect to the weight function $|x|^{2\mu}(1-x^2)^{\lambda-1/2}$ on $[-1,1]$. It follows that $C_n^{(\lambda,0)} = C_n^\lambda$, the usual Gegenbauer polynomial. Let $\widetilde{C}_n^{(\lambda,\mu)}$ denote the corresponding orthonormal polynomial. For $d=1$ and $h_\kappa(x) = |x_1|^{\kappa_1}|x_2|^{\kappa_2}$ an orthonormal basis for $\mathcal{H}_n^2(h_\kappa^2)$ is given by

$$Y_n^1(x) = r^n\widetilde{C}_n^{(\kappa_2,\kappa_1)}(\cos\theta), \quad Y_n^2(x) = r^n\sqrt{\frac{\kappa_1+\kappa_2+1}{\kappa_2+\frac12}}\sin\theta\,\widetilde{C}_{n-1}^{(\kappa_2+1,\kappa_1)}(\cos\theta).$$

For $d > 1$, we use the following notation: associated to $\kappa = (\kappa_1,\dots,\kappa_{d+1})$, define

$$\kappa^j = (\kappa_j,\dots,\kappa_{d+1}), \qquad 1 \le j \le d+1.$$

Since κ^{d+1} consists of only the last element of κ, write $\kappa^{d+1} = \kappa_{d+1}$. Similarly define α^j for $\alpha \in \mathbb{N}_0^d$.

Theorem 3.3. *Let $d \ge 1$. In spherical coordinates an orthonormal basis of $\mathcal{H}_n^{d+1}(h_\kappa^2)$ is given by*

$$Y_\alpha^{n,i}(x) = [A_\alpha^n]^{-1} r^n \prod_{j=1}^{d-1} \Big[\widetilde{C}_{\alpha_j}^{(a_j,\kappa_j)}(\cos\theta_{d-j+1})(\sin\theta_{d-j+1})^{|\alpha^{j+1}|} \Big] Y_{\alpha_d}^i(\cos\theta_1, \sin\theta_1),$$

where $\alpha \in \mathbb{N}_0^{d+1}$, $|\alpha| = n$, $a_j = |\alpha^{j+1}| + |\kappa^{j+1}| + \frac{d-j}{2}$, $Y_{\alpha_d}^i$ with $i = 1, 2$ are two-dimensional h-harmonics with parameters (κ_{d-1}, κ_d) and

$$[A_\alpha^n]^2 = \frac{1}{\big(|\kappa| + \frac{d+1}{2}\big)_n} \prod_{j=1}^{d} \Big(|\alpha^{j+1}| + |\kappa^j| + \frac{d-j+2}{2} \Big)_{\alpha_j}.$$

This can be verified by straightforward computation, using the integral

$$\int_{S^d} f(x) d\omega_d(x) = \int_0^\pi \int_{S_{d-1}} f(\cos\theta, \sin\theta\, x') d\omega_{d-1}(x') \sin^{d-1}\theta d\theta$$

inductively and the orthogonality of $C_n^{(\lambda,\mu)}$.

3.2.2. *h-harmonics.* There is another way of describing the space $\mathcal{H}_n^{d+1}(h_\kappa^2)$ using a second order differential-difference operator that plays the role of the Laplace operator for the ordinary harmonics. For $\kappa_i \ge 0$, define Dunkl's operators $\mathcal{D}_j$ by

$$\mathcal{D}_j f(x) = \partial_j f(x) + \kappa_j \frac{f(x) - f(x_1, \ldots, -x_j, \ldots, x_{d+1})}{x_j}, \qquad 1 \le j \le d+1.$$

It is easy to see that these first order differential-difference operators map $\mathcal{P}_n^d$ into $\mathcal{P}_{n-1}^d$. A remarkable fact is that these operators commute, which can be verified by an easy computation.

Theorem 3.4. *The operators $\mathcal{D}_i$ commute: $\mathcal{D}_i\mathcal{D}_j = \mathcal{D}_j\mathcal{D}_i$, $1 \le i, j \le d+1$.*

Proof. Let $x\sigma_j = (x_1, \ldots, -x_j, \ldots, x_{d+1})$. A simple computation shows that

$$\begin{aligned}\mathcal{D}_i\mathcal{D}_j f(x) =& \partial_i\partial_j f(x) + \frac{\kappa_i}{x_i}(\partial_j f(x) - \partial_j f(x\sigma_j)) + \frac{\kappa_j}{x_j}(\partial_i f(x) - \partial_i f(x\sigma_i)) \\ &+ \frac{\kappa_i\kappa_j}{x_i x_j}(f(x) - f(x\sigma_j) - f(x\sigma_i) - f(x\sigma_j\sigma_i)),\end{aligned}$$

from which $\mathcal{D}_i\mathcal{D}_j = \mathcal{D}_j\mathcal{D}_i$ is evident. ∎

The operator $\mathcal{D}_i$ plays the role of ∂_i. The h-Laplacian is defined by

$$\Delta_h = \mathcal{D}_1^2 + \cdots + \mathcal{D}_{d+1}^2.$$

It's a second order differential-difference operator. If all $\kappa_i = 0$ then Δ_h becomes the classical Laplacian Δ. A quick computation shows that

$$\Delta_h f(x) = \Delta f(x) + 2\sum_{j=1}^{d+1} \frac{\kappa_j}{x_j}\frac{\partial}{\partial_j} f(x) - \sum_{j=1}^{d+1} \kappa_j \frac{f(x) - f(x_1, \dots, -x_j, \dots, x_d)}{x_j^2}.$$

Let us write $\Delta_h = L_h + D_h$, where L_h is the differential part and D_h is the difference part of the above equation. The following theorem shows that h-harmonics are homogeneous polynomials P satisfying $\Delta_h P = 0$.

Theorem 3.5. *Suppose f and g are homogeneous polynomials of different degrees satisfying $\Delta_h f = 0$ and $\Delta_h g = 0$, then $\int_{S^d} f(x)g(x)h_\kappa^2(x)d\omega = 0$.*

Proof. Assume $\kappa_i \geq 1$ and use analytic continuation to extend the range of validity to $\kappa \geq 0$. The following formula can be proved using Green's identity:

$$\int_{S^d} \frac{\partial f}{\partial n} g h_\kappa^2 d\omega = \int_{B^{d+1}} (gL_h f + \langle \nabla f, \nabla g\rangle) h_\kappa^2 dx,$$

where $\partial f/\partial n$ denotes the normal derivative of f. If f is homogeneous, then Euler's equation shows that $\partial f/\partial n = (\deg f)f$. Hence,

$$\begin{aligned}(\deg f - \deg g)\int_{S^d} fgh_\kappa^2 d\omega &= \int_{B^{d+1}} (gL_h f - fL_h g)h_\kappa^2 dx \\ &= \int_{B^{d+1}} (gD_h f - fD_h g)h_\kappa^2 dx = 0,\end{aligned}$$

since the explicit formula of D_h shows that it is a symmetric operator. ∎

There is a linear operator V_κ, called the intertwining operator, which acts between ordinary harmonics and h-harmonics. It is defined by the properties

$$\mathcal{D}_i V_\kappa = V_\kappa \partial_i, \qquad V1 = 1, \qquad V\mathcal{P}_n \subset \mathcal{P}_n.$$

It follows that $\Delta_h V_\kappa = V_\kappa \Delta$ so that if P is an ordinary harmonic polynomial, then $V_\kappa P$ is an h-harmonic. In the case $\mathbb{Z}_2^{d+1}$, V_κ is given by an integral operator:

Theorem 3.6. *For $\kappa_i \geq 0$,*

$$V_\kappa f(x) = \int_{[-1,1]^{d+1}} f(x_1 t_1, \dots, x_{d+1}t_{d+1}) \prod_{i=1}^{d+1} c_{\kappa_i}(1+t_i)(1-t_i^2)^{\kappa_i - 1} dt,$$

where $c_\lambda = \Gamma(\lambda + 1/2)/(\sqrt{\pi}\Gamma(\lambda))$ and if any one of $\kappa_i = 0$, the formula holds under the limit

$$\lim_{\mu \to 0} c_\mu \int_{-1}^{1} f(t)(1-t^2)^{\mu - 1} d\mu(t) = [f(1) + f(-1)]/2.$$

Proof. Denote the difference part of $\mathcal{D}_i$ as $\widetilde{\mathcal{D}}_i$ so that $\mathcal{D}_i = \partial_i + \widetilde{\mathcal{D}}_i$. Clearly,

$$\partial_j V_\kappa f(x) = \int_{[-1,1]^{d+1}} \partial_j f(x_1 t_1, \ldots, x_{d+1} t_{d+1}) t_j \prod_{i=1}^{d+1} c_{\kappa_i}(1+t_i)(1-t_i^2)^{\kappa_i - 1} dt.$$

Taking into account the parity of the integrand, an integration by parts shows

$$\widetilde{\mathcal{D}}_j V_\kappa f(x) = \int_{[-1,1]^{d+1}} \partial_j f(x_1 t_1, \ldots, x_{d+1} t_{d+1})(1-t_j) \prod_{i=1}^{d+1} c_{\kappa_i}(1+t_i)(1-t_i^2)^{\kappa_i - 1} dt.$$

Adding the last two equations gives $\mathcal{D}_i V_\kappa = V_\kappa \partial_i$. ∎

As one important application, a compact formula of the reproducing kernel can be given in terms of V_κ. The reproducing kernel $P_n(h_\kappa^2; x, y)$ of $\mathcal{H}_n^{d+1}(h_\kappa^2)$ is defined uniquely by the property

$$\int_{S^d} P_n(h_\kappa^2; x, y) Q(y) h_\kappa^2(y) d\omega(y) = Q(x), \quad \forall Q \in \mathcal{H}_n^{d+1}(h_\kappa^2).$$

If Y_α is an orthonormal basis of $\mathcal{H}_n^{d+1}(h_\kappa^2)$, then $P_n(h_\kappa^2; x, y) = \sum Y_\alpha(x) Y_\alpha(y)$.

Theorem 3.7. *For $\kappa_i \ge 0$, and $\|y\| \le \|x\| = 1$,*

$$P_n(h_\kappa^2; x, y) = \frac{n + |\kappa| + \frac{d-1}{2}}{|\kappa| + \frac{d-1}{2}} V_\kappa \Big[C_n^{|\kappa| + \frac{d-1}{2}} \Big(\Big\langle \cdot, \frac{y}{\|y\|} \Big\rangle \Big) \Big](x) \|y\|^n.$$

Proof. Let $K_n(x, y) = V_\kappa^{(x)}(\langle x, y\rangle^n)/n!$. Using the defining property of V_κ, it is easy to see that $K_n(x, \mathcal{D}^{(y)}) f(y) = f(x)$ for $f \in \mathcal{P}_n^{d+1}$. Fixing y let $p(x) = K_n(x, y)$; then $P_n(h_\kappa^2; x, y) = 2^n(|\kappa| + d/2)_n \operatorname{proj}_n p(x)$, where proj_n is the projection operator from $\mathcal{P}_n^{d+1}$ onto $\mathcal{H}_n^{d+1}(h_\kappa^2)$. The projection operator can be computed explicitly, which gives

$$P_n(h_\kappa^2; x, y) = \sum_{0 \le j \le n/2} \frac{\big(\gamma_\kappa + \frac{d}{2}\big)_n 2^{n-2j}}{(2 - n - \gamma_\kappa - d/2)_j j!} \|x\|^{2j} \|y\|^{2j} K_{n-2j}(x, y).$$

When $\|x\| = 1$, we can write the right hand side as $V_\kappa(L_n(\langle \cdot, y/\|y\|\rangle))(x)$, where the polynomial L_n is a constant multiple of the Gegenbauer polynomial. ∎

For the classical harmonics ($\kappa = 0$), these are the so-called zonal harmonics,

$$P_n(x, y) = \frac{n + \frac{d-1}{2}}{\frac{d-1}{2}} C_n^{\frac{d-1}{2}}(\langle x, y\rangle), \qquad x, y \in S^d.$$

As one more application of the intertwining operator, we mention an analogue of the Funk-Hecke formula for ordinary harmonics. Denote by w_λ the normalized weight function

$$w_\lambda(t) = \frac{\Gamma(\lambda+1)}{\sqrt{\pi}\,\Gamma(\lambda + 1/2)} (1 - t^2)^{\lambda - 1/2}, \qquad t \in [-1, 1],$$

whose corresponding orthogonal polynomials are the Gegenbauer polynomials.

Theorem 3.8. *Let f be a continuous function on $[-1,1]$. Let $Y_n^h \in \mathcal{H}_n^d(h_\kappa^2)$. Then*

$$\int_{S^d} V_\kappa f(\langle x,\cdot\rangle)(y) Y_n^h(y) h_\kappa^2(y) d\omega(y) = \lambda_n(f) Y_n^h(x), \qquad x \in S^d,$$

where $\lambda_n(f)$ is a constant defined by

$$\lambda_n(f) = \frac{1}{C_n^{|\kappa|+(d-1)/2}(1)} \int_{-1}^{1} f(t) C_n^{|\kappa|+\frac{d-1}{2}}(t) w_{|\kappa|+(d-1)/2}(t) dt.$$

The case $\kappa = 0$ is the classical Funk-Hecke formula.

3.2.3. *Monomial basis.* Another interesting orthogonal basis of $\mathcal{H}_n^{d+1}(h_\kappa^2)$ can be given explicitly. Let V_κ be the intertwining operator.

Definition 3.1. *Let P_α be polynomials defined by*

$$V_\kappa\Big[(1-2\langle b,\cdot\rangle + \|b\|^2)^{-|\kappa|-\frac{d-1}{2}}\Big](x) = \sum_{\alpha\in\mathbb{N}_0^{d+1}} b^\alpha P_\alpha(x), \qquad b \in \mathbb{R}^{d+1}.$$

The polynomials P_α are indeed homogeneous polynomials and they can be given explicitly in terms of the Lauricella function of type B which is defined by

$$F_B(\alpha,\beta;c;x) = \sum_\gamma \frac{(\alpha)_\gamma(\beta)_\gamma}{(c)_{|\gamma|}\gamma!} x^\gamma, \qquad \alpha,\beta \in \mathbb{N}_0^{d+1}, \quad c \in \mathbb{R},$$

where the summation is taken over $\gamma \in \mathbb{N}_0^{d+1}$. For $\alpha \in \mathbb{N}_0^{d+1}$, let $[\alpha/2]$ denote the multi-index whose elements are $[\alpha_i/2]$ where $[a]$ denotes the integer part of a. The notation $(\alpha)_\gamma$ abbreviates the product $(\alpha_0)_{\gamma_0}\cdots(\alpha_{d+1})_{\gamma_{d+1}}$.

