BATH '23 — MINICOURSE ON WEBS

Goal of minicourse: try to understand the diagrammatic toolkit.

Modern diagram revolution (Kuperberg, Khovanov, ...): There are some really complex categories we want to understand where it is hard to do computations (e.g. category $\mathcal{O}$, Perv (quiver vty), etc.). But if you choose carefully a combinatorial monoidal subcategory, then you can completely describe it (if you're lucky) by gens + relns using diagrams. Suddenly you can compute!

Ex: Webs, KLR algebras, Soergel diagrammatics, ... (ignore + for the moment.)

We'll focus on most approachable example: $\mathrm{Rep}^+ GL_n(\mathbb{C})$. Try to explain philosophy + methodology. Go present a new category of your own (I have ideas.)

If you took a class on Rep Thry (if you didn't, don't worry) then maybe you think you know everything about $\mathrm{Rep}\,GL_n$ (f.d. $\mathbb{C}$ (smooth) repns of $GL_n(\mathbb{C})$)

- Classification of irred repns by highest weight

$$\mathrm{Irr}\,GL_n \longleftrightarrow \Lambda^{\mathrm{dom}}_{\mathrm{wt}} \quad \leftarrow \text{later}$$

$$L_\lambda \longleftrightarrow \lambda \in \Lambda$$

OOPS

- Formulas for $\dim L_\lambda$ and character $\chi_{L_\lambda}: G_n \to \mathbb{C}$, $g \mapsto \mathrm{Tr}_{L_\lambda}(g)$ (or equiv, wt space multiplicity)

- Even formulas for $\otimes$ decomp: $L_\lambda \otimes L_\mu \cong \bigoplus_\nu L_\nu^{\oplus c^\nu_{\lambda\mu}}$.
(it's got a $\otimes$!)

- Semisimple! Any $X \in \mathrm{Rep}\,GL_n$ is $X \cong \bigoplus_\lambda L_\lambda^{\oplus k_\lambda}$ for some $k_\lambda \in \mathbb{N}$.

All these statements are about <u>objects</u>. What about morphisms?

- <u>Schur's lemma</u> $\quad \mathrm{Hom}(L_\lambda, L_\mu) = \begin{cases} 0 & \text{if } \lambda \neq \mu \\ \mathbb{C}\cdot \mathrm{id} & \text{if } \lambda = \mu \end{cases}$

So if $X = \bigoplus_\lambda L_\lambda^{\oplus x_\lambda}$, $Y = \bigoplus_\mu L_\mu^{\oplus y_\mu}$ then

$$\mathrm{Hom}(X,Y) = \mathrm{Hom}\left(\oplus L_\lambda^{\oplus x_\lambda}, \oplus L_\mu^{\oplus y_\mu}\right) = \bigoplus_{\lambda,\mu} \mathrm{Hom}\left(L_\lambda^{\oplus x_\lambda}, L_\mu^{\oplus y_\mu}\right)$$

$$= \bigoplus_\lambda \underbrace{\mathrm{Hom}\left(L_\lambda^{\oplus x_\lambda}, L_\lambda^{\oplus y_\lambda}\right)}_{=: \mathrm{Hom}_\lambda(X,Y)}.$$

This is <u>canonical</u>, $L_\lambda^{\oplus x_\lambda} \subset X$ is <u>isotypic component</u>, this canonical. (i.e. functorial)

Whereas $\left.\begin{array}{l} L_\lambda \overset{\oplus}{\subset} L_\lambda^{\oplus x_\lambda} \\ \text{or } L_\lambda \oplus L_\lambda \oplus \cdots \oplus L_\lambda \cong L_\lambda^{\oplus x_\lambda} \end{array}\right\} \circledast$ isn't canonical, like a $\left.\begin{array}{l}\text{line} \\ \text{or choice of basis}\end{array}\right\}$ inside $\mathbb{C}^n$.

Now $\mathrm{Hom}_\lambda(X,Y) = \mathrm{Hom}(L_\lambda^{\oplus x_\lambda}, L_\lambda^{\oplus y_\lambda}) \cong \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\oplus x_\lambda}, \mathbb{C}^{\oplus y_\lambda}) = \mathrm{Mat}_{y_\lambda \times x_\lambda}(\mathbb{C})$.

