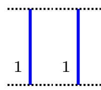
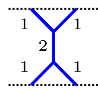
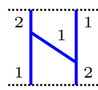
Lecture 1 main exercises

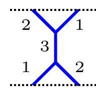
Exercise 1.1. Using the state sum model, verify the rung squash relation.

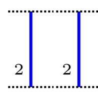
Exercise 1.2. For any representations X, Y of GL_n, the map $X \otimes Y \to Y \otimes X$, $x \otimes y \mapsto y \otimes x$ is a GL_n-intertwiner. We typically draw this map as a crossing. For each choice of X and Y below, describe the crossing as a linear combination of the given webs.

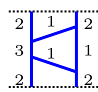
(a) $X = Y = L_1$. Webs: 

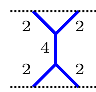
, 

(b) $X = L_1$ and $Y = L_2$. Webs: 

, 

(c) $X = L_2$ and $Y = L_2$. Webs: 

, 



Lecture 1 supplementary exercises

Let X and Y be representations of GL_n. To verify that a linear map $\varphi\colon X \to Y$ is a G-intertwiner, it can be easier to verify that it commutes with the action of the lie algebra $\mathfrak{gl}_n$. More precisely, one need only verify that it intertwines with the generating elements $\{x_i, y_i\}_{i=1}^{n-1}$ of $\mathfrak{gl}_n$. Let V be the standard representation $\mathbb{C}^n$. Then

$$x_i(e_{i+1}) = e_i, \qquad x_i(e_j) = 0 \text{ otherwise}, \qquad y_i(e_i) = e_{i+1}, \qquad y_i(e_j) = 0 \text{ otherwise}.$$

When an element x in a lie algebra acts on tensor products (or exterior products, etcetera), it acts by the formula

$$x(v \otimes w) = x(v) \otimes w + v \otimes x(w).$$

So for example, acting on $V \otimes V \otimes V$ we have

$$y_1(e_1 \otimes e_3 \otimes e_1) = e_2 \otimes e_3 \otimes e_1 + e_1 \otimes e_3 \otimes e_2.$$

Exercise 1.3. Verify directly that x_i and y_i commute with the multiplication and comultiplication maps between exterior products of V.

Exercise 1.4. (This is not the easiest exercise, but it is very worthwhile!) Using the state sum model, verify the square flop relation. You will need the Chu-Vandermonde identity, which states that, for any given $0 \le k, m \le n$ we have

$$\binom{n}{k} = \sum_{a+b=k} \binom{m}{a}\binom{n-m}{b}. \tag{1.1}$$

Exercise 1.5. Find a combinatorial proof of the Chu-Vandermonde identity.

Exercise 1.6. Try to generalize Example 1.2, and find a formula for the crossing in terms of webs, when $X = L_k$ and $Y = L_m$ for all k and m. Use the state sum model and the inclusion/exclusion principle to justify your answer.

For the remaining exercises, we examine the q-deformation of Webs.

Recall that $[n] = q^{-n+1} + q^{-n+3} + \ldots + q^{n-3} + q^{n-1}$, a Laurent polynomial which evaluates to n at $q = 1$. For example, $[2] = q^{-1} + q$ and $[3] = q^{-2} + 1 + q^2$.

Let $\mathbb{B} = \{1, \ldots, n\}$, so that n is the number of size 1 subsets of $\mathbb{B}$. In other words,

$$n = \sum_{T \subset \mathbb{B}, \#T = 1} |T|.$$

Above we used both $\#T$ and $|T|$ to indicate the size of T. We continue to use $\#T$ to indicate the size of T below. Meanwhile, the quantum number $[n]$ is a weighted count of size 1 subsets of $\mathbb{B}$, and henceforth we use $|T|$ for the weighted count of a subset of $\mathbb{B}$. Let us define $|\{i\}| = q^{2i}$. Then

$$[n] = q^{-n-1} \sum_{T \subset \mathbb{B}, \#T = 1} |T|.$$

The power of q at the beginning is just a renormalization factor, to make the Laurent polynomial symmetric around q^0.

Exercise 1.7. Define the weight of a subset of size k, and prove an analogous formula which describes the quantum binomial number $\begin{bmatrix} n \\ k \end{bmatrix}$ as the weighted sum of subsets of $\mathbb{B}$ of size k.

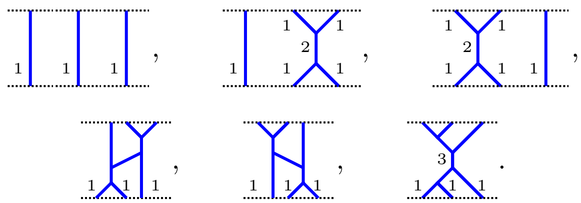
Exercise 1.8. Use a q-deformed state sum model to prove the bigon relation and the rung squash relation in Webs_q.

Exercise 1.9. Formulate and prove the q-Chu-Vandermonde identity.

Exercise 1.10. (Don't actually do this exercise!) Use the state sum model to prove the square flop relation in Webs_q.

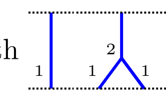
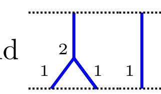
Lecture 2 main exercises

Exercise 2.1. This exercise computes the clasp $e_{3\varpi_1}$ associated to $3\varpi_1$. If desired, you can assume the direct sum decomposition $L_1 \otimes L_1 \otimes L_1 \cong L_{3\varpi_1} \oplus L_{\varpi_1+\varpi_2}^{\oplus 2} \oplus L_{\varpi_3}$. You can also assume the following basis for $\mathrm{End}(L_1 \otimes L_1 \otimes L_1)$:



(a) Justify why $e_{3\varpi_1}$ is the unique morphism in $\mathrm{End}(L_1 \otimes L_1 \otimes L_1)$ satisfying the following two properties:

(i) The coefficient of the identity is 1 (with respect to the basis above), and

(ii) $e_{3\varpi_1}$ is killed by postcomposition with  and  .