Theorem 3.9. *For $\alpha \in \mathbb{N}_0^{d+1}$,*

$$P_\alpha(x) = \frac{2^{|\alpha|}(|\kappa| + \frac{d-1}{2})_\alpha}{\alpha!} \frac{(\frac{1}{2})_{[\frac{\alpha+1}{2}]}}{(\kappa+\frac{1}{2})_{[\frac{\alpha+1}{2}]}} Y_\alpha(x),$$

where Y_α are given by

$$\begin{aligned} Y_\alpha(x) &= x^\alpha \times \\ &\times F_B\Big(-\alpha + \Big[\tfrac{\alpha+1}{2}\Big], -\Big[\tfrac{\alpha+1}{2}\Big] - \kappa + \tfrac{1}{2}; -|\alpha| - |\kappa| - \tfrac{d-3}{2}; \tfrac{1}{x_1^2}, \ldots, \tfrac{1}{x_{d+1}^2} \Big). \end{aligned}$$

Proof. Let $\lambda = |\kappa| + \frac{d-1}{2}$. The multinomial and binomial formulae show that

$$(1-2\langle b,x\rangle + \|b\|^2)^{-\lambda} = \sum_\alpha b^\alpha 2^{|\alpha|} \sum_\gamma \frac{(\lambda)_{|\alpha|-|\gamma|}(-\alpha+\gamma)_\gamma}{(\alpha-\gamma)!\gamma!} 2^{-2|\gamma|} x^{\alpha-2\gamma}.$$

Applying the intertwining operator and using the formula

$$V_\kappa x^{\alpha-2\gamma} = \frac{(\frac{1}{2})_{[\frac{\alpha+1}{2}]}}{(\kappa+\frac{1}{2})_{[\frac{\alpha+1}{2}]}} \frac{\left(-\left[\frac{\alpha+1}{2}\right]-\kappa+\frac{1}{2}\right)_\gamma}{\left(-\left[\frac{\alpha+1}{2}\right]+\frac{1}{2}\right)_\gamma} x^{\alpha-2\gamma}$$

completes the proof. ■

Since $(-a)_k = 0$ if $a < k$, $Y_\alpha(x)$ is a homogeneous polynomial of degree $|\alpha|$. Furthermore, when restricted to S^d, $Y_\alpha(x) = x^\alpha+$ lower order terms. The following theorem says that we can call Y_α monomial orthogonal polynomials.

Theorem 3.10. *For $\alpha \in \mathbb{N}_0^{d+1}$, Y_α are elements of $\mathcal{H}_{|\alpha|}^{d+1}(h_\kappa^2)$.*

Proof. We show that Y_α are orthogonal to x^β for $\beta \in \mathbb{N}_0^{d+1}$ and $|\beta| \le n-1$. If one of the components of $\alpha-\beta$ is odd, then the orthogonality follows from changing sign of that component in the integral. Hence, we can assume that all components of $\alpha-\beta$ are even and we only have to work with x^β for $|\beta| = |\alpha|-2$, since every polynomial of degree n on the sphere is the restriction of a homogeneous polynomial of degree n. Using the beta integral on the sphere, a tedious computation shows that

$$\begin{aligned}\int_{S^d} P_\alpha(x) x^\beta h_\kappa^2(x) d\omega(x) &= \frac{(\kappa+\frac{1}{2})_{\frac{\alpha+\beta}{2}}}{(|\kappa|+\frac{d+1}{2})_{|\alpha|}} \sum_\gamma \frac{(-\alpha+[\frac{\alpha+1}{2}])_\gamma(-[\frac{\alpha+1}{2}]-\kappa+\frac{1}{2})_\gamma}{(-\frac{\alpha+\beta}{2}-\kappa+\frac{1}{2})_\gamma \gamma!} \\ &= \frac{(\kappa+\frac{1}{2})_{\frac{\alpha+\beta}{2}}}{(|\kappa|+\frac{d+1}{2})_{|\alpha|}} \prod_{i=1}^{d+1} {}_2F_1\left(\begin{matrix}-\alpha_i+[\frac{\alpha_i+1}{2}], -[\frac{\alpha_i+1}{2}]-\kappa_i+\frac{1}{2} \\ -\frac{\alpha_i+\beta_i}{2}-\kappa_i+\frac{1}{2}\end{matrix}; 1\right).\end{aligned}$$

Since at least one $\beta_i < \alpha_i$, this last term is zero using the Chu-Vandermonde identity for ${}_2F_1$. ■

A standard Hilbert space argument shows that among all polynomials of the form $x^\alpha+$ polynomials of lower degrees, Y_α has the smallest $L^2(h_\kappa^2 d\omega)$-norm and Y_α is the orthogonal projection of x^α onto $\mathcal{H}_n^{d+1}(h_\kappa^2)$. Let us mention another expression of Y_α. For any $\alpha \in \mathbb{N}^d$, define homogeneous polynomials H_α (cf. [44])

$$H_\alpha(x) = \|x\|^{2|\kappa|+d-2+2|\alpha|} \mathcal{D}^\alpha \left\{ \|x\|^{-2|\kappa|-d+2} \right\},$$

where $\mathcal{D}^\alpha = \mathcal{D}_1^{\alpha_1} \ldots \mathcal{D}_d^{\alpha_d}$.

Theorem 3.11. *For $\alpha \in \mathbb{N}_0^{d+1}$ and $|\alpha| = n$,*

$$H_\alpha(x) = (-1)^n 2^n \left(|\kappa| + \frac{d-1}{2}\right)_n Y_\alpha(x).$$

For $\kappa = 0$, the polynomials H_α are called Maxwell's representation. That they are constant multiples of monomial polynomials follows from the recursive relation

$$H_{\alpha+\varepsilon_i}(x) = -(2|\kappa|+d-2+2|\alpha|) x_i H_\alpha(x) + \|x\|^2 \mathcal{D}_i H_\alpha,$$

where $\varepsilon_i = (0, \ldots, 1, \ldots, 0)$ is the i-th standard unit vector in $\mathbb{R}^{d+1}$.

3.3. ***h*-harmonics for a general reflection group.** The theory of h-harmonics is established for a general reflection group ([6, 7, 8]). For a nonzero vector $v \in \mathbb{R}^{d+1}$ define the reflection σ_v by $x\sigma_v := x - 2\langle x, v\rangle v/\|v\|^2$, $x \in \mathbb{R}^{d+1}$, where $\langle x, y\rangle$ denotes the usual Euclidean inner product. A finite reflection group G is described by its root system R, which is a finite set of nonzero vectors in $\mathbb{R}^{d+1}$ such that $u, v \in R$ implies $u\sigma_v \in R$, and G is the subgroup of the orthogonal group generated by the reflections $\{\sigma_u : u \in R\}$. If R is not the union of two nonempty orthogonal subsets, the corresponding reflection group is called irreducible. Note, that $\mathbb{Z}_2^{d+1}$ is reducible, a product of $d+1$ copies of the irreducible group $\mathbb{Z}_2$. There is a complete classification of irreducible finite reflection groups. The list consists of root systems of infinite families A_{d-1} with G being the symmetric group of d objects, B_d with G being the symmetry group of the hyper-octahedron $\{\pm\varepsilon_1, \ldots, \pm\varepsilon_{d+1}\}$ of $\mathbb{R}^{d+1}$, D_d with G being a subgroup of the hyper-octahedral group for $d \ge 4$, the dihedral systems $I_2(m)$ with G being the symmetric group of regular m-gons in $\mathbb{R}^2$ for $m \ge 3$, and several other individual systems H_3, H_4, F_4 and E_6, E_7, E_8.

Fix $u_0 \in \mathbb{R}^{d+1}$ such that $\langle u, u_0\rangle \neq 0$. The set of positive roots R_+ with respect to u_0 is defined by $R_+ = \{u \in R : \langle u, u_0\rangle > 0\}$ so that $R = R_+ \cup (-R_+)$. A multiplicity function $v \mapsto \kappa_v$ of $R_+ \mapsto \mathbb{R}$ is a function defined on R_+ with the property that $\kappa_u = \kappa_v$ if σ_u is conjugate to σ_v; in other words, the function is G-invariant. Fix a positive root system R_+. Then the function h_κ defined by

$$h_\kappa(x) = \prod_{v \in R_+} |\langle x, v\rangle|^{\kappa_v}, \qquad x \in \mathbb{R}^{d+1},$$

is a positive homogeneous function of degree $\gamma_\kappa := \sum_{v \in R_+} \kappa_v$ and $h_\kappa(x)$ is invariant under G. The h-harmonics are homogeneous orthogonal polynomials on S^d with respect to $h_\kappa^2 d\omega$. Beside the product weight function $h_\kappa = \prod_{i=1}^{d+1} |x_i|^{\kappa_i}$, the most interesting to us are the case A_d for which $R_+ = \{\varepsilon_i - \varepsilon_j : i > j\}$ and

$$h_\kappa(x) = \prod_{1 \le i,j \le d+1} |x_i - x_j|^\kappa, \quad \kappa \ge 0,$$

which is invariant under the symmetric group S_d, and the case B_{d+1} for which $R_+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j : i < j\} \cup \{\varepsilon : 1 \le i \le d+1\}$ and

$$h_\kappa(x) = \prod_{i=1}^{d+1} |x_i|^{\kappa_0} \prod_{1 \le i,j \le d+1} |x_i^2 - x_j^2|^{\kappa_1}, \quad \kappa_0, \kappa_1 \ge 0,$$

which is invariant under the hyper-octahedral group.

For a finite reflection group G with positive roots R_+ and a multiplicity function, Dunkl's operators are defined by

$$\mathcal{D}_i f(x) := \partial_i f(x) + \sum_{v \in R_+} \kappa(v) \frac{f(x) - f(x\sigma_v)}{\langle x, v\rangle} \langle v, \varepsilon_i \rangle, \qquad 1 \le i \le d+1,$$

where $\varepsilon_1, \ldots, \varepsilon_{d+1}$ are the standard unit vectors of $\mathbb{R}^{d+1}$. The remarkable fact that these are commuting operators, $\mathcal{D}_i \mathcal{D}_j = \mathcal{D}_j \mathcal{D}_i$, holds for every reflection group. The h-Laplacian is defined again by $\Delta_h = \mathcal{D}_1^2 + \cdots + \mathcal{D}_{d+1}^2$, which plays the role of the Laplacian in the theory of ordinary harmonics. The h-harmonics are the homogeneous polynomials satisfying the equation $\Delta_h P = 0$ and Theorem 3.5 holds also for a general reflection group.

There again exists an intertwining operator V_κ between the algebra of differential operators and the commuting algebra of Dunkl's operators, and it is the unique linear operator defined by

$$V_\kappa \mathcal{P}_n \subset \mathcal{P}_n, \qquad V1 = 1, \qquad \mathcal{D}_i V_\kappa = V_\kappa \partial_i, \qquad 1 \le i \le d.$$

The representation of the reproducing kernel in Theorem 3.7 in terms of the intertwining operator holds for a general reflection group. It was proved by Rösler [27] that V_κ is a nonnegative operator; that is, $V_\kappa p \ge 0$ if $p \ge 0$.

However, unlike the case of $G = \mathbb{Z}_2^{d+1}$, there are few explicit formulae known for h-harmonics with respect to a general reflection group. In fact, there is no orthonormal basis of $\mathcal{H}_n^{d+1}(h_\kappa^2)$ known for $d > 0$. Recall, that the basis given in Theorem 3.3 depends on the product nature of the weight function there. For the orthogonal basis, one can still show that H_α defined in the previous section are h-harmonics and $\{H_\alpha : |\alpha| = n, \alpha \in \mathbb{N}_0^{d+1}, \alpha_{d+1} = 0 \,\text{or}\, 1\}$ is an orthogonal basis, but explicit formulae for H_α and its $L^2(h_\kappa^2 d\omega)$-norm are unknown. Another basis for $\mathcal{H}_n^{d+1}(h_\kappa^2)$ consists of $V_\kappa Y_\alpha$, where Y_α is a basis for the space $\mathcal{H}_n^{d+1}$ of ordinary harmonics. However, other than $\mathbb{Z}_2^{d+1}$, an explicit formula of V_κ is known only in the case of the symmetric group S_3 on $\mathbb{R}^3$, and the formula is complicated and likely not in its final form. In fact, even in the case of dihedral groups on $\mathbb{R}^2$ the formula of V_κ is unknown. The first non-trivial case should be the group $I_2(4) = B_2$ for which $h_\kappa(x) = |x_1 x_2|^{\kappa_1} |x_1^2 - x_2^2|^{\kappa_2}$.

4. Orthogonal polynomials on the unit ball

As we have seen in the Section 3.1, the orthogonal polynomials on the unit ball are closely related to orthogonal polynomials on the sphere. This allows us to derive, working with $G \times \mathbb{Z}_2$ on $\mathbb{R}^{d+1}$, various properties for orthogonal polynomials with respect to the weight function $h_\kappa(x)(1 - \|x\|)^{\mu - 1/2}$, where h_κ is a reflection invariant function defined on $\mathbb{R}^d$. Again we will work with the case of h_κ being a product weight function,

$$W_{\kappa,\mu}^B(x) = \prod_{i=1}^{d+1} |x_i|^{2\kappa_i}(1 - \|x\|)^{\mu-1/2}, \quad \kappa_i \ge 0, \quad \mu > -1/2$$

for which various explicit formulae can be derived from the results in Section 3.2. In the case $\kappa = 0$, $W_\mu^B = W_{0,\mu}^B$ is the classical weight function on B^d, which is invariant under rotations.

4.1. **Differential-difference equation.** For $y \in \mathbb{R}^{d+1}$ we use the polar coordinates $y = r(x, x_{d+1})$, where $r = \|y\|$ and $(x, x_{d+1}) \in S^d$. The relation in Section 3.1 states that if P_α are orthogonal polynomials with respect to $W_{\kappa,\mu}^B$, then $Y_\alpha(y) = r^n P_\alpha(x)$ are h-harmonics with respect to $h_{\kappa,\mu}(y) = \prod_{i=1}^d |y_i|^{\kappa_i}|y_{d+1}|^\mu$. Since the polynomials Y_α defined above are even in y_{d+1}, we only need to deal with the upper half space $\{y \in \mathbb{R}^{d+1} : y_{d+1} \ge 0\}$. In order to write the operator for P_α^n in terms of $x \in B^d$, we choose the following mapping:

$$y \mapsto (r, x) : \quad y_1 = rx_1, \ldots, y_d = rx_d, \quad y_{d+1} = r\sqrt{1 - x_1^2 - \cdots - x_d^2},$$

which is one-to-one from $\{y \in \mathbb{R}^{d+1} : y_{d+1} \ge 0\}$ to itself. We rewrite the h-Laplacian in terms of the new coordinates (r, x). Let $\Delta_h^{\kappa,\mu}$ denote the h-Laplacian associated with the weight function $h_{\kappa,\mu}$, and preserve the notation h_κ for the h-Laplacian associated with the weight function $h_\kappa(x) = \prod_{i=1}^d |x_i|^{\kappa_i}$ for $x \in \mathbb{R}^d$.

Proposition 4.1. *Acting on functions on $\mathbb{R}^{d+1}$ that are even in y_{d+1}, the operator $\Delta_h^{\kappa,\mu}$ takes the form*

$$\Delta_h^{\kappa,\mu} = \frac{\partial^2}{\partial r^2} + \frac{d + 2|\kappa| + 2\mu}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{h,0}^{\kappa,\mu}$$

in terms of the coordinates (r, x) in $\{y \in \mathbb{R}^{d+1} : y_{d+1} \ge 0\}$, where the spherical part $\Delta_{h,0}^{\kappa,\mu}$, acting on functions in the variables x, is given by

$$\Delta_{h,0}^{\kappa,\mu} = \Delta_h - \langle x, \nabla\rangle^2 - (2|\kappa| + 2\mu + d - 1)\langle x, \nabla\rangle,$$

in which the operators Δ_h and $\nabla = (\partial_1, \ldots, \partial_d)$ are all acting on x variables.

Proof. Since $\Delta_h^{\kappa,\mu}$ is just $\Delta_h^{(y)}$ acting on functions defined on $\mathbb{R}^{d+1}$ for $h_\kappa(y) = \prod_{i=1}^{d+1} |y_i|^{\kappa_i}$ (with $\kappa_{d+1} = \mu$), its formula is given in Section 3.2. Writing r and x_i in terms of y under the change of variables $y \mapsto (r, x)$ and computing the partial derivatives $\partial x_i/\partial y_i$, the chain rule implies that

$$\frac{\partial}{\partial y_i} = x_i\frac{\partial}{\partial r} + \frac{1}{r}\Big(\frac{\partial}{\partial x_i} - x_i\langle x, \nabla^{(x)}\rangle\Big), \qquad 1 \le i \le d+1,$$

where for $i = d+1$ we use the convention that $x_{d+1} = \sqrt{1-\|x\|_2^2}$ and define $\partial/\partial x_{d+1} = 0$. A tedious computation gives the second order derivatives

$$\frac{\partial^2}{\partial y_i^2} = x_i^2 \frac{\partial^2}{\partial r^2} + \frac{1-x_i^2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Big[\frac{\partial^2}{\partial x_i^2} - (1-x_i^2)\langle x, \nabla^{(x)}\rangle - x_i \langle x, \nabla^{(x)}\rangle \frac{\partial}{\partial x_i}\Big]$$
$$+ \Big(x_i \frac{\partial}{\partial x_i} - x_i^2 \langle x, \nabla^{(x)}\rangle\Big)\Big(\frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\Big) - \Big(x_i \frac{\partial}{\partial x_i} - x_i^2 \langle x, \nabla^{(x)}\rangle\Big)\langle x, \nabla^{(x)}\rangle.$$

Using these formulae and the fact that $f(y_1, \ldots, y_{d+1}) - f(y_1, \ldots, -y_{d+1}) = 0$ for f even in y_{d+1}, the stated equation follows from a straightforward computation. ∎

We use the notation $\mathcal{V}_n^d(W)$ to denote the space of orthogonal polynomials of degree exactly n with respect to the weight function W.

Theorem 4.1. *The orthogonal polynomials in $\mathcal{V}_n^d(W_{\kappa,\mu}^B)$ satisfy the differential–difference equation*

$$\Big[\Delta_h - \langle x, \nabla\rangle^2 - (2|\kappa| + 2\mu + d - 1)\langle x, \nabla\rangle\Big] P = -n(n+d+2|\kappa|+2\mu-1)P.$$

Proof. Let $P \in \mathcal{V}_n^d(W_{\kappa,\mu}^B)$. The formula in the Proposition 4.1 applied to the homogeneous polynomial $Y_\alpha(y) = r^n P_\alpha(x)$ gives

$$0 = \Delta_h^{\kappa,\mu} Y_\alpha(y) = r^{n-2}[n(n+d+2|\kappa|+2\mu-1)P_\alpha(x) + \Delta_{h,0}^{\kappa,\mu} P_\alpha(x)].$$

The stated result follows from the formula for $\Delta_{h,0}^{\kappa,\mu}$. ∎

For $\kappa = 0$, Δ_h becomes the ordinary Laplacian and the equation becomes a differential equation, which is the classical differential equation in [2]; note, that in this case the weight function W_μ^B is rotation invariant. A similar differential-difference equation holds for the weight functions of the form $h_\kappa^2(x)(1-\|x\|^2)^{\mu-1/2}$ for h_κ associated with a general reflection group.

4.2. **Orthogonal bases and reproducing kernels.** From the correspondence $Y_\alpha(y) = r^n P_\alpha(x)$, $y = r(x, x_{d+1}) \in \mathbb{R}^d$ and $x \in B^d$, several orthogonal bases for $\mathcal{V}_n^d(W_{\kappa,\mu}^B)$ follow from the bases for the h-harmonics in Section 3.2. All orthonormal bases are with respect to the weight function normalized to have unit integral. Associated with $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, define $\mathbf{x}_j = (x_1, \ldots, x_j)$ for $1 \le j \le d$ and $\mathbf{x}_0 = 0$.

Theorem 4.2. *An orthonormal basis of $\mathcal{V}_n^d(W_{\kappa,\mu}^B)$ is given by $\{P_\alpha : \alpha \in \mathbb{N}_0^d, |\alpha| = n\}$ defined by*

$$P_\alpha(x) = [h_\alpha^B]^{-1} \prod_{j=1}^d (1-\|\mathbf{x}_{j-1}\|^2)^{\frac{\alpha_j}{2}} \widetilde{C}_{\alpha_j}^{(a_j,\kappa_j)}\Big(\frac{x_j}{\sqrt{1-\|\mathbf{x}_{j-1}\|^2}}\Big),$$

where $a_j = \mu + |\alpha^{j+1}| + |\kappa^j| + \frac{d-j}{2}$ *and* h_α^B *are given by*

$$[h_\alpha^B]^2 = \frac{1}{(|\kappa| + \mu + \frac{d+1}{2})_n} \prod_{j=1}^{d} \Big(\mu + |\alpha^{j+1}| + |\kappa^j| + \frac{d-j+2}{2}\Big)_{\alpha_j}.$$

Proof. In the spherical coordinates of $(x, x_{d+1}) \in S^d$, $\cos\theta_{d-j} = x_{j+1}/\sqrt{1 - \|\mathbf{x}_j\|^2}$ and $\sin\theta_{d-j} = \sqrt{1 - \|\mathbf{x}_{j+1}\|^2}/\sqrt{1 - \|\mathbf{x}_j\|^2}$. Hence, this basis is obtained from the h-harmonic basis in Theorem 3.3 using the correspondence. ■

Another orthonormal basis can be given in the polar coordinates. Using

$$\int_{B^d} f(x)dx = \int_0^1 r^{d-1} \int_{S^{d-1}} f(rx')d\omega(x')dr,$$

the verification is a straightforward computation.

Theorem 4.3. *For* $0 \le j \le n/2$ *let* $\{S^h_{n-2j,\beta}\}$ *denote an orthonormal basis of* $\mathcal{H}^d_{n-2j}(h_\kappa^2)$*; then the polynomials*

$$P_{\beta,j}(x) = [c^B_{j,n}]^{-1} \widetilde{C}_{2j}^{(\mu, n-2j+|\kappa|+\frac{d-1}{2})}(\|x\|) S^h_{\beta,n-2j}(x)$$

form an orthonormal basis of $\mathcal{V}_n^d(W^B_{\kappa,\mu})$*, in which the constants are given by*

$$[c^B_{j,n}]^2 = \frac{\Gamma(|\kappa| + \mu + \frac{d+1}{2})\Gamma(n - 2j + |\kappa| + \frac{d}{2})}{\Gamma(|\kappa| + \frac{d}{2})\Gamma(n - 2j + |\kappa| + \mu + \frac{d+1}{2})}.$$

Another interesting basis, the monomial basis, can be derived from the monomial h-harmonics in Section 3.2.3. This is a basis for which $P_\alpha(x) = c_\alpha x^\alpha + \ldots$, corresponding to $\alpha_{d+1} = 0$ of the basis in Definition 3.1. We denote this basis by P_α^B, they are defined by the generating function

$$\int_{[-1,1]^d} \frac{1}{(1 - 2(b_1x_1t_1 + \ldots + b_dx_dt_d) + \|b\|^2)^\lambda} \prod_{i=1}^{d} c_{\kappa_i}(1+t_i)(1-t_i^2)^{\kappa_i - 1}dt$$
$$= \sum_{\alpha \in \mathbb{N}_0^d} b^\alpha P_\alpha^B(x),$$

where $\lambda = |\kappa| + \mu + \frac{d-1}{2}$. This corresponds to the case of $b_{d+1} = 0$ in Definition 3.1. The explicit formulae of these polynomials follow from Theorem 3.9,

$$P_\alpha^B(x) = \frac{2^{|\alpha|}(|\kappa| + \mu + \frac{d-1}{2})_\alpha}{\alpha!} \frac{(\frac{1}{2})_{[\frac{\alpha+1}{2}]}}{(\kappa + \frac{1}{2})_{[\frac{\alpha+1}{2}]}} x^\alpha \times$$
$$F_B\Big(-\alpha + \Big[\frac{\alpha+1}{2}\Big], -\Big[\frac{\alpha+1}{2}\Big] - \kappa + \frac{1}{2}; -|\alpha| - |\kappa| - \mu - \frac{d-3}{2}; \frac{1}{x_1^2}, \ldots, \frac{1}{x_{d+1}^2}\Big).$$

Clearly the highest degree of $P_\alpha^B(x)$ is a multiple of x^α, and these are monomial polynomials. In the case of the classical weight function W_μ^B, $\kappa = 0$, and using the fact that

$$\Big(-\alpha+\Big[\frac{\alpha+1}{2}\Big]\Big)_\gamma\Big(-\Big[\frac{\alpha+1}{2}\Big]+\frac{1}{2}\Big)_\gamma = \Big(-\frac{\alpha}{2}\Big)_\gamma\Big(-\frac{\alpha}{2}+\frac{1}{2}\Big)_\gamma,$$

the formula of P_α^B can be rewritten as

$$\begin{aligned} V_\alpha(x) \;=\; P_\alpha^B(x) \;&=\; \frac{2^{|\alpha|}(\mu+\frac{d-1}{2})_{|\alpha|}}{\alpha!}\,x^\alpha\times \\ &\quad \times F_B\Big(-\frac{\alpha}{2}, -\frac{\alpha+1}{2}; -|\alpha|-\mu-\frac{d-3}{2}; \frac{1}{x_1^2},\ldots,\frac{1}{x_{d+1}^2}\Big), \end{aligned}$$

which are Appell's monomial orthogonal polynomials. In this case, there is another basis defined by

$$U_\alpha(x) = \frac{(-1)^{|\alpha|}(2\mu)_{|\alpha|}}{2^{|\alpha|}(\mu+\frac{1}{2})_{|\alpha|}\alpha!}(1-\|x\|^2)^{-\mu+\frac{1}{2}}\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\ldots\partial x_d^{\alpha_d}}(1-\|x\|^2)^{|\alpha|+\mu-\frac{1}{2}},$$

which is biorthogonal to the monomial basis in the following sense:

Theorem 4.4. *The polynomials $\{U_\alpha\}$ and $\{V_\alpha\}$ are biorthogonal,*

$$w_\mu^B\int_{B^d} V_\alpha(x)U_\beta(x)W_\mu^B(x)dx = \frac{\mu+\frac{d-1}{2}}{|\alpha|+\mu+\frac{d-1}{2}}\cdot\frac{(2\mu)_{|\alpha|}}{\alpha!}\delta_{\alpha,\beta}.$$

Proof. Since $\{V_\alpha\}$ form an orthogonal basis, we only need to consider the case $|\beta|\ge|\alpha|$. Using the Rodrigues formula and integration by parts yields

$$\begin{aligned} &w_\mu^B\int_{B^d} V_\alpha(x)U_\beta(x)W_\mu^B(x)dx \\ &= w_\mu^B\frac{(2\mu)_{|\alpha|}}{2^{|\alpha|}(\mu+\frac{1}{2})_{|\alpha|}\alpha!}\int_{B^d}\Big[\frac{\partial^{|\beta|}}{\partial x_1^{\beta_1}\ldots\partial x_d^{\beta_d}}V_\alpha(x)\Big](1-\|x\|^2)^{|\alpha|+\mu-\frac{1}{2}}dx. \end{aligned}$$

However, since V_α is a constant multiple of a monomial orthogonal polynomial, $\frac{\partial^{|\beta|}}{\partial x^\beta}V_\alpha(x) = 0$ for $|\beta|>|\alpha|$, which proves the orthogonality. A simple computation gives the constant for the case $|\beta|=|\alpha|$. ■

Among other explicit formulae that we get, the compact formula for the reproducing kernel is of particular interest. Let us denote the reproducing kernel of $\mathcal{V}_n^d(W)$ by $\mathbf{P}_n(W;x,y)$ as defined in Section 1.5.

Theorem 4.5. *For $x, y \in B^d$, the reproducing kernel can be written as an integral*

$$\mathbf{P}_n(W^B_{\kappa,\mu}; x, y) = \frac{n + |\kappa| + \mu + \frac{d-1}{2}}{|\kappa| + \mu + \frac{d-1}{2}} \times$$
$$\int_{-1}^1 \int_{[-1,1]^d} C_n^{|\kappa|+\mu+\frac{d-1}{2}} \big(t_1 x_1 y_1 + \cdots + t_d x_d y_d + s\sqrt{1-\|x\|^2}\sqrt{1-\|y\|^2}\,\big)$$
$$\times \prod_{i=1}^d c_{\kappa_i}(1+t_i)(1-t_i^2)^{\kappa_i - 1} dt\; c_\mu\ (1-s^2)^{\mu-1} ds.$$

Proof. Let $h_\kappa(y) = \prod_{i=1}^{d+1} |y_i|^{\kappa_i}$ with $\kappa_{d+1} = \mu$. Then the correspondence in Section 3.1 can be used to show that

$$\mathbf{P}_n(W^B_{\kappa,\mu}; x, y) = \frac{1}{2}\Big[P_n(h_\kappa^2; x, (y, \sqrt{1-|y|^2})) + P_n(h_\kappa^2; x, (y, -\sqrt{1-|y|^2}))\Big],$$

from which the stated formula follows from Theorem 3.6. ∎

In particular, taking the limit $\kappa_i \to 0$ for $i = 1, \ldots, d$, we conclude that for the classical weight function W^B_μ,

$$\mathbf{P}_n(W^B_\mu; x, y) = c_\mu \frac{n + \mu + \frac{d-1}{2}}{\mu + \frac{d-1}{2}} \tag{4.1}$$
$$\times \int_{-1}^1 C_n^{\mu+\frac{d-1}{2}} \big(\langle x, y\rangle + t\sqrt{1-\|x\|^2}\sqrt{1-\|y\|^2}\big)(1-t^2)^{\mu-1} dt.$$

Even in this case the formula has been discovered only recently. For $d = 1$, this reduces to the classical product formula of the Gegenbauer polynomials:

$$\frac{C_n^\mu(x) C_n^\mu(y)}{C_n^\mu(1)} = c_\mu \int_{-1}^1 C_n^\mu(xy + t\sqrt{1-x^2}\sqrt{1-y^2})(1-t^2)^{\mu-1} dt.$$

There is also an analogue of the Funk-Hecke formula for orthogonal polynomials on B^d. The most interesting case is the formula for the classical weight function:

Theorem 4.6. *Let f be a continuous function on $[-1, 1]$. Let $P \in \mathcal{V}_n^d(W^B_\mu)$. Then*

$$\int_{B^d} f(\langle x, y\rangle) P(y) W^B_\mu(y) dy = \lambda_n(f) P(x), \qquad \|x\| = 1,$$

where $\lambda_n(f)$ is the same as in Theorem 3.8 with $|\kappa|$ replaced by μ.

As a consequence, it follows that the polynomial $C_n^{(\mu+(d-1)/2)}(\langle x, \eta\rangle)$ with η satisfying $\|\eta\| = 1$ is an element of $\mathcal{V}_n^d(W^B_\mu)$. Furthermore, if ξ also satisfies $\|\xi\| = 1$, then

$$\int_{B^d} C_n^{\mu+\frac{d-1}{2}}(\langle x, \xi\rangle) C_n^{\mu+\frac{d-1}{2}}(\langle x, \eta\rangle) W^B_\mu(x) dx = \lambda_n C_n^{\mu+\frac{d-1}{2}}(\langle \eta, \xi\rangle),$$

where $\lambda_n = (\mu + (d-1)/2)/(n + \mu + (d-1)/2)$. The basis in Subsection 1.2.3 is derived from this integral.

Several results given above hold for the weight functions $h_\kappa^2(x)(1-\|x\|^2)^{\mu-1/2}$, where h_κ is one of the weight functions in Section 3.3 that are invariant under reflection groups. Most notably is Theorem 4.2, which holds with $|\kappa|$ replaced by $\gamma_\kappa = \sum_{v \in R_+} \kappa_v$. Analogous of Theorems 4.4 and 4.5, in which the intertwining operator is used, also hold, but the formulae are not really explicit since neither an explicit basis for $\mathcal{H}_n^d(h_\kappa^2)$ nor the formula for V_κ are known for h_κ^2 associated with the general reflection groups.

4.3. **Rotation invariant weight function.** If $\rho(t)$ is a nonnegative even function on $\mathbb{R}$ with finite moments, then the weight function $W(x) = \rho(\|x\|)$ is a rotation invariant weight function on $\mathbb{R}^d$. Such a function is call a radial function. The classical weight function $W_\mu^B(x)$ corresponds to $\rho(t) = (1-t^2)^{\mu-1/2}$ for $|t| < 1$ and $\rho(t) = 0$ for $|t| > 1$. The orthonormal basis in Theorem 4.3 can be extended to such a weight function.

Theorem 4.7. *For $0 \le j \le n/2$ let $\{S_{n-2j,\beta}\}$ denote an orthonormal basis for $\mathcal{H}_{n-2j}^d$ of ordinary spherical harmonics. Let $p_{2n}^{(2n-4j+d-1)}$ denote the orthonormal polynomials with respect to the weight function $|t|^{2n-4j+d-1}\rho(t)$. Then the polynomials*

$$P_{\beta,j}(x) = p_{2j}^{(2n-4j+d-1)}(\|x\|) S_{\beta,n-2j}(x)$$

form an orthonormal basis of $\mathcal{V}_n^d(W)$ with $W(x) = \rho(\|x\|)$.

Since $|t|^{2n-4j+d-1}\rho(t)$ is an even function, the polynomials $p_{2j}^{(2n-4j+d-1)}(t)$ are even with respect to W. Hence, $P_{\beta,j}(x)$ are indeed polynomials of degree n in x. The proof is an simple verification upon writing the integral in polar coordinates.

As one application, we consider the partial sums of the Fourier orthogonal expansion $S_n(W; f)$ defined in Section 1.5. Let $s_n(w, g)$ denote the n-th partial sum of the Fourier orthogonal expansion with respect to the weight function $w(t)$ on $\mathbb{R}$. The following theorem states that the partial sum of a radial function with respect to a radial weight function is also a radial function.

Theorem 4.8. *Let $W(x) = \rho(\|x\|)$ be as above and $w(t) = |t|^{d-1}\rho(t)$. If $g : \mathbb{R} \mapsto \mathbb{R}$ and $f(x) = g(\|x\|)$, then*

$$S_n(W; f, x) = s_n(w; f, \|x\|), \qquad x \in \mathbb{R}^d.$$

Proof. The orthonormal basis in the previous theorem gives a formula for the reproducing kernel,

$$\mathbf{P}_n(x,y) = \sum_{0\le 2j\le n} p_{2j}^{(2n-4j+d-1)}(\|x\|)p_{2j}^{(2n-4j+d-1)}(\|y\|) \\ \times \frac{n+\frac{d-1}{2}}{\frac{d-1}{2}}\|x\|^{n-2j}\|y\|^{n-2j}C_{n-2j}^{\frac{d-1}{2}}(\langle x,y\rangle),$$

where we have used the fact that $S_{n-2j,\beta}$ are homogeneous and $\sum_\beta S_{m,\beta}(x)S_{m,\beta}(y)$ for $x, y \in S^{d-1}$ is a zonal polynomial. Using polar coordinates,

$$S_n(W; f, x) = \int_0^\infty g(r)\int_{S^{d-1}} \mathbf{P}_n(x, ry')d\omega(y')r^{d-1}\rho(r)dr.$$

Since $C_n^{\frac{d-1}{2}}(\langle x, y\rangle)$ is a zonal harmonic, its integral over S^{d-1} is zero for $n > 0$ and is equal to the surface measure σ_{d-1} of S^{d-1} if $n = 0$. Hence, we get

$$S_n(W; f, x) = \sigma_{d-1}\int_0^\infty g(r)p_n^{(d-1)}(r)w(r)dr\, p_n^{(d-1)}(\|x\|).$$

Since orthonormal bases are assumed to be with respect to the normalized weight function, setting $g = 1$ shows that there is no constant in front of $s_n(w; g)$. ■

5. Orthogonal polynomials on the simplex

Orthogonal polynomials on the simplex $T^d = \{x \in \mathbb{R}^d : x_1 \ge 0, \dots, x_d \ge 0, 1 - |x| > 0\}$ are closely related to those on the unit ball. The relation depends on the basic formula

$$\int_{B^d} f(y_1^2, \dots, y_d^2)dy = \int_{T^d} f(x_1, \dots, x_d)\frac{dx}{\sqrt{x_1 \dots x_d}}.$$

Let $W^B(x) = W(x_1^2, \dots, x_d^2)$ be a weight function defined on B^d. Associated with W^B define a weight function W^T on T^d by

$$W^T(y) = W(y_1, \dots, y_d)\big/\sqrt{y_1 \cdots y_d}, \qquad y = (y_1, \dots, y_d) \in T^d.$$

Let $\mathcal{V}_{2n}^d(W^B; \mathbb{Z}_2^d)$ denote the subspace of orthogonal polynomials in $\mathcal{V}_{2n}^d(W^B)$ that are even in each of its variables (that is, invariant under $\mathbb{Z}_2^d$). If $P_\alpha \in \mathcal{V}_{2n}^d(W^B; \mathbb{Z}_2^d)$, then there is a polynomial $R_\alpha \in \Pi_n^d$ such that $P_\alpha(x) = R_\alpha(x_1^2, \dots, x_d^2)$. The polynomial R_α is in fact an element of $\mathcal{V}_n^d(W^T)$.

Theorem 5.1. *The relation $P_\alpha(x) = R_\alpha(x_1^2, \dots, x_d^2)$ defines a one-to-one correspondence between an orthonormal basis of $\mathcal{V}_{2n}(W^B, \mathbb{Z}_2^d)$ and an orthonormal basis of $\mathcal{V}_n^d(W^T)$.*

Proof. Assume that $\{R_\alpha\}_{|\alpha|=n}$ is an orthonormal polynomial in $\mathcal{V}_n^d(W^T)$. If $\beta \in \mathbb{N}_0^d$ has one odd component, then the integral of $P_\alpha(x)x^\beta$ with respect to W^B over B^d is zero. If all components of β are even and $|\beta| < 2n$, then it can be written as $\beta = 2\tau$ with $\tau \in \mathbb{N}_0^d$ and $|\tau| \le n-1$. The basic integral can be used to convert the integral of $P_\alpha(x)x^{2\tau}$ over B^d to the integral over T^d, so that the orthogonality of R_α implies that P_α is orthogonal to x^β. ■

The classical weight function W_κ^T on the simplex T^d is defined by

$$W_\kappa^T(x) = \prod_{i=1}^d |x_i|^{\kappa_i - 1/2}(1-|x|)^{\kappa_{d+1}-1/2}, \qquad x \in T^d, \qquad \kappa_i > -1.$$

Sometimes we write W_κ^T as $W_{\kappa,\mu}^T$ with $\mu = \kappa_{d+1}$, as the orthogonal polynomials with respect to $W_{\kappa,\mu}^T$ on T^d are related to the orthogonal polynomials with with respect to $W_{\kappa,\mu}^B$ on B^d. Below we give explicit formulas for these classical orthogonal polynomials on the simplex.

Theorem 5.2. *The orthogonal polynomials in $\mathcal{V}_n^d(W_\kappa^T)$ satisfy the partial differential equation*

$$\sum_{i=1}^d x_i(1-x_i)\frac{\partial^2 P}{\partial x_i^2} - 2\sum_{1\le i<j\le d} x_i x_j \frac{\partial^2 P}{\partial x_i \partial x_j}$$
$$+ \sum_{i=1}^d \left(\Big(\kappa_i + \frac{1}{2}\Big) - \Big(|\kappa| + \frac{d-1}{2}\Big)x_i \right)\frac{\partial P}{\partial x_i} = -n\Big(n + |\kappa| + \frac{d-1}{2}\Big)P.$$

Proof. For functions that are even in each of its variables, the h-Laplacian Δ_h for the product weight function becomes a differential operator (see the formula in Section 3.2.2). Consequently, for the orthogonal polynomials $P_\alpha \in \mathcal{V}_{2n}^d(W_{\kappa,\mu}^B; \mathbb{Z}_2^d)$, the differential-difference equation in Theorem 4.1 become a differential equation. Changing variables $x_i \to \sqrt{z_i}$ gives

$$\frac{\partial}{\partial x_i} = 2\sqrt{z_i}\frac{\partial}{\partial z_i} \qquad \text{and} \qquad \frac{\partial^2}{\partial x_i^2} = 2\Big[\frac{\partial}{\partial z_i} + 2z_i\frac{\partial^2}{\partial z_i^2}\Big],$$

from which the equation for $P_\alpha(x) = R_\alpha(x_1^2, \ldots, x_d^2)$ translates into an equation satisfied by $R_\alpha \in \mathcal{V}_n^d(W_{\kappa,\mu}^T)$. ■

The theorem and its proof can be extended to the case of the weight function

$$\prod_{i=1}^d x_i^{\kappa_0 - 1/2} \prod_{1\le i<j\le d} |x_i - x_j|^{\kappa_1}(1-|x|)^{\mu - 1/2},$$

and the differential equation becomes a differential-difference equation.

Next we give explicit formulae for orthogonal bases. Let $\{P_\alpha\}$ be the orthonormal basis with respect to $W_{\kappa,\mu}^B$ given in Theorem 4.2; then it is easy to check that

$\{P_{2\alpha} : \alpha \in \mathbb{N}_0^d,\ |\alpha| = n\}$ forms an orthonormal basis for $\mathcal{V}_{2n}^d(W_{\kappa,\mu}^B; \mathbb{Z}_2^d)$. Hence, using the fact that $C_{2n}^{(\lambda,\mu)}$ is given in terms of Jacobi polynomial, we get

Theorem 5.3. *With respect to W_κ^T, the polynomials*

$$P_\alpha(x) = [h_\alpha^T]^{-1} \prod_{j=1}^d \Big(\frac{1-|\mathbf{x}_j|}{1-|\mathbf{x}_{j-1}|}\Big)^{|\alpha^{j+1}|} p_{\alpha_j}^{(a_j,b_j)}\Big(\frac{2x_j}{1-|\mathbf{x}_{j-1}|} - 1\Big),$$

where $a_j = 2|\alpha^{j+1}| + |\kappa^{j+1}| + \frac{d-j-1}{2}$ and $b_j = \kappa_j - \frac{1}{2}$, are orthonormal and the normalization constants h_α^T are given by

$$[h_\alpha^T]^{-2} = \frac{(|\kappa| + \frac{d+1}{2})_{2|\alpha|}}{\prod_{j=1}^d (2|\alpha^{j+1}| + |\kappa^j| + \frac{d-j+2}{2})_{2\alpha_j}}.$$

On the simplex T^d, it is often convenient to define $x_{d+1} = 1 - |x|$ and work with the homogeneous coordinates $(x_1, \ldots, x_{d+1})$. One can also derive a basis of orthogonal polynomials that are homogeneous in the homogeneous coordinates. In fact, the relation between orthogonal polynomials on B^d and S^d allows us to work with h-harmonics. Let us denote by $\mathcal{H}_{2n}^{d+1}(h_\kappa^2, \mathbb{Z}_2^{d+1})$ the subspace of h-harmonics in $\mathcal{H}_{2n}^{d+1}(h_\kappa^2)$ that are even in each of its variables. Let $\{S_\alpha^n(x_1^2, \ldots, x_{d+1}^2) : |\alpha| = n,\ \alpha \in \mathbb{N}_0^d\}$ be an orthonormal basis of $\mathcal{H}_{2n}^d(h_\kappa^2, \mathbb{Z}_2^{d+1})$. Then $\{S_\alpha^n(x_1, \ldots, x_{d+1}) : |\alpha| = n,\ \alpha \in \mathbb{N}_0^d\}$ forms an orthonormal homogeneous basis of $\mathcal{V}_n^d(W_\kappa^T)$.

For $x \in T^d$ let $X = (x_1, \ldots, x_d, x_{d+1})$ denote the homogeneous coordinates. Let Y_α be the monomial basis of h-harmonics in Section 3.2.3. Then $Y_{2\alpha}$ are even in each of their variables, which gives monomial orthogonal polynomials in $\mathcal{V}_n^d(W_\kappa^T)$ in the homogeneous coordinates $X := (x_1, \ldots, x_{d+1})$ with $x_{d+1} = 1 - |x|$,

$$P_\alpha(x) = X^\alpha F_B\Big(-\alpha, -\alpha-\kappa+\frac{1}{2}; -2|\alpha| - |\kappa| - \frac{d-3}{2}; \frac{1}{x_1}, \ldots, \frac{1}{x_{d+1}}\Big).$$

Furthermore, changing summation index shows that the above Lauricella function of type B can be written as a constant multiple of the Lauricella function of type A defined by

$$F_A(c, \alpha; \beta; x) = \sum_\gamma \frac{(c)_{|\gamma|}(\alpha)_\gamma}{(\beta)_\gamma \gamma!} x^\gamma, \qquad \alpha, \beta \in \mathbb{N}_0^{d+1}, \quad c \in \mathbb{R},$$

where the summation is taken over $\gamma \in \mathbb{N}_0^{d+1}$. This gives the following:

Theorem 5.4. *For each $\alpha \in \mathbb{N}_0^{d+1}$ with $|\alpha| = n$, the polynomials*

$$R_\alpha(x) = F_A(|\alpha| + |\kappa| + d, -\alpha; \kappa + \mathbf{1}; X), \qquad x \in T^d$$

are orthogonal polynomials in $\mathcal{V}_n^d(W_\kappa^T)$ and

$$R_\alpha = (-1)^n \frac{(n + |\kappa| + d)_n}{(\kappa + \mathbf{1})_\alpha} X^\alpha + p_\alpha^\kappa, \qquad p_\alpha^\kappa \in \Pi_{n-1}^d,$$

where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^{d+1}$.

The polynomial R_α is a polynomial whose leading coefficient is a constant multiple of X^α for $\alpha \in \mathbb{N}_0^{d+1}$. The set $\{R_\alpha : \alpha \in \mathbb{N}_0^{d+1}, |\alpha| = n\}$ clearly has more elements than it is necessary for a basis of $\mathcal{V}_n^d(W_\kappa^T)$; its subset with $\alpha_{d+1} = 0$ forms a basis of $\mathcal{V}_n^d(W_\kappa^T)$. Let us write $V_\beta^T(x) = R_{(\beta,0)}(x)$. Then $\{V_\beta^T : \beta \in \mathbb{N}_0^d, |\beta| = n\}$ is the monomial basis of $\mathcal{V}_n^d(W_\kappa^T)$. The notation V_β goes back to [2], we write a superscript T to distinguish it from the notation for the intertwining operator. Associated with V_β^T is its biorthogonal basis, usually denoted by U_β.

Theorem 5.5. *For $\alpha \in \mathbb{N}^d$, $|\alpha| = n$, the polynomials U_α defined by*

$$\begin{aligned} U_\alpha(x) &= x_1^{-\kappa_1+1/2} \dots x_d^{-\kappa_d+1/2}(1-|x|)^{-\kappa_{d+1}+1/2} \\ &\times \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}} x_1^{\alpha_1+\kappa_1-1/2} \dots x_d^{\alpha_d+\kappa_d-1/2}(1-|x|)^{|\alpha|+\kappa_{d+1}-1/2}. \end{aligned}$$

are polynomials in $\mathcal{V}_n^d(W_\kappa^T)$ and they are biorthogonal to polynomials V_α,

$$\int_{T^d} V_\beta(x) U_\alpha(x) W_\kappa^T(x) dx = \frac{(\kappa+1/2)_\alpha (\kappa_{d+1}+1/2)_{|\alpha|}}{(|\kappa| + (d+1)/2)_{2|\alpha|}} \alpha! \, \delta_{\alpha,\beta}.$$

Proof. It follows from the definition that U_α is a polynomial of degree n. Integrating by parts leads to

$$\begin{aligned} &w_\kappa^T \int_{T^d} V_\beta(x) U_\alpha(x) W_\kappa^T(x) dx \\ &= w_\kappa^T \int_{T^d} \Big[\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}} V_\beta(x)\Big] \prod_{i=1}^d x_i^{\alpha_i+\kappa_i-\frac12}(1-|x|)^{|\alpha|+\kappa_{d+1}-\frac12} dx. \end{aligned}$$

Since V_α is an orthogonal polynomial with respect to W_κ^T, the left integral is zero for $|\beta| > |\alpha|$. For $|\beta| \le |\alpha|$ the fact that V_β is a monomial orthogonal polynomial gives

$$\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}} V_\beta(x) = \alpha! \, \delta_{\alpha,\beta},$$

from which the stated formula follows. The fact that U_α are biorthogonal to V_β also shows that they are orthogonal polynomials with respect to W_κ^T. ∎

The correspondence between orthogonal polynomials in Theorem 5.1 also gives an explicit formula for the reproducing kernel associated with W_κ^T.

Theorem 5.6. *Let $\lambda = |\kappa| + (d-1)/2$. Then*

$$\mathbf{P}_n(W^T_{\kappa,\mu}, x, y) = \frac{2n+\lambda}{\lambda} \int_{[-1,1]^{d+1}} C^\lambda_{2n}\Big(\sqrt{x_1 y_1}\, t_1 + \cdots + \sqrt{x_{d+1} y_{d+1}} t_{d+1}\Big) \times \prod_{i=1}^{d+1} c_{\kappa_i}(1+t_i)(1-t_i^2)^{\kappa_i - 1} dt,$$

where $x_{d+1} = 1 - |x|$ and $y_{d+1} = 1 - |y|$.

Proof. Recall that we write $W^T_{\kappa,\mu}$ for W^T_κ with $\kappa_{d+1} = \mu$. Using the correspondence in Theorem 5.1, it can be verified that

$$\mathbf{P}_n(W^T_{\kappa,\mu}; x, y) = \frac{1}{2^d} \sum_{\varepsilon \in \mathbb{Z}_2^d} \mathbf{P}_{2n}\big(W^B_{\kappa,\mu}; (\varepsilon_1 \sqrt{x_1}, \ldots, \varepsilon_d \sqrt{x_d}\,), (\sqrt{y_1}, \ldots, \sqrt{y_d})\big).$$

Hence the stated formula follows from Theorem 4.5. ∎

6. Classical type product orthogonal polynomials

As mentioned in the introduction, if $W(x)$ is a product weight function

$$W(x) = w_1(x_1) \ldots w_d(x_d), \qquad x \in \mathbb{R}^d,$$

then an orthonormal basis with respect to W is given by the product orthogonal polynomials $P_\alpha(x) = p_{\alpha_1,1}(x_1) \ldots p_{\alpha_d,d}(x_d)$, where $p_{m,i}$ is the orthogonal polynomial of degree m with respect to the weight function w_i. In this section we discuss the product classical polynomials and some of their extensions.

6.1. **Multiple Jacobi polynomials.** Recall that the product Jacobi weight functions is denoted by $W_{a,b}$ in Section 1.3. One orthonormal basis is given by

$$P_\alpha(x) = p^{(a_1,b_1)}_{\alpha_1}(x_1) \ldots p^{(a_d,b_d)}_{\alpha_d}(x_d),$$

where $p^{(a,b)}_m$ is the m-th orthonormal Jacobi polynomial. Although this basis of multiple Jacobi polynomials are simple, there is no close formula for the reproducing kernel $\mathbf{P}_n(W_{a,b}; x, y)$ in general. There is, however, a generating function for $\mathbf{P}_n(W_{a,b}; x, \mathbf{1})$, where $\mathbf{1} = (1, 1, \ldots, 1) \in [-1,1]^d$. It is based on the generating function (Poisson formula) of the Jacobi polynomials,

$$\begin{aligned} G^{(a,b)}(r; x) &:= \sum_{k=0}^\infty p^{(a,b)}_k(1) p^{(a,b)}_k(x) r^n \\ &= \frac{1-r}{(1+r)^{a+b+2}}\, {}_2F_1\left(\begin{matrix} \frac{a+b+2}{2}, \frac{a+b+3}{2} \\ b+1 \end{matrix}; \frac{2r(1+x)}{(1+r)^2}\right), \qquad 0 \le r < 1, \end{aligned}$$

which gives a generating function for the multiple Jacobi polynomials that can be written in term of the reproducing kernel $\mathbf{P}_n(W_{a,b})$ as

$$\sum_{n=0}^{\infty} P_\alpha(W_{a,b}; x, y) r^n = \prod_{i=1}^{d} G^{(a_i,b_i)}(r; x_i) := G_d^{(a,b)}(r; x).$$

Moreover, in some special cases, we can derive an explicit formula for $\mathbf{P}_n(W; x, \mathbf{1})$.

Let us consider the reproducing kernel for the case that $a = b$ and a_i are half-integers. This corresponds to the multiple Gegenbauer polynomials with respect to the weight function

$$W_\lambda(x) = \prod_{i=1}^{d} (1 - x_i)^{\lambda_i - 1/2}, \qquad \lambda_i > -1/2, \qquad x \in [-1, 1]^d,$$

Recall that the generating function of the Gegenbauer polynomials is given by

$$\frac{1 - r^2}{(1 - 2rx + r^2)^{\lambda+1}} = \sum_{n=0}^{\infty} \frac{\lambda + n}{\lambda} C_n^\lambda(x) r^n.$$

Since $\frac{\lambda+n}{\lambda} C_n^\lambda(x) = \widetilde{C}_n^\lambda(1) \widetilde{C}_n^\lambda(x)$, it follows that $\mathbf{P}_n(W; x, \mathbf{1})$ satisfies a generating function relation

$$\frac{(1 - r^2)^d}{\prod_{i=1}^{d} (1 - 2rx_i + r^2)^{\lambda_i + 1}} = \sum_{n=0}^{\infty} \mathbf{P}_n(W_\lambda; x, \mathbf{1}) r^n.$$

If $\lambda \in \mathbb{N}_0$, then the reproducing kernel can be given in terms of the divided difference, defined inductively by

$$[x_0]f = f(x_0), \quad [x_0, \ldots x_m]f = \frac{[x_1, \ldots x_m]f - [x_0, \ldots x_{m-1}]f}{x_m - x_0}.$$

The divided difference $[x_1, \ldots, x_d]f$ is a symmetric function of $x_1, \ldots, x_d$.

Theorem 6.1. *Let $\lambda_i \in \mathbb{N}_0^d$. Then*

$$\mathbf{P}_n(W_\lambda; x, \mathbf{1}) = [\overbrace{x_1, \ldots, x_1}^{\lambda_1+1}, \ldots, \overbrace{x_d, \ldots, x_d}^{\lambda_d+1}] G_n,$$

with

$$G_n(t) = (-1)^{[\frac{d+1}{2}]} 2 (1 - t^2)^{\frac{d-1}{2}} \begin{cases} T_n(t) & \textit{for } d \textit{ even}, \\ U_{n-1}(t) & \textit{for } d \textit{ odd}. \end{cases}$$

Proof. In the case of $\lambda_i = 0$, the left hand side of the generating function for $\mathbf{P}_n(W_0; x, \mathbf{1})$ can be expanded as a power series using the formula

$$[x_1, \ldots, x_d] \frac{1}{a - b(\cdot)} = \frac{b^{d-1}}{\prod_{i=1}^{d} (a - bx_i)},$$

which can be proved by induction on the number of variables; the result is

$$\frac{(1-r^2)^d}{\prod_{i=1}^d(1-2rx_i+r^2)} = \frac{(1-r^2)^d}{(2r)^{d-1}}[x_1,\dots,x_d]\frac{1}{1-2r(\cdot)+r^2}$$
$$= \frac{(1-r^2)^d}{(2r)^{d-1}}[x_1,\dots,x_d]\sum_{n=d-1}^{\infty}U_n(\cdot)r^n$$

using the generating function of the Chebyshev polynomials of the second kind. Using the binomial theorem to expand $(1-r^2)^d$ and the fact that $U_{m+1}(t) = \sin m\theta/\sin\theta$ and $\sin m\theta = (e^{im\theta}-e^{-im\theta})/(2i)$ with $t=\cos\theta$, the last term can be shown to be equal to $\sum_{r=0}^{\infty}[x_1,\dots,x_d]G_n r^n$.

For $\lambda_i > 1$, we use the fact that $(d^k/dx^k)C_n^\lambda(x) = 2^k(\lambda)_k C_{n-k}^{\lambda+k}(x)$ and $\lambda+n = (\lambda+k)+(n-k)$, which implies

$$\frac{d^k}{dx^k}\mathbf{P}_n(W_\lambda;x,\mathbf{1}) = 2^k(\lambda+1)_k\mathbf{P}_{n-k}(W_{\lambda+k};x,\mathbf{1}),$$

so that the formula

$$\frac{d}{dx_1}[x_1,x_2,\dots,x_d]g = [x_1,x_1,x_2,x_3,\dots,x_d]g$$

and the fact that the divided difference is a symmetric function of its knots can be used to finish the proof. ■

For multiple Jacobi polynomials, there is a relation between $P_n(W_{a,b};x,y)$ and $P_n(W_{a,b};x,\mathbf{1})$. This follows from the product formula of the Jacobi polynomials,

$$\frac{P_n^{(\alpha,\beta)}(x_1)P_n^{(\alpha,\beta)}(x_2)}{P_n^{(\alpha,\beta)}(1)} = \int_0^\pi\int_0^1 P_n^{(\alpha,\beta)}(2A^2(x_1,x_2,r,\phi)-1)dm_{\alpha,\beta}(r,\phi),$$

where $\alpha > \beta > -1/2$,

$$A(x_1,x_2,r,\phi) = \frac{1}{2}\Big((1+x_1)(1+x_2)+(1-x_1)(1-x_2)r^2+2\sqrt{1-x_1^2}\sqrt{1-x_2^2}\,r\cos\phi\Big)^{1/2}$$

and

$$dm_{\alpha,\beta}(r,\phi) = c_{\alpha,\beta}(1-r^2)^{\alpha-\beta-1}r^{2\beta+1}(\sin\phi)^{2\beta}drd\phi,$$

in which $c_{\alpha,\beta}$ is a constant so that the integral of $dm_{\alpha,\beta}$ over $[0,1]\times[0,\pi]$ is 1. Sometimes the precise meaning of the formula is not essential, and the following theorem of [12] is useful.

Theorem 6.2. *Let $a,b > -1$. There is an integral representation of the form*

$$p_n^{(a,b)}(x)p_n^{(a,b)}(y) = p_n^{(a,b)}(1)\int_{-1}^1 p_n^{(a,b)}(t)d\mu_{x,y}^{(a,b)}(t), \qquad n\ge 0,$$

with the real Borel measures $d\mu_{x,y}^{(a,b)}$ *on* $[-1,1]$ *satisfying*

$$\int_{-1}^{1} |d\mu_{x,y}^{(a,b)}(t)|dt \le M, \qquad -1 < x, y < 1,$$

for some constant M *independent of* x, y*, if and only if* $a \ge b$ *and* $a + b \ge -1$*. Moreover, the measures are nonnegative, i.e.,* $d\mu_{x,y}^{(a,b)}(t) \ge 0$*, if and only if* $b \ge -1/2$ *or* $a + b \ge 0$*.*

The formula can be extended to the multiple Jacobi polynomials in an obvious way, which gives a relation between $\mathbf{P}_n(W_{a,b}; x, y)$ and $\mathbf{P}_n(W_{a,b}; x, \mathbf{1})$. Hence the previous theorem can be used to give a formula of $\mathbf{P}_n(W_\lambda, x, y)$. In the simplest case of $\lambda_i = 0$, $1 \le i \le d$, we have [5]

$$\mathbf{P}_n(W_0; x, y) = \sum_{\tau \in \mathbb{Z}_2^d} \mathbf{P}_n(W_0; \cos(\Phi + \tau\Theta), \mathbf{1}),$$

where $x = \cos\Theta = (\cos\theta_1, \ldots, \cos\theta_d)$ and $y = \cos\Phi = (\cos\phi_1, \ldots, \cos\phi_d)$. The vector $\Phi + \tau\Theta$ has components $\phi_i + \tau_i\theta_i$.

6.2. **Multiple Laguerre polynomials.** The multiple Laguerre polynomials are orthogonal with respect to $W_\kappa^L(x) = x^\kappa e^{-|x|}$ with $x \in \mathbb{R}_+^d$. One orthonormal basis is given by the multiple Laguerre polynomials $P_\alpha(x) = L_{\alpha_1}^{\kappa_1} \ldots L_{\alpha_d}^{\kappa_d}$. Let us denote this basis by $P_\alpha(W_\kappa^L; x)$ to emphasis the weight function. Recall the classical relation

$$\lim_{b\to\infty} P_n^{(a,b)}(1 - 2x/b) = L_n^b(x);$$

there is an extension of this relation to several variables. Let us denote the orthogonal basis with respect to the weight function $W_{\kappa,\mu}^T$ in Theorem 5.3 by $P_\alpha^n(W_{\kappa,\mu}^T; x)$ (set $\kappa_{d+1} = \mu$).

Theorem 6.3. *The multiple Laguerre polynomials associated to* W_κ^L *are the limit of the product type polynomials in Theorem 5.3,*

$$\lim_{\mu\to\infty} P_\alpha^n(W_{\kappa+1/2,\mu}^T; x/\mu) = \widetilde{L}_{\alpha_1}^{\kappa_1}(x_1) \ldots \widetilde{L}_{\alpha_d}^{\kappa_d}(x_d),$$

where $\widetilde{L}_\alpha^\kappa$ *denotes the normalized Laguerre polynomials.*

Proof. As $\mu \to \infty$, $1 - |\mathbf{x}_j|/\mu \to 1$ so that the orthogonal polynomials in Theorem 5.3 converge to the multiple Laguerre polynomials, and the normalization constants also carry over under the limit. ■

As a consequence of this limit relation, the differential equation for the classical polynomials on the simplex leads to a differential equation for the multiple Laguerre polynomials.

Theorem 6.4. *The multiple Laguerre polynomials associated to W_κ^L satisfy the partial differential equation*

$$\sum_{i=1}^d x_i \frac{\partial^2 P}{\partial x_i^2} + \sum_{i=1}^d \Big((\kappa_i + 1) - x_i \Big) \frac{\partial P}{\partial x_i} \Big) = -nP.$$

Proof. We make a change of variables $x \mapsto x/\mu$ in the equation satisfied by the orthogonal polynomials with respect to $W_{\kappa,\mu}^T$ and then divide by μ and take the limit $\mu \to \infty$. ■

The limit relation, however, does not give an explicit formula for the reproducing kernel. Just as in the case of multiple Jacobi polynomials, there is no explicit formula for the kernel $\mathbf{P}_n(W_\kappa^L; x, y)$ in general. There is, however, a simple formula if one of the arguments is 0.

Theorem 6.5. *The reproducing kernel for $\mathcal{V}_n^d(W_\kappa^L)$ satisfies*

$$\mathbf{P}_n(W_\kappa^L; x, 0) = L_n^{|\kappa|+d-1}(|x|), \qquad x \in \mathbb{R}_+^d.$$

Proof. The generating function of the Laguerre polynomials is

$$(1-r)^{-a-1} \exp\left(\frac{-xr}{1-r} \right) = \sum_{n=0}^\infty L_n^a(x) r^n, \qquad |r| < 1.$$

Since $L_n^a(0) = (a+1)_n/n! = \|L_n^a\|_2^2$, where the norm is taken with respect to the normalized weight function $x^a e^{-x}$, multiplying the formula gives a generating function for $\mathbf{P}_n(W_\kappa^L; x, 0)$,

$$\sum_{n=0}^\infty \mathbf{P}_n(W_\kappa^L; x, 0) r^n = \sum_{n=0}^\infty \sum_{|\alpha|=n} \tilde{L}_\alpha^\kappa(0) \tilde{L}_\alpha^\kappa(x) r^n = (1-r)^{-|\kappa|-d} e^{-|x|r/(1-r)},$$

which gives the stated formula. ■

There is also a product formula for the multiple Laguerre polynomials which gives a relation between $\mathbf{P}_n(W_\kappa^L; x, y)$ and $\mathbf{P}_n(W_\kappa^L; x, 0)$. The relation follows from the product formula

$$L_n^\lambda(x) L_n^\lambda(y) = \frac{\Gamma(n+\lambda+1) 2^\lambda}{\Gamma(n+1)\sqrt{2\pi}} \int_0^\pi L_n^\lambda(x + y + 2\sqrt{xy}\cos\theta) e^{-\sqrt{xy}\cos\theta} \times j_{\lambda-\frac{1}{2}}(\sqrt{xy}\sin\theta) \sin^{2\lambda}\theta \, d\theta,$$

where j_μ is the Bessel function of fractional order.

More interesting, however, is the fact that the limiting relation in Theorem 6.3 and the differential equations can be extended to hold for orthogonal polynomials with respect to the weight functions

$$\prod_{i=1}^{d} x_i^{\kappa_0} \prod_{1\le i<j\le d} |x_i - x_j|^{\kappa_1} e^{-|x|}, \qquad x \in \mathbb{R}_+^d.$$

6.3. **Multiple generalized Hermite polynomials.** By generalized Hermite polynomials we mean orthogonal polynomials with respect to the weight function $|x|^{\mu} e^{-x^2}$ on $\mathbb{R}$. For $\mu \ge 0$ the generalized Hermite polynomial $H_n^{\mu}(x)$ is defined by

$$H_{2n}^{\mu}(x) = (-1)^n 2^{2n} n! L_n^{\mu-1/2}(x^2),$$
$$H_{2n+1}^{\mu}(x) = (-1)^n 2^{2n+1} n! x L_n^{\mu+1/2}(x^2).$$

The normalization is chosen such that the leading coefficient of H_n^{μ} is 2^n. For several variables, we consider the multiple generalized Hermite polynomials with respect to $W_\kappa^H(x) = \prod_{i=1}^d |x_i|^{\kappa_i} e^{-\|x\|^2}$, $\kappa_i \ge 0$; evidently an orthogonal basis is given by $H_{\alpha_1}^{\kappa_1}(x_1)\dots H_{\alpha_d}^{\kappa_d}(x_d)$, and another basis can be given in polar coordinates in terms of h-spherical harmonics associated with $h_\kappa(x) = \prod_{i=1}^d |x_i|^{\kappa_i}$. One can also define analogous of Appell's biorthogonal bases.

Much of the information about these polynomials can be derived from the orthogonal polynomials on the unit ball, since

$$\lim_{\lambda\to\infty} \lambda^{-n/2} C_n^{\lambda}\Big(\frac{x}{\sqrt{\lambda}}\Big) = \frac{1}{n!} H_n(x).$$

Indeed, denote the orthonormal polynomials with respect to $W_{\kappa,\mu}^B$ on B^d in Theorem 4.2 by $P_\alpha^n(W_{\kappa,\mu}^B; x)$, then it is easy to see the following:

Theorem 6.6. *Let $\widetilde{H}_n^{\mu}$ denote the orthonormal generalized Hermite polynomials. Then*

$$\lim_{\mu\to\infty} P_\alpha^n(W_{\kappa,\mu}^B; x/\sqrt{\mu}) = \tilde{H}_{\alpha_1}^{\kappa_1}(x_1)\dots \tilde{H}_{\alpha_d}^{\kappa_d}(x_d).$$

Using this limit relation, it follows from the equation in Theorem 4.1 that the polynomials in $\mathcal{V}_n^d(W_\kappa^H)$ satisfy the differential equation

$$(\Delta - 2\langle x, \nabla\rangle)P = -2nP.$$

There is, however, no explicit formula for the reproducing kernel $\mathbf{P}_n(W_\kappa^H; x, y)$, not even when y takes a special value. In fact, there is no special point to be taken for $\mathbb{R}^d$ and no convolution structure. What can be proved is a generating function for the reproducing kernel (Mehler type formula):

Theorem 6.7. *For* $0 < z < 1$ *and* $x, y \in \mathbb{R}^d$

$$\sum_{n=0}^{\infty} P_n(W_\kappa^H; x, y) z^n = \frac{1}{(1-z^2)^{\gamma_\kappa + d/2}} \times \\ \times \exp\Big\{ -\frac{z^2(\|x\|^2 + \|y\|^2)}{1-z^2} \Big\} V_\kappa \Big[\exp\Big\{ \frac{2z\langle x, \cdot\rangle}{1-z^2} \Big\} \Big](y),$$

where V_κ *is the formula given in Theorem 3.6 with* $d+1$ *there replaced by* d.

We should also mention the relation between the multiple generalized Hermite and the multiple Laguerre polynomials, defined by $P_\alpha(x) = R_\alpha(x_1^2, \ldots, x_d^2)$, just as the relation between the orthogonal polynomials on B^d and those on T^d. Much of the properties for the multiple Laguerre polynomials can be derived from this relation and properties of the multiple Hermite polynomials. For example, there is a counterpart of Mehler's formula for the multiple Laguerre polynomials.

The limiting relation similar to that in Theorem 6.6 holds for the orthogonal basis in polar coordinates, which implies that the differential equations also hold for orthogonal polynomials with respect to the weight functions $h_\kappa^2(x) e^{-\|x\|^2}$. For example, it holds for the type A weight functions

$$\prod_{1 \le i < j \le d} |x_i - x_j|^{2\kappa} e^{-\|x\|^2}, \qquad x \in \mathbb{R}^d$$

and the type B weight functions

$$\prod_{i=1}^{d} |x_i|^{2\kappa_0} \prod_{1 \le i < j \le d} |x_i^2 - x_j^2|^{2\kappa} e^{-\|x\|^2}, \qquad x \in \mathbb{R}^d.$$

These two cases are related to the Schrödinger equations of the Calogero-Sutherland systems; these are exactly solvable models of quantum mechanics involving identical particles in a one dimensional space. The eigenfunctions can be expressed in terms of a family of homogeneous polynomials, called the nonsymmetric Jack polynomials, which are simultaneous eigenfunctions of a commuting set of self-adjoint operators. Although there is no explicit orthogonal basis known for these weight functions, there is a uniquely defined basis of orthogonal polynomials for which the L^2-norm of the polynomials can be computed explicitly. The elements of this remarkable family are labeled by partitions, and their normalizing constants are proved using the recurrence relations and algebraic techniques.

7. Fourier orthogonal expansion

The n-th partial sums of the Fourier orthogonal expansion do not converge for continuous functions pointwisely or uniformly. It is necessary to consider summability methods of the orthogonal expansions, such as certain means of the partial sums. One important method are the Cesàro (C, δ) means.

Definition 7.1. *Let $\{c_n\}_{n=0}^{\infty}$ be a given sequence. For $\delta > 0$, the Cesàro (C, δ) means are defined by*

$$s_n^\delta = \sum_{k=0}^{n} \frac{(-n)_k}{(-n-\delta)_k} c_k.$$

The sequence $\{c_n\}$ is (C, δ) summable by Cesàro's method of order δ to s if s_n^δ converges to s as $n \to \infty$.

If $\delta = 0$, then s_n^δ is the n-th partial sum of $\{c_n\}$ and we write s_n^0 as s_n.

7.1. h-harmonic expansions.

We start with the h-harmonics. Using the reproducing kernel $P_n(h_\kappa^2; x, y)$, the n-th (C, δ) means of the h-harmonic expansion $S_n^\delta(h_\kappa^2; f)$ can be written as an integral

$$S_n^\delta(h_\kappa^2; f, x) = c_h \int_{S^d} f(y) K_n^\delta(h_\kappa^2; x, y) h_\kappa^2(y) d\omega(y),$$

where c_h is the normalization constant of h_κ^2 and $K_n^\delta(h_\kappa^2; x, y)$ are the Cesàro (C, δ) means of the sequence $\{P_n(h_\kappa^2; x, y)\}$. If $\delta = 0$, then $K_n(h_\kappa^2)$ is the n-th partial sum of $P_k(h_\kappa^2)$.

Proposition 7.1. *Let $f \in C(S^d)$. Then the (C, δ) means $S_n^\delta(h_\kappa^2; f, x)$ converge to $f(x)$ if*

$$I_n(x) := c_h \int_{S^d} |K_n^\delta(h_\kappa^2; x, y)| h_\kappa^2(y) d\omega < \infty;$$

the convergence is uniform if $I_n(x)$ is uniformly bounded.

Proof. First we show that if p is a polynomial then $S_n^\delta(h_\kappa^2; p)$ converge uniformly to p. Indeed, let $S_n(h_\kappa^2; f)$ denote the n-th partial sum of the h-harmonic expansion of f ($\delta = 0$ of S_n^δ). It follows that

$$S_n^\delta(h_\kappa^2; f) = \frac{\delta}{\delta + n} \sum_{k=0}^{n} \frac{(-n)_k}{(1-\delta-n)_k} S_k(h_\kappa^2; f).$$

Assume $p \in \Pi_m^d$. By definition, $S_n(h_\kappa^2; p) = p$ if $n \geq m$. Hence,

$$S_n^\delta(h_\kappa^2; p, x) - p(x) = \frac{\delta}{\delta + n} \sum_{k=0}^{m-1} \frac{(-n)_k}{(1-\delta-n)_k} \big(S_k(h_\kappa^2; p, x) - p(x)\big),$$

which is of size $\mathcal{O}(n^{-1})$ and converges to zero uniformly as $n \to \infty$. Now the definition of $I_n(x)$ shows that $|S_n^\delta(h_\kappa^2; f, x)| \le I_n(x)\|f\|_\infty$, where $\|f\|_\infty$ is the uniform norm of f taken over S^d. The triangular inequality implies

$$|S_n^\delta(h_\kappa; f, x) - f(x)| \le (1 + I(x))\|f - p\|_\infty + |S_n^\delta(h_\kappa; p, x) - p(x)|.$$

Since $f \in C(S^d)$, we can choose p such that $\|f - p\|_\infty < \epsilon$. ∎

Recall that the explicit formula of the reproducing kernel is given in terms of the intertwining operator. Let $p_n^\delta(w_\lambda; x, y)$ be the (C, δ) means of the reproducing kernels of the Gegenbauer expansion with respect to $w_\lambda(x) = (1-x^2)^{\lambda-1/2}$. Then $p_n^\delta(w_\lambda, x, 1)$ are the (C, δ) means of $\frac{n+\lambda}{\lambda}C_n^\lambda(x)$. Let $\lambda = |\kappa| + (d-1)/2$, it follows from Theorem 3.7 that

$$K_n^\delta(h_\kappa^2; x, y) = V_\kappa\big[p_n^\delta(w_\lambda; \langle x, \cdot\rangle, 1)\big](y).$$

Theorem 7.1. *If $\delta \ge 2|\kappa| + d$, then the (C, δ) means of the h-harmonic expansion with respect to h_κ^2 define a positive linear operator.*

Proof. The (C, δ) kernel of the Gegenbauer expansion with respect to w_λ is positive if $\delta \ge 2\lambda + 1$ (cf [3, p. 71]), and V_κ is a positive operator. ∎

The positivity shows that $I_n(x) = 1$ for all x, hence it implies that $S_n^\delta(h_\kappa^2; f)$ converges uniformly to the continuous function f. For convergence, however, positivity is not necessary. First we state an integration formula for the intertwining operator.

Theorem 7.2. *Let V_κ be the intertwining operator. Then*

$$\int_{S^d} V_\kappa f(x) h_\kappa^2(x) d\omega(x) = A_\kappa \int_{B^{d+1}} f(x)(1 - \|x\|^2)^{|\kappa|-1} dx,$$

for $f \in L^2(h_\kappa^2; S^d)$ such that both integrals are finite. In particular, if $g : \mathbb{R} \mapsto \mathbb{R}$ is a function such that all integrals below are defined, then

$$\int_{S^d} V_\kappa g(\langle x, \cdot\rangle)(y) h_\kappa^2(y) d\omega(y) = B_\kappa \int_{-1}^{1} g(t\|x\|)(1 - t^2)^{|\kappa| + \frac{d-2}{2}} dt,$$

where A_κ and B_κ are constants whose values can be determined by setting $f(x) = 1$ and $g(t) = 1$, respectively.

To prove this theorem, we can work with the Fourier orthogonal expansion of $V_\kappa f$ in terms of the orthogonal polynomials with respect to the classical weight function $W^B_{|\kappa|-1/2}$ on the unit ball B^{d+1}. It turns out that if $P_n \in \mathcal{V}_n^d(W^B_{|\kappa|-1/2})$, then the normalized integral of $V_\kappa P_n$ with respect to h_κ^2 over S^d is zero for $n > 0$, and 1 for $n = 0$, so that the integral of $V_\kappa f$ over S^d is equal to the constant term in the orthogonal expansion on B^{d+1}. Of particular interest to us is the second formula, which can also be derived from the Funk-Hecke formula in Theorem 3.8. It should be mentioned that this theorem holds for the intertwining operator with respect to every reflection group (with $|\kappa|$ replaced by γ_κ), even though an explicit formula for the intertwining operator is unknown in general. The formula plays an essential role in the proof of the following theorem.

Theorem 7.3. *Let $f \in C(S^d)$. Then the Cesàro (C,δ) means of the h-harmonic expansion of f converge uniformly on S^d provided $\delta > |\kappa| + (d-1)/2$.*

Proof. Using the fact that V_κ is positive and Theorem 7.2, we conclude that

$$I_n(x) \le c_h \int_{S^d} V_\kappa\big[|p_n^\delta(w_\lambda; \langle x, \cdot\rangle, 1)|\big](y) h_\kappa^2(y) d\omega(y) = b_\lambda \int_{-1}^1 |p_n^\delta(w_\lambda; 1, t)| w_\lambda(t) dt,$$

where b_λ is a constant (in fact, the normalization constant of w_λ). The fact that the (C,δ) means of the Gegenbauer expansion with respect to w_λ converge if and only if $\delta > \lambda$ finishes the proof. ∎

The above theorem and its proof in fact hold for h-harmonics with respect to any reflection group. It reduces the convergence of the h-harmonics to that of the Gegenbauer expansion. Furthermore, since $\sup_x |I_n(x)|$ is also the L^1 norm of $S_n^\delta(h_\kappa^2; f)$, the Riesz interpolation theorem shows that $S_n^\delta(h_\kappa^2; f)$ converges in $L^p(h_\kappa^2; S^d)$, $1 \le p < \infty$, in norm if $\delta > |\kappa| + (d-1)/2$.

It is natural to ask if the condition on δ is sharp. With the help of Theorem 7.2, the above proof is similar to the usual one for the ordinary harmonics in the sense that the convergence is reduced to the convergence at just one point. For the ordinary harmonics, the underlying group is the orthogonal group and S^d is its homogeneous space, so reduction to one point is to be expected. For the weight function $h_\kappa(x) = \prod_{i=1}^{d+1} |x_i|^{\kappa_i}$, however, the underlying group $\mathbb{Z}_2^{d+1}$ is a subgroup of the the orthogonal group, which no longer acts transitively on S^d. In fact, in this case, the condition on δ is not sharp. The explicit formula of the reproducing kernel for the product weight function allows us to derive a precise estimate for the kernel $K_n^\delta(h_\kappa^2; x, y)$: for $x, y \in S^d$ and $\delta > (d-2)/2$,

$$|K_n^\delta(h_\kappa^2; x, y)| \le c\Big[\frac{\prod_{j=1}^{d+1}(|x_j y_j| + n^{-1}|\bar{x} - \bar{y}| + n^{-2})^{-\kappa_j}}{n^{\delta-(d-2)/2}(|\bar{x} - \bar{y}| + n^{-1})^{\delta + \frac{d+1}{2}}} + \frac{\prod_{j=1}^{d+1}(|x_j y_j| + |\bar{x} - \bar{y}|^2 + n^{-2})^{-\kappa_j}}{n(|\bar{x} - \bar{y}| + n^{-1})^{d+1}}\Big],$$

where $\bar{x} = (|x_1|, \ldots, |x_{d+1}|)$ and $\bar{y} = (|y_1|, \ldots, |y_{d+1}|)$. This estimate allows us to prove the sufficient part of the following theorem:

Theorem 7.4. *The (C,δ) means of the h-harmonic expansion of every continuous function for $h_\kappa(x) = \prod_{i=1}^{d+1} |x_i|^{\kappa_i}$ converge uniformly to f if and only if*

$$\delta > (d-1)/2 + |\kappa| - \min_{1 \le i \le d+1} \kappa_i.$$

The sufficient part of the proof follows from the estimate, the necessary part follows from that of the orthogonal expansion with respect to $W^B_{\kappa,\mu}$ in Theorem 7.7, see below. For $\kappa = 0$, the order $(d-1)/2$ is the so-called classical index for the ordinary harmonics. The proof of the theorem shows that the maximum of $I_n(x)$ appears on the great circles defined by the intersection of S^d and the coordinate planes. An estimate that takes the relative position of $x \in S^d$ into consideration proves the following result:

Theorem 7.5. *Let $f \in C(S^d)$. If $\delta > (d-1)/2$, then $S^\delta_n(h^2_\kappa, f; x)$ converges to $f(x)$ for every $x \in S^d_{\rm int}$ defined by*

$$S^d_{\rm int} = \{x \in S^d : x_i \neq 0, \quad 1 \le i \le d+1\}.$$

This shows that the great circles $\{x \in S^d : x_i = 0\}$ are like a boundary on the sphere for summability with respect to h^2_κ; concerning convergence the situation is the same as in the classical case where we have a critical index $(d-1)/2$ at those points away from the boundary.

The sequence of the partial sums $S_n(h^2_\kappa; f)$ does not converge to f if f is merely continuous, since the sequence is not bounded. The sequence may converge for smooth functions, and the necessary smoothness of the functions depends on the order of $S_n(h^2_\kappa; f)$ as $n \to \infty$.

Theorem 7.6. *Let $\lambda = |\kappa| + (d-1)/2$. Then as $n \to \infty$,*

$$\|S_n(h^2_\kappa; \cdot)\| = \sup_{\|f\|_\infty \le 1} \|S_n(h^2_\kappa; f)\|_\infty = \mathcal{O}\big(n^\lambda\big).$$

In particular, if $f \in C^{[\lambda]+1}(S^d)$, then $S_n(h^2_\kappa; f)$ converge to f uniformly.

Proof. By the definition of $S_n(h^2_\kappa; f)$ and Theorem 7.2, we get

$$|S_n(h^2_\kappa; f, x)| \le b_\lambda \int_{-1}^1 \Big| \sum_{k=0}^n \frac{k+\lambda}{\lambda} C^\lambda_k(t) \Big| w_\lambda(t) dt \, \|f\|_\infty.$$

The integral of the partial sum of the Gegenbauer polynomials is known to be bounded by $\mathcal{O}(n^\lambda)$. The smoothness of f shows that there is a polynomial P of degree n such that $\|f - P\|_\infty \le c n^{-[\lambda]-1}$, from which the convergence follows from the fact that $S_n(h_\kappa; P) = P$. ∎

Again, this theorem holds for h_κ associated with every reflection group. For the product weight function $h_\kappa(x) = \prod_{i=1}^{d+1} |x_i|^{\kappa_i}$, the statement in Theorem 7.6 suggests the following conjecture:

$$\|S_n(h^2_\kappa; \cdot)\| = \mathcal{O}\big(n^\sigma\big) \quad \text{with} \quad \sigma = \frac{d-1}{2} + |\kappa| - \min_{1 \le i \le d+1} \kappa_i;$$

furthermore, for $x \in S^d_{\rm int}$, $|S_n(h^2_\kappa; f, x)|$ is of order $\mathcal{O}(n^{\frac{d-1}{2}})$. If the estimate of $K^\delta_n(h^2_\kappa; x, y)$ could be extended to $\delta = 0$, then the conjecture would be proved.

However, in the proof given in [21], the restriction $\delta > (d-1)/2$ seems to be essential.

7.2. **Orthogonal expansions on B^d and on T^d.** For a weight function W on B^d or T^d we denote the (C,δ) means of the orthogonal expansion with respect to W as $S_n^\delta(W;f)$. It follows that

$$S_n^\delta(W;f,x) = c\int_\Omega \mathbf{K}_n^\delta(W;x,y)f(y)W(y)dy,$$

where $\Omega = B^d$ or T^d, c is the constant defined by $c^{-1} = \int_\Omega W(y)dy$, and $\mathbf{K}_n^\delta(W)$ is the (C,δ) means of the sequence $\mathbf{P}_n(W)$. Using the correspondence in Theorem 3.1, most of the results for h-harmonics can be extended to the orthogonal expansions with respect to $W_{\kappa,\mu}^B$ on the ball B^d.

Theorem 7.7. *The (C,δ) means of the orthogonal expansion of every continuous function with respect to $W_{\kappa,\mu}^B$ converge uniformly to f if and only if*

$$\delta > (d-1)/2 + |\kappa| + \mu - \min\{\kappa_1,\ldots,\kappa_d,\mu\}. \tag{7.1}$$

Furthermore, for f continuous on B^d, $S_n^\delta(W_{\kappa,\mu}^B,f;x)$ converges to $f(x)$ for every $x \in B_{\mathrm{int}}^d$, where

$$B_{\mathrm{int}}^d = \{x \in B^d : \|x\| \le 1 \quad \textit{and} \quad x_i \ne 0, \quad 1 \le i \le d\}$$

provided $\delta > (d-1)/2$.

Proof. The sufficient part of the first and the second statement follow from Theorems 7.4 and 7.5, upon using the fact that

$$\mathbf{K}_n^\delta(W_{\kappa,\mu}^B;x,y) = \frac{1}{2}\Big[K_n^\delta(h_\kappa^2;x,(y,\sqrt{1-|y|^2})) + K_n^\delta(h_\kappa^2;x,(y,-\sqrt{1-|y|^2}))\Big]$$

and the elementary integration formula in the proof of Theorem 3.1. The necessary part of the first statement uses the fact that the expansion reduces to the generalized Gegenbauer expansion at certain points. Let $w_{\lambda,\mu}(t) = |t|^{2\mu}(1-t^2)^{\lambda-1/2}$. Denote by $K_n^\delta(w_{\lambda,\mu};t,s)$ the (C,δ) means of the kernel for the generalized Gegenbauer expansion. Let $\gamma = |\kappa| + \mu + (d-1)/2$. From the explicit formula of the reproducing kernel, it follows that

$$\mathbf{K}_n^\delta(W_{\kappa,\mu}^B;x,0) = K_n(w_{\mu,\gamma-\mu};\|x\|,0) \quad \text{and} \quad \mathbf{K}_n^\delta(W_{\kappa,\mu}^B;x,\varepsilon_i) = K_n(w_{\gamma-\kappa_i,\kappa_i};x_i,1)$$

for $i = 1,2,\ldots,d$, so that the necessary condition can be derived from the convergence of the (C,δ) means of the generalized Gegenbauer expansion $s_n^\delta(w_{\lambda,\mu};f)$ at 0 and 1. In fact, for continuous functions g on $[-1,1]$, $s_n^\delta(w_{\lambda,\mu};g)$ converge uniformly to g if and only if $\delta > \max\{\lambda,\mu\}$. ∎

We can also state that the order of growth of $\|S_n(W_{\kappa,\mu}^B;\cdot)\|_\infty$ is bounded by n^λ with $\lambda = |\kappa| + \mu + (d-1)/2$, just as in the Theorem 7.6, and conjecture that the sharp order is $\sigma = \frac{d-1}{2} + |\kappa| + \mu - \min_{1\le i\le d+1} \kappa_i$ with $\kappa_{d+1} = \mu$. For the classical orthogonal polynomials with respect to $W_\mu^B = (1-\|x\|^2)^{\mu-1/2}$, we have

$$\|S_n(W_\mu^B;\cdot)\| = \sup_{\|f\|_\infty\le 1} \|S_n(W_\mu^B;f)\|_\infty \sim n^{\mu+\frac{d-1}{2}}, \quad \mu \ge 0.$$

The operator $f \mapsto S_n(W_\mu^B;f)$ is a projection operator from the space of continuous functions onto Π_n^d. It is proved in [31] that every such projection operator L_n satisfies

$$\|L_n\|_\infty \ge c\, n^{\frac{d-1}{2}}, \qquad d \ge 2,$$

where c is a constant depending only on d. It turns out that the minimal norm is obtained for the weight function W_μ^B with $\mu < 0$ ([48]):

Theorem 7.8. *For $-1 < \mu < 0$ and $d \ge 3$,*

$$\|S_n(W_\mu^B;\cdot)\| \sim n^{\frac{d-1}{2}}.$$

That is, $S_n(W_\mu^B;\cdot)$ has the smallest possible rate of growth for $\mu \le 0$. The proof of this theorem is quite involved, and does not follow from h-harmonics. In fact, the explicit formula for the reproducing kernel in (4.1) holds only for $\mu \ge 0$. A formula that works also for $\mu < 0$ can be derived from it using integration by parts and analytic continuation. Then a careful estimate of the kernel is derived to prove the above theorem.

Similar questions can be asked for the weight function $W_{\kappa,\mu}^B$ or h_κ, $\kappa_i < 0$. It is easy to conjecture, but likely hard to prove, that $\|S_n(W_{\kappa,\mu}^B;\cdot)\| \sim n^{\frac{d-1}{2}}$ if $\mu \le 0$ and $\kappa_i \le 0$, $1 \le i \le d$.

For orthogonal expansions with respect to W_κ^T, we can state similar results for the convergence of the Cesàro means, but the results do not follow directly from those for orthogonal expansions. In fact, from the relation between the reproducing kernels of $\mathcal{V}_n(W_{\kappa,\mu}^B)$ and $\mathcal{V}_n(W_{\kappa,\mu}^T)$,

$$\mathbf{P}_n(W_{\kappa,\mu}^T;x,y) = \frac{1}{2^d} \sum_{\varepsilon\in\mathbb{Z}_2^d} \mathbf{P}_{2n}\big(W_{\kappa,\mu}^B;(\varepsilon_1\sqrt{x_1},\ldots,\varepsilon_d\sqrt{x_d}\,),(\sqrt{y_1},\ldots,\sqrt{y_d})\big),$$

the (C,δ) means of the left hand side does not relate directly to that of the (C,δ) means of the right hand side. Much of the difference can be seen already in the case of $d=1$, for which the weight function $W_{\kappa,\mu}^B(t) = |t|^{2\kappa}(1-t)^{\mu-1/2}$ for $t \in [-1,1]$ and $W_{\kappa,\mu}^T(t) = t^{\kappa-1/2}(1-t)^{\mu-1/2}$ for $t\in[0,1]$, and the latter one is the classical Jacobi weight function $(1+t)^{\kappa-1/2}(1-t)^{\mu-1/2}$ when converting to $t\in[-1,1]$; thus, it is the difference between the generalized Gegenbauer expansion and the Jacobi expansion on $[-1,1]$.

Theorem 7.9. *Suppose the parameters of $W^T_{\kappa,\mu}$ satisfy the conditions*

$$\sum_{i=1}^{d+1} \big(2\kappa_i - [\kappa_i]\big) \ge 1 + \min_{1\le i\le d+1} \kappa_i \qquad \textit{with} \quad \mu = \kappa_{d+1}, \tag{7.2}$$

where $[x]$ stands for the largest integer part of x. Then the (C,δ) means of the orthogonal expansion of every continuous function with respect to $W^T_{\kappa,\mu}$ converge uniformly to f on T^d if and only if (7.1) holds.

The necessary part of the theorem follows from the (C,δ) means of the Jacobi expansion without the additional condition (7.2). The proof of the sufficient part uses an explicit estimate of the kernel $\mathbf{K}_n^\delta(W^T_{\kappa,\mu}; x, y)$ just as in the proof of Theorem 7.7. However, there is an additional difficulty for the estimate of $\mathbf{K}_n^\delta(W^T_{\kappa,\mu}; x, y)$, and condition (7.2) is used to simplify the matter. We note, that the condition excludes only a small range of parameters. Indeed, if one of the parameters, say κ_1 or μ, is $1/2$, or if one of the parameters is ≥ 1, then the condition holds. In particular, it holds for the unit weight function ($\kappa_1 = \ldots = \kappa_{d+1} = 1/2$). Naturally, we expect that the theorem holds for all $\kappa_i \ge 0$ without the condition.

For pointwise convergence, a theorem similar to that of Theorem 7.7 can be stated for the interior of the simplex, but the proof in [21] puts a stronger restriction on the parameters. We only state the case for the unit weight function.

Theorem 7.10. *If $f \in C(T^d)$, then the (C,δ) means of the orthogonal expansion of f converge to f uniformly on each compact set contained in the interior of T^d if $\delta > (d-1)/2$.*

For the unit weight function, the uniform convergence of the (C,δ) means on T^d holds if and only if $\delta > d - 1/2$.

7.3. **Product type weight functions.** The Fourier expansions for the product type weight functions are quite different from those on the ball and on the simplex. As we pointed out in Section 6, there is no explicit formula for the kernel function. In the case of multiple Jacobi polynomials and multiple Laguerre polynomials, the product formulae of the orthogonal polynomials lead to a convolution structure that can be used to study the Fourier expansions.

Let us consider the multiple Jacobi polynomials. Denote the kernel function of the Cesàro means by $\mathbf{K}_n^\delta(W_{a,b}; x, y)$.

Theorem 7.11. *In order to prove the uniform convergence of the (C,δ) means of the multiple Jacobi expansions for a continuous function, it is sufficient to prove* *that, for $a_j \geq b_j > -1$, $a_j + b_j > -1$,*

$$\int_{[-1,1]^d} |\mathbf{K}_n^\delta(W_{a,b}; \mathbf{1}, y)| W_{a,b}(y) dy \leq c,$$

where c is a constant independent of n.

Proof. We know that the convergence of the (C, δ) means follows from

$$\int_{[-1,1]^d} |\mathbf{K}_n^\delta(W_{a,b}; x, y)| W_{a,b}(y) dy \leq c, \qquad x \in [-1,1]^d, \quad n \geq 0.$$

The product formula in Theorem 6.2 shows that that

$$\mathbf{K}_n^\delta(W_{a,b}; x, y) = \int_{[-1,1]^d} \mathbf{K}_n^\delta(W_{a,b}; t, \mathbf{1}) d\mu_{x,y}^{(a,b)}(t),$$

where the measure $\mu_{x,y}^{(a,b)}$ is the product measure given in Theorem 6.2. This leads to a convolution structure which gives the stated result using the corresponding result for one variable. ■

This shows that uniform convergence is reduced to convergence at one point. Multiplying the generating function of the multiple Jacobi polynomials by $(1 - r)^{-\delta-1} = \sum_{n=0}^\infty \binom{n+\delta}{n} r^n$ gives

$$\sum_{n=0}^\infty \binom{n+\delta}{n} \mathbf{K}_n^\delta(W_{a,b}; x, \mathbf{1}) r^n = (1-r)^{-\delta-1} G_d^{(a,b)}(r; x).$$

This is the generating function of $\mathbf{K}_n^\delta(W_{a,b}; x, \mathbf{1})$, which does not give the explicit formula. What can be used to study the orthogonal expansion is the following:

Theorem 7.12. *For $d \geq 1$ and $0 \leq r < 1$,*

$$\mathbf{K}_n^\delta(W_{a,b}; x, \mathbf{1}) = \binom{n+\delta}{n}^{-1} \frac{1}{\pi r^n} \int_{-\pi}^{\pi} (1 - re^{i\theta})^{-\delta-1} G_d^{(a,b)}(re^{i\theta}; x) e^{-in\theta} d\theta.$$

Proof. Since both sides are analytic functions of r for $|r| < 1$, the generating function for $\mathbf{K}_n^\delta(W_{a,b}; x, y)$ holds for r being complex numbers. Replacing r by $re^{i\theta}$, we get

$$\sum_{n=0}^\infty \binom{n+\delta}{n} K_n^\delta(W_{a,b}; x, \mathbf{1}) r^n e^{in\theta} = (1 - re^{i\theta})^{-\delta-1} G_d^{(a,b)}(re^{i\theta}; x).$$

Hence, we see that $\binom{n+\delta}{n} K_n^\delta(W_{a,b}; x, \mathbf{1}) r^n$ is the n-th Fourier coefficient of the function (of θ) in the right hand side. ■

With $r = 1 - n^{-1}$, this expression allows us to derive a sharp estimate for the kernel $\mathbf{K}_n^\delta(W_{a,b}; x, \mathbf{1})$, which can be used to show that the integral in Theorem 7.13 is finite for δ larger than the critical index. One result is as follows:

Theorem 7.13. *Let $a_j, b_j \geq -1/2$. The Cesàro (C,δ) means of the multiple Jacobi expansion with respect to $W_{a,b}$ are uniformly convergent in the norm of $C([-1,1]^d)$ provided $\delta > \sum_{j=1}^d \max\{a_j, b_j\} + \frac{d}{2}$.*

Similar results also hold for the case $a_j > -1$, $b_j > -1$ and $a_j + b_j \geq -1$, $1 \leq j \leq d$, with a properly modified condition on δ. In particular, if $a_j = b_j = -1/2$, then the convergence holds for $\delta > 0$.

The difference between the Fourier expansion on $[-1,1]^d$ and the expansion on B^d or T^d is best explained from the behavior of the multiple Fourier series on $\mathbb{T}^d$,

$$f \sim \sum_{\alpha} a_\alpha(f) e^{i\alpha\cdot x}, \quad \text{where} \quad a_\alpha(f) = \int_{\mathbb{T}^d} f(x) e^{i\alpha\cdot x} dx,$$

On the one hand, we have seen that the orthogonal expansion with respect to the weight function $(1-\|x\|^2)^{-1/2}$ on B^d is closely related to the spherical harmonic expansion, which is known to behave like summability of spherical multiple Fourier series; that is, sums are taken over the ℓ-2 ball,

$$S_n^{(2)}(f;x) = \sum_{\|\alpha\|\leq n} a_\alpha(f) e^{i\alpha\cdot x} = (D_n^{(2)} * f)(x),$$

where $f * g$ means the convolution of f and g and the Dirichlet kernel $D_n^{(2)}(x) = g_n(\|x\|)$ is a radial function (g_n is given in terms of the Bessel function). In this case, it is known that the (C,δ) means converge if $\delta > (d-1)/2$, the so-called critical index.

On the other hand, the usual change of variables $x_i = \cos\theta_i$ shows that the summability in the case of $\prod_{i=1}^d (1-x_i^2)^{-1/2}$ on $[-1,1]^d$ corresponds to summability of multiple Fourier series in the ℓ-1 sense; that is,

$$S_n^{(1)}(f;x) = \sum_{|\alpha|_1\leq n} a_\alpha(f) e^{i\alpha\cdot x} = (D_n^{(1)} * f)(x),$$

the Dirichlet kernel $D_n^{(1)}$ is given by (recall Theorem 6.1)

$$D_n^{(1)}(x) = [\cos x_1, \ldots, \cos x_d] G_n.$$

In this case, the (C,δ) means converge if $\delta > 0$, independent of the dimension.

For the multiple Laguerre polynomials, there is also a convolution structure which allows us to reduce the convergence of the (C,δ) means to just one point, $x = 0$; the proof is more involved since the measure is not positive. The result is as follows:

Theorem 7.14. *Let $\kappa_i \geq 0$, $1 \leq i \leq d$, and $1 \leq p \leq \infty$. The Cesàro (C,δ) means of the multiple Laguerre expansion are uniformly convergent in the norm of $C(\mathbb{R}_+^d)$ if and only if $\delta > |\alpha| + d - 1/2$.*

For both multiple Jacobi expansions and multiple Laguerre expansions, the uniform convergence is reduced to a single point, the corner point of the support set of the weight function. In the case of the multiple Hermite expansions, the support set is $\mathbb{R}^d$ and there is no finite corner point. In fact, the convergence in this case cannot be reduced to just a single point. Only the situation of the classical Hermite expansions, that is, the case $\kappa_i = 0$, is studied, see [32].

8. Notes and Literature

Earlier books on the subject are mentioned at the end of the Section 1. Many historical notes on orthogonal polynomials of two variables can be found in Koornwinder [15] and in Suetin [30]. The references given below are for the results in the text. We apologize for any possible omission and refer to [10] for more detailed references.

Section 2: The study of the general properties of orthogonal polynomials in several variables appeared in Jackson [14] of 1936. In the paper [18] of 1967, Krall and Sheffer suggested that some of the properties can be restored if orthogonality is taken in terms of orthogonal subspaces instead of a particular basis. The first vector-matrix form of the three-term relation and Favard's theorem appeared in Kowalski [16, 17]; the present form and the theorem appeared in Xu [34, 35]. This form adopted the point of view of Krall and Sheffer. Further studies have been conducted in a series of papers; see the survey in [37] and the book [10]. The study of Gaussian cubature formulae started with the classical paper of Radon [26]. Significant results on cubature formulae and common zeros of orthogonal polynomials were obtained by Mysovskikh and his school [24] and Möller [23]. Further study appeared in [36, 43]. The problem can be studied using the language of polynomial ideals and varieties.

Section 3: Section 3.1 is based on [40]. Ordinary spherical harmonics appeared in many books, for example, [1, 28, 33]. The h-harmonics are introduced and studied by Dunkl in a number of papers; see [6, 7, 8] and the references in [10]. A good reference for reflection groups is [13]. The account of the theory of h-harmonics given in [10] is self-contained. The case of the product weight function in Section 3.2 is studied in [38], while the monomial basis contained in Subsection 3.2.3 is new [49].

Section 4, 5 and 6: The relation between orthogonal polynomials with respect to $(1-\|x\|^2)^{(m-1)/2}$ on B^d and spherical harmonics on S^{d+m} can be traced back to the work of Hermite, Didon, Appell and Kampé de Fériet; see Chapt. XII, Vol. II, of [11]. In the general setting, the relation is studied in [40] and further properties are given in [45, 46]. In various special cases the explicit formulae for the classical orthogonal polynomials on B^d and on T^d have appeared in the literature. The relation between orthogonal polynomials on the simplex and those on the ball or on the sphere has also appeared in special cases. It is studied in the general setting in [41]. Except for the multiple Jacobi polynomials, all other classical type orthogonal polynomials can be studied using h-harmonics; see [47]. Apart from some two dimensional examples (cf. [15]), classical and product type orthogonal polynomials are the only cases for which explicit formulae are available.

The Hermite type polynomials of type A and type B are studied by Baker and Forrester [4], Lassalle [19], Dunkl [9], and several other people. The commuting self-adjoint operators that are used to define the nonsymmetric Jack polynomials are due to Cherednik. They are related to Dunkl operators. The nonsymmetric Jack polynomials are defined by Opdam [25]. There are many other papers studying these polynomials and Calogero-Sutherland models.

Section 7: Summability of orthogonal expansion is an old topic, but most of the results in this section are obtained only recently. See [42] for the expansion of classical orthogonal polynomials on the unit ball, [20] for the product Jacobi polynomials, [46] and [21] for h-harmonics expansions and expansions on the unit ball and on the simplex. The integration formula of the intertwining operator and its application to summability appeared in [39]. The topic is still in its initial stage, apart from the problems on the growth rate of the partial sums, many questions such as those on L^p and almost everywhere convergence have not been studied.

Acknowledgement: I would like to thank the organizers of the workshop for inviting me to give these lectures, and especially Wolfgang zu Castell for carefully reading through these notes and making numerous corrections.

References

[1] G.E. Andrews, R. Askey, and R. Roy, *Special functions*, Encyclopedia of Mathematics and its Applications **71**, Cambridge University Press, Cambridge, 1999.

[2] P. Appell and J.K. de Fériet, *Fonctions hypergéométriques et hypersphériques, Polynomes d'Hermite*, Gauthier-Villars, Paris, 1926.

[3] R. Askey, *Orthogonal polynomials and special functions*, Regional Conference Series in Applied Mathemathics **21**, SIAM, Philadelphia, 1975.

[4] T.H. Baker and P.I. Forrester, Nonsymmetric Jack polynomials and integral kernels, *Duke Math. J.* **95** (1998), 1–50.

[5] H. Berens and Yuan Xu, Fejér means for multivariate Fourier series, *Math. Z.* **221** (1996), 449–465.

[6] C.F. Dunkl, Reflection groups and orthogonal polynomials on the sphere, *Math. Z.* **197** (1988), 33-60.

[7] ———, Differential-difference operators associated to reflection groups, *Trans. Amer. Math. Soc.* **311** (1989), 167–183.

[8] ———, Integral kernels with reflection group invariance, *Canad. J. Math.* **43** (1991), 1213-1227.

[9] ———, Orthogonal polynomials of types A and B and related Calogero models, *Comm. Math. Phys.* **197** (1998), 451–487.

[10] C.F. Dunkl and Yuan Xu, *Orthogonal Polynomials of Several Variables*, Encyclopedia of Mathematics and its Applications **81**, Cambridge University Press, Cambridge, 2001.

[11] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher transcendental functions*, Vol. 2, McGraw-Hill, New York, 1953.

[12] G. Gasper, Banach algebras for Jacobi series and positivity of a kernel, *Ann. Math.* **95** (1972), 261–280.

[13] J.E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics **29**, Cambridge University Press, Cambridge, 1990.

[14] D. Jackson, Formal properties of orthogonal polynomials in two variables, *Duke Math. J.* **2** (1936), 423–434.

[15] T. Koornwinder, Two-variable analogues of the classical orthogonal polynomials, In: R. Askey (ed.), *Theory and applications of special functions*, 435–495, Academic Press, New York, 1975.

[16] M.A. Kowalski, The recursion formulas for orthogonal polynomials in n variables, *SIAM J. Math. Anal.* **13** (1982), 309–315.

[17] ———, Orthogonality and recursion formulas for polynomials in n variables, *SIAM J. Math. Anal.* **13** (1982), 316–323.

[18] H.L. Krall and I.M. Sheffer, Orthogonal polynomials in two variables, *Ann. Mat. Pura Appl.* **76**(4) (1967), 325-376.

[19] M. Lassalle, Polynômes de Hermite généralisés. (French) *C. R. Acad. Sci. Paris Sér. I Math.* **313** (1991), 579–582.

[20] Zh.-K. Li and Yuan Xu, Summability of product Jacobi expansions, *J. Approx. Theory* **104** (2000), 287–301.

[21] ———, Summability of orthogonal expansions on spheres, balls and simplices, *J. Approx. Theory* **122** (2003), 267-333.

[22] B. Logan and I. Shepp, Optimal reconstruction of a function from its projections, *Duke Math. J.* **42** (1975), 649–659.

[23] H.M. Möller, Kubaturformeln mit minimaler Knotenzahl *Numer. Math.* **25** (1976), 185–200.

[24] I.P. Mysovskikh, Interpolatory cubature formulas (in Russian), “Nauka”, Moscow, 1981.

[25] E.M. Opdam, Harmonic analysis for certain representations of graded Hecke algebras, *Acta Math.* **175** (1995), 75–121.

[26] J. Radon, Zur mechanischen Kubatur, *Monatsh. Math.* **52** (1948), 286–300.

[27] M. Rösler, Positivity of Dunkl's intertwining operator, *Duke Math. J.*, **98** (1999), 445–463.

[28] E.M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Univ. Press, Princeton, NJ, 1971.

[29] A. Stroud, *Approximate calculation of multiple integrals*, Prentice-Hall, Englewood Cliffs, NJ, 1971.

[30] P.K. Suetin, *Orthogonal polynomials in two variables*, translated from the 1988 Russian original by E. V. Pankratiev, Gordon and Breach, Amsterdam, (1999).

[31] B. Sündermann, On projection constants of polynomial space on the unit ball in several variables, *Math. Z.* **188** (1984), 111-117.

[32] S. Thangavelu, *Lectures on Hermite and Laguerre expansions*, Princeton University Press, Princeton, NJ, 1993.

[33] N.J. Vilenkin, *Special functions and the theory of group representations*, American Mathematical Society Translation of Mathematics Monographs **22**, American Mathematical Society, Providence, RI, 1968.

[34] Yuan Xu, On multivariate orthogonal polynomials, *SIAM J. Math. Anal.* **24** (1993), 783–794.

[35] ______, Multivariate orthogonal polynomials and operator theory, *Trans. Amer. Math. Soc.* **343** (1994), 193–202.

[36] ______, *Common zeros of polynomials in several variables and higher dimensional quadrature*, Pitman Research Notes in Mathematics Series **312**, Longman, Harlow, 1994.

[37] ______, On orthogonal polynomials in several variables, in *Special functions, q-series and related topics*, 247–270, The Fields Institute for Research in Mathematical Sciences, Communications Series **14**, American Mathematical Society, Providence, RI, 1997.

[38] ______, Orthogonal polynomials for a family of product weight functions on the spheres, *Canad. J. Math.* **49** (1997), 175-192.

[39] ______, Integration of the intertwining operator for h-harmonic polynomials associated to reflection groups, *Proc. Amer. Math. Soc.* **125** (1997), 2963–2973.

[40] ______, Orthogonal polynomials and cubature formulae on spheres and on balls, *SIAM J. Math. Anal.* **29** (1998), 779–793.

[41] ______, Orthogonal polynomials and cubature formulae on spheres and on simplices, *Methods Anal. Appl.* **5** (1998), 169–184.

[42] ______, Summability of Fourier orthogonal series for Jacobi weight on a ball in $\mathbb{R}^d$, *Trans. Amer. Math. Soc.* **351** (1999), 2439–2458.

[43] ______, Cubature formulae and polynomial ideals, *Adv. Appl. Math.* **23** (1999), 211–233.

[44] ______, Harmonic polynomials associated with reflection groups, *Canad. Math. Bull.* **43** (2000), 496-507.

[45] ______, Funk-Hecke formula for orthogonal polynomials on spheres and on balls, *Bull. London Math. Soc.* **32** (2000), 447–457.

[46] ______, Orthogonal polynomials and summability in Fourier orthogonal series on spheres and on balls, *Math. Proc. Cambridge Phil. Soc.* **31** (2001), 139–155.

[47] ______, Generalized classical orthogonal polynomials on the ball and on the simplex, *Constr. Approx.*, **17** (2001), 383-412.

[48] ______, Representation of reproducing kernels and the Lebesgue constants on the ball, *J. Approx. Theory*, **112** (2001), 295-310.

[49] ______, Monomial orthogonal polynomials of several variables, submitted.