If you fix a decomposition $\circledast \leftrightarrow$ choice of basis, then a matrix entry $i\begin{pmatrix} & & j \\ 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ on RHS matches $L_\lambda^{\oplus x_\lambda} \xrightarrow{j^{\text{th}} \text{ proj}} L_\lambda \xrightarrow{i^{\text{th}} \text{ inclusion}} L_\lambda^{\oplus y_\lambda}$.

Composition is matrix mult. Artin-Wedderburn basis of $\mathrm{Hom}(X,Y)$.

BUT there's also $\otimes$ on morphisms. Oh no! A-W basis is "incompatible" with $\otimes$ in that tensoring matrix entries gives difficult + uncontrollable (?) nonsense.

The $\otimes$ has ~~a~~ surprising amount of structure, and describing $\mathrm{Rep}\,\mathfrak{S}_n$ as a <u>monoidal</u> <u>additive</u> cat is much more interesting than describing as just an ~~additive~~ abelian one.

Think: $\mathrm{Vect}_{\mathbb{C}} \cong \mathrm{Mat}_{n\times n}(\mathbb{C})\text{-Mod} \cong$ Subcat of $\mathrm{Rep}\, GL_n$ w/ objects $L_\lambda^{\otimes m}$ for fixed λ. ③

$$\mathbb{C} \leftrightarrow \mathbb{C}^n \longleftrightarrow L_\lambda$$

Morita equivalence, an equiv of additive cats. $\mathrm{Hom}(L,L) = \mathbb{C}\,\mathrm{id}$ for simple L

It doesn't matter what L is! This is good in some ways!

Indeed, thinking categorically, you lose all info about objects!! What is $\dim L_\lambda$?

What kind of thing is an object? "Who cares – I know morphisms" says scarecrow

cat. theorist. But $\mathrm{Rep}\, GL_2 \cong \mathrm{Rep}\, GL_{15} \cong \mathrm{Rep}\, Sp_{26} \cong \ldots$ as abelian cats! countable set of Irreps

This is fixed by keeping track of monoidal structure. $\mathrm{Rep}\, G \ncong \mathrm{Rep}\, G'$ as

monoidal cats.

That course on rep theory is only the beginning! Gotta work monoidally.

Basics of $\mathrm{Rep}\, GL_n$

My favorite rep: $V = \mathbb{C}^n \circlearrowleft GL_n$.

Basis $\{e_1, e_2, \ldots, e_n\}$

From V we can get lots of other reps!

Recall: G a group, $G \circlearrowright V, V'$. Then $G \circlearrowright V \otimes V'$ where $g(v\otimes v') = g(v) \otimes g(v')$.

So $G \circlearrowright \underbrace{V \otimes V \otimes V}_{= V^{\otimes 3}}$, etc. This action commutes w/ action of S_3 on V, permuting the tensor factors.

$\rightsquigarrow G \circlearrowright S^3 V$ and $G \circlearrowright \Lambda^3 V$. Eg: $g(v_1 \wedge v_2 \wedge v_3) = g(v_1) \wedge g(v_2) \wedge g(v_3)$

So we have reps $L_i = \Lambda^i V$ for all $0 \le i \le n$.

$L_0 = \mathbb{C}$ where G acts trivially, monoidal identity $L_0 \otimes X \cong X \quad \forall X$. (canonically) i.e. functorially.

$L_1 = V$. $L_n = \mathrm{Det}$ is also 1D, g acts by mult by $\det(g)$.

$L_k = 0$ for $k > n$.

Also have $\Lambda^3 V \otimes \Lambda^6 V \otimes \Lambda^2 V = L_3 \otimes L_6 \otimes L_2 =: L_{(3,6,2)} = L_{\underline{i}}$ for $\underline{i} = (3,6,2)$

<u>Note:</u> L_i is irreducible, but $L_i \otimes L_j$ usually isn't.

How do I tell if a linear map $f: V \otimes V \to V \otimes V$ is a G-intertwiner?

Lots of elements of $G \rightsquigarrow$ lots of work? Often better to rephrase using

lie algebra $\mathrm{Lie}\, GL_n = \mathfrak{gl}_n \underset{\text{v.s.}}{\cong} \mathrm{Mat}_{n \times n}$. acting on V in usual way.

Enough to check f commutes w/ generators of $\mathfrak{gl}_n$, namely

$$x_i = \begin{pmatrix} 0 & & & \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{pmatrix} \qquad x_i(e_{i+1}) = e_i \quad , \; x_i(e_j) = 0 \text{ else}$$

$$y_i = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & 1 & 0 & \\ & & & 0 \end{pmatrix} \qquad y_i(e_i) = e_{i+1} \quad y_i(e_j) = 0 \text{ else.}$$

$1 \le i \le n-1$

$\mathfrak{gl}_n$ action on $V \otimes V'$ via $x_i(v \otimes v') = x_i(v) \otimes v' + v \otimes x_i(v')$.

So e.g. $y_1(e_1 \otimes e_3 \otimes e_1) = e_2 \otimes e_3 \otimes e_1 + \cancel{e_1 \otimes 0 \otimes e_1} + e_1 \otimes e_3 \otimes e_2$.

Similarly $\underset{\overset{\shortparallel}{0}}{y_1(e_1 \wedge e_3 \wedge e_1)} = e_2 \wedge e_3 \wedge e_1 + \underset{\overset{\shortparallel}{0}}{e_1 \wedge e_3 \wedge e_2}$ whew.

Now, $\Lambda^* V$ is a graded algebra w/ mult $\wedge$. Gives

$L_i \otimes L_j \xrightarrow{M} L_{i+j}$. Let's give more tools to compute.

$v \otimes w' \mapsto v \wedge w'$ ← G-intertwiner by defn of action on Λ^*.

For $S \subset \{1,2,\dots,n\}$ let $e_S := e_{s_1}\wedge e_{s_2}\wedge \dots \wedge e_{s_k} \in \Lambda^k V$.
$\quad\quad\;\; \| $
$\{s_1, s_2, \dots, s_k\}$

Then $\{e_S\}_{|S|=k}$ is a basis for L_k.

$$e_S \wedge e_{S'} = \begin{cases} 0 & \text{if } S \not\sqcup S' \\ (-1)^{\ell(S,S')}\, e_{S\sqcup S'} & \text{if } S \sqcup S' \end{cases}$$

where $\ell(S,S')$ = # of flips needed to put SS' in order
$= \#\{(s,s')\in S\times S' \mid s'<s\}$

$\boxed{\ell(S,S') \neq \ell(S',S)}$

Exercise: From this formula verify that $\wedge$ is a $\mathfrak{gl}_n$-intertwiner.

Now $\Lambda^* V$ is also a graded coalgebra! $\quad \Delta : L_{i+j} \to L_i \otimes L_j$ is adjoint to mult.

$$\Delta(e_T) = \sum_{\substack{S \sqcup S' = T \\ |S| = i \\ |S'| = j}} (-1)^{\ell(S,S')}\, e_S \otimes e_{S'} .$$

Exercise: Same, less obvious.

First real computation: $\quad m(\Delta(e_T)) = \sum_{S\sqcup S' = T} (-1)^{\ell(S,S')} (-1)^{\ell(S,S')}\, e_T = \binom{i+j}{i} e_T.$

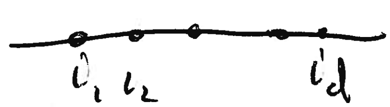
So $\quad m \circ \Delta = \binom{i+j}{i} \cdot \mathrm{id}_{L_{i+j}}$.

Indication: set combinatorics governs intertwiners between $\otimes$ of $\Lambda^k V$!!

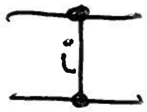
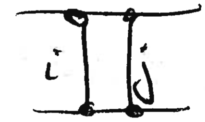
Now let's use diagrammatics to encode these morphisms. $\quad i \leftrightarrow L_i = \Lambda^i V$.

Notation: $\underline{i} = (i_1, i_2, \dots, i_d)$ a sequence w/ $i_k \in \mathbb{N}$.

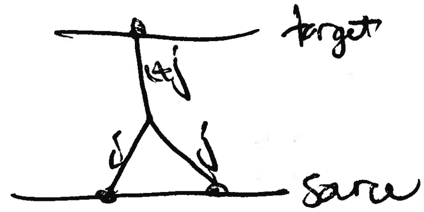
$L_{\underline{i}} = L_{i_1} \otimes \dots \otimes L_{i_d}$

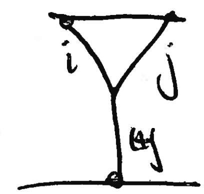
Draw $\underline{i}$ as 

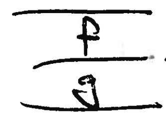
, $\otimes$ is horiz concat.

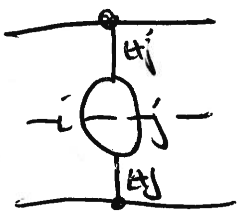
Draw id_{L_i} as  and $\mathrm{id}_{L_i \otimes L_j} = \mathrm{id}_{L_i} \otimes \mathrm{id}_{L_j}$ as 

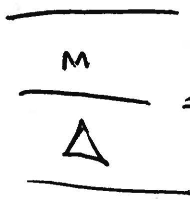
$\otimes$ is still horiz concat, now on morphisms.

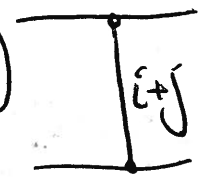
Draw $M : L_i \otimes L_j \to L_{i+j}$ as 

Δ as 

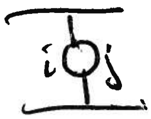
composition is vertical concat, $f \circ g \leftrightarrow$  target(g) = source(f)

Ex: 

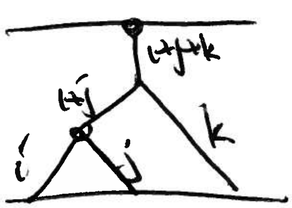
$=$ 

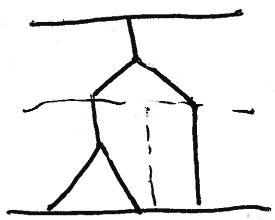
$= M \circ \Delta = \binom{i+j}{i} \mathrm{id}_{L_{i+j}} = \binom{i+j}{i}$ 

note: If a picture represents a morphism, so does a linear combo of pictures like RHS (w/ same source + target)

Call 

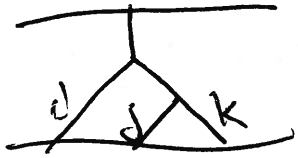
$= \binom{i+j}{i}$  the bigon relation.

Ex: 

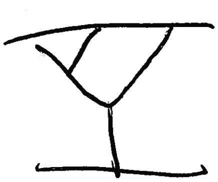
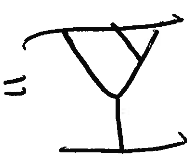
$=$  $\leftarrow M_{i+j,k} \circ (M_{i,j} \otimes \mathrm{id}_k)$

split into rectangles to parse!

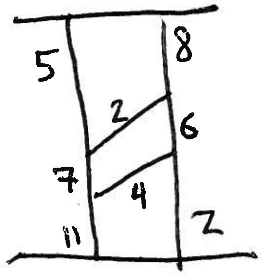
$\|$



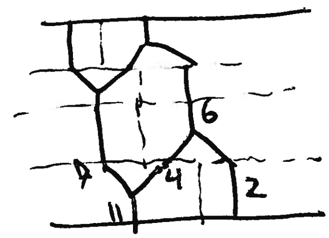
equality is associativity of $\wedge$.

Ex:  $=$  coassociativity

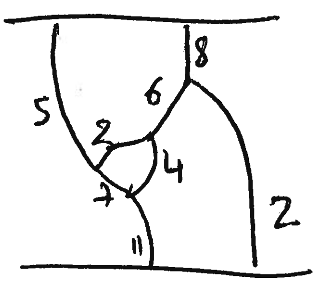
Ex of a diagrammatic computation: "parallelogram squash"



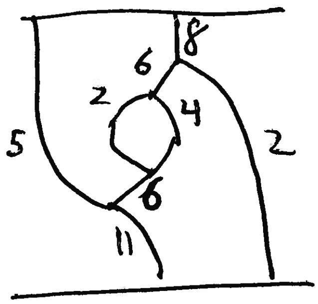
It's a map $\Lambda^{11}V\otimes\Lambda^{2}V \xrightarrow{f} \Lambda^{5}V\otimes\Lambda^{8}V$



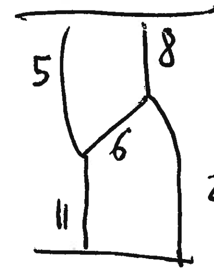
assoc $\|$



loassoc $=$



bigon $= \binom{6}{2}$



You could do this w/ vectors, but it wouldn't be nearly so easy, and there would be a lot of case by case analysis!

Idea: the easy diagram  efficiently encodes an enormous matrix and obfuscates a complicated operation on vectors. Can do hard matrix mult w/ easy diagrammatic rules. It's nice NOT knowing what the objects are...

Here's how to do the computation with vectors though! Matrix coeff by matrix coeff.

Pick S_1 of size 11, S_2 of size 2, T_1 of size 5, T_2 of size 8.

$f(e_{S_1}\otimes e_{S_2}) \in \Lambda^5V\otimes\Lambda^8V$, coeff of $e_{T_1}\otimes e_{T_2}$ is a matrix entry.

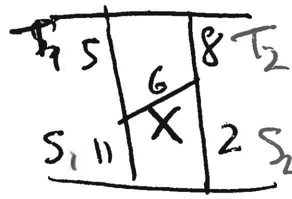
Lemma: Coeff of $e_{T_1}\otimes e_{T_2}$ in $f(e_{S_1}\otimes e_{S_2})$ is a signed count of # of ways to label all non-boundary strands w/ subsets of $\{1\ldots n\}$ s.t.

and

at each vertex. Sign is $(-1)^{\ell(A,B)}$.

To check equality we'll check all matrix coeffs.

RHS:



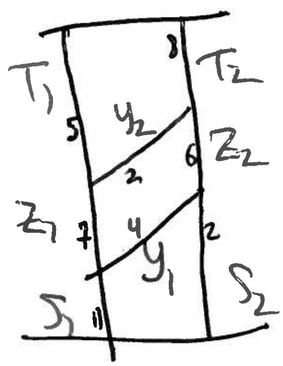
need $X \sqcup S_2 = T_2$, $X \sqcup T_1 = S_1$.

If ~~$T_2 \setminus S_2 \not\subset S_1 \setminus T_1$~~ not ⊛, coeff = zero.

If $S_2 \subset T_2$ $T_1 \subset S_1$ $T_2 \setminus S_2 = S_1 \setminus T_1$ ⊛ then only option is $X = T_2 \setminus S_2$

and coeff is $(-1)^{\ell(X,S_2) + \ell(T_1,X)}$.

LHS:



How many ways to label Y_1, Y_2, Z_1, Z_2?

Claim: If not ⊛, no valid label, so coeff 0.

Claim: If ⊛, then $Y_1 \sqcup Y_2 = T_2 \setminus S_2$ in any valid label, and any splitting of $X = T_2 \setminus S_2$ into sets of size 2 and 4 (Y_2, Y_1) gives a valid labeling (and !-ly determines Z_1, Z_2).

$X = T_2 \setminus S_2$ ← size 6

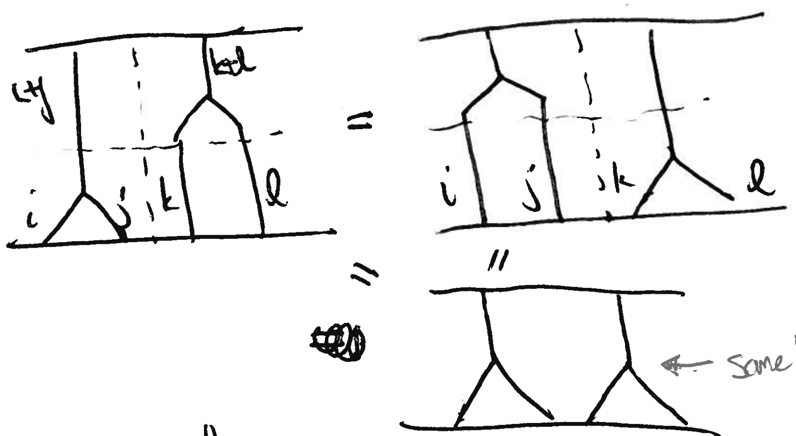
Moreover, $(-1)^{\ell(Z_1,Y_1) + \ell(T_1,Y_2) + \ell(Y_1,S_2) + \ell(Y_2,Z_2)} = (-1)^{\ell(X,S_2) + \ell(T_1,X)}$

It's a worthy exercise to do! "State sum model" for evaluating webs on vectors.

(But diagrammatic is much easier!)

More relns:

Ex:

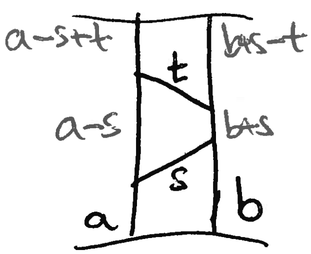


This is one of the axioms of $\otimes$: (interchange law)

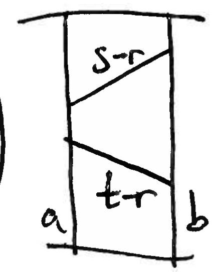
$$f\otimes g = (f\otimes \mathrm{id})\circ(\mathrm{id}\otimes g)$$
$$= (\mathrm{id}\otimes g)\circ(f\otimes \mathrm{id})$$

"rectilinear isotopy" move rectangles past each other like robot arms.

Ex: The hardest one! Square flop



$$= \sum_{r\geq 0} \binom{(a+t)-(b+s)}{r}$$

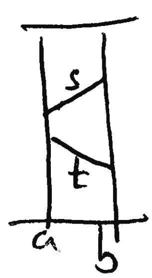


so really $r \leq \min(s,t)$

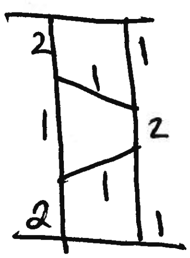
Convention: Negative label on any strand $\Rightarrow$ diagram $=0$. (No valid state sum

Convention: 0 label can be removed (monoidal identity)

Note: $r=0$ gives $1\cdot$



Ex:



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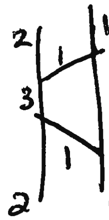
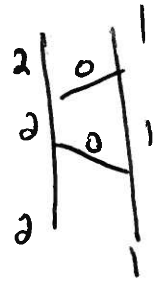


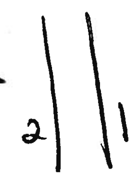
$+ \binom{3-2}{1}$



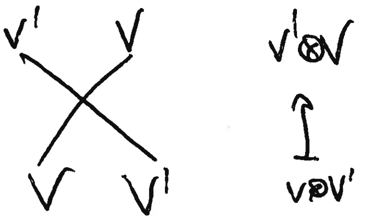
$r=0$ $\quad$ $r=1$

$=$

$+$

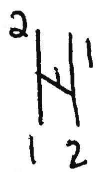
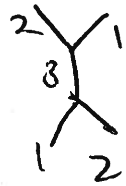


Beautiful + difficult exercise: Use the state sum model to verify the square flop relation. You'll need Chu-Vandermonde.

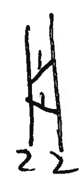
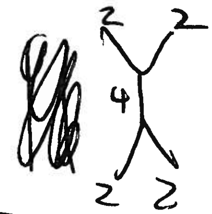
Exercise: 

When $V=V'=L_1$ find as linear combo of  and .

When $V=L_1$, $V'=L_2$

 and 

When $V=V'=L_2$

Generalize and prove w/ state sum model.

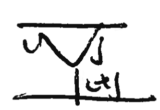
Now let's state a theorem.

Defn: Fund RepGL$_n$ is the full subcategory whose objects have the form $L_{\underline{i}}$ for words $\underline{i}$ in $\{1,..,n\}$.

This is the category whose morphisms we're modeling. It's a strict monoidal cat: objects are words and $\otimes$ is concatenation. Diagrams are for strict monoidal cats.
Soon: how much of RepGL$_n$ does Fund know.

Defn: Let Webs$^+$ be the (strict) monoidal $\mathbb{Z}$-linear category w/ presentation:

Ob: gen by $i\in\mathbb{N}_{\geq 1}$, i.e. objects are words $\underline{i}$

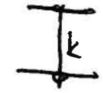
Mor: gen by  . Diagrams built from these = Webs.

Relns: Assoc., Coassoc., Bigon, Square flop.

Note: Interchange law is free as part of what a monoidal presentation means.

(rectilinear isotopy classes of)

That is, $\mathrm{Hom}_{\mathrm{Webs}^+}(\underline{i}, \underline{j})$ = linear combo of diagrams w/ bottom $\underline{i}$, top $\underline{j}$, modulo relations applied locally to subdiagrams.

Def: Webs_n^+ is quotient of Webs^+ by further relation that  $= 0$ for all $k > n$. (Amazing: other relations independent of n!!!)

Def: $\mathrm{eval}_n : \mathrm{Webs}^+ \longrightarrow \mathrm{Fund}\, GL_n$ the obvious functor ($\otimes$, $\mathbb{Z}$-linear)

obviously, eval_n descends to Webs_