(b) Compute $e_{3\varpi_1}$ using these two properties above.

(c) (Big Challenge - try supplementary exercises first) Alternatively, compute the two orthogonal idempotents projecting to $L_{\varpi_1+\varpi_2}^{\oplus 2}$, and the idempotent projecting to L_{ϖ_3}, and subtract them from the identity to compute $e_{3\varpi_1}$. Hopefully your answers agree!

Lecture 2 supplementary exercises

Exercise 2.2. Compute the clasp $e_{2\varpi_2}$. It will help to know the direct sum decomposition $L_2 \otimes L_2 \cong L_{2\varpi_2} \oplus L_{\varpi_1+\varpi_3} \oplus L_{\varpi_4}$.

Exercise 2.3. Formal nonsense. To what extent are clasps unique? Let φ and ψ be two clasps for the same irreducible λ.

(a) Prove that ${}_X\varphi_X = {}_X\psi_X$ for all $X \in P(\lambda)$, i.e. the idempotents in a clasp are unique.

(b) Prove that there exist scalars κ_X for all $X \in P(\lambda)$ such that ${}_X\varphi_Y = \kappa_X \kappa_Y^{-1} \, {}_X\psi_Y$.

Exercise 2.4. Formal nonsense. Suppose that $\{ {}_X\varphi_Y \}$ is a family of maps between objects in $P(\lambda)$ such that

Alph Each map in the family is orthogonal to $\mathrm{Hom}_{<\lambda}$.

Blph For all $X \in P(\lambda)$, ${}_X\varphi_X$ agrees with id_X modulo $\mathrm{Hom}_{<\lambda}$.

Clph φ satisfies compatibility modulo $\mathrm{Hom}_{<\lambda}$.

Then φ is a clasp.

Exercise 2.5. Check that the computation of the clasp $\varphi_{\varpi_1+\varpi_2}$ from class is correct. You can use one of two methods:

(a) Check that φ satisfies the compatibility axiom. (This would be a ton of work. Please don't do this! Maybe check one or two compositions to get the flavor.)

(b) Use the criteria of Exericse 2.4. (Yes, do this!)

Exercise 2.6. If you're new to weights for GL_n, do this exercise! Let $n = 3$ and $L_1 = V = \mathbb{C}^n$.

(a) Give a weight basis for $V \otimes V$ and for each basis vector give its weight.

(b) Give a weight basis of S^2V, and for each basis vector give its weight.

(c) Argue using only the multiplicities of weights that $L_1 \otimes L_1 \cong S^2V \oplus L_2$. Why is $S^2V = L_{2\varpi_1}$?

(d) Compute the multiplicities of $L_1 \otimes L_2$. Why is L_3 a direct summand? (Hint: there's a nonzero map.) What is the weight decomposition of the complementary direct summand? Indeed, this is $L_{\varpi_1+\varpi_2}$.

(e) Enumerate the weights of $S^2V \otimes V$ with multiplicity. Show using weights that $S^2V \otimes V \cong L_{\varpi_1+\varpi_2} \oplus S^3V$.

(f) Verify the decomposition of $L_1 \otimes L_1 \otimes L_1$ stated in Exercise 2.1.

(g) Now repeat the whole process with $n = 4$. The dimensions grow very large, so one will need to figure out how to enumerate things more cleverly and abstractly, without just writing down the weights one by one.

Lecture 3 main exercises

Exercise 3.1. Writing down elementary light ladders.

(a) Write down all the elementary light ladders for L_2 when $n = 4$. (There are only 6.)

(b) Write down all the elementary light ladders for L_2 when $n = 6$. (Ok, there are a lot more now, but after some examination, many of them start to look alike. How many different graphs are there, ignoring labels? How do you know what the graph will be from the weight?)

(c) Write down all the elementary light ladders for L_3 when $n = 6$. (There is one new graph which didn't appear before.)

Exercise 3.2. Drawing branching graphs.

(a) Draw the branching graph for $L_1 \otimes L_1 \otimes L_1 \otimes L_1$.

(b) Draw the branching graph for $L_2 \otimes L_3 \otimes L_2$.

(c) Do some more of your choosing.

Lecture 3 supplementary exercises

Exercise 3.3. Consider the elementary light ladder for $\nu = (0110010)$ given in class, a map from $L_{(1,5,3)}$ to $L_{(3,6)}$. Find a vector $x_\nu \in L_3$ such that $v_+ \otimes v_+ \otimes x_\nu \mapsto v_+ \otimes v_+$ under the light ladder. Deduce that the light ladder descends to a nonzero map $L_{\varpi_1+\varpi_5} \otimes L_3 \to L_{\varpi_3+\varpi_6}$.

Exercise 3.4. Consider any sequence $0 \leq a_1 < b_1 < a_2 < b_2 < \ldots < b_d < a_{d+1}$ and let $k = \sum a_i - \sum b_i$. Find a weight ν for L_k such that the corresponding light ladder is a map from $L_{\underline{b}} \otimes L_k \to L_{\underline{a}}$.

Exercise 3.5. Verify that the maps given in Exercise 2.1 form a basis for $\mathrm{End}(L_1 \otimes L_1 \otimes L_1)$.

Exercise 3.6. For each of the branching paths in Exercise 3.2, construct the light leaves.

Exercise 3.7. Construct a basis for $\mathrm{End}(L_2 \otimes L_3 \otimes L_2)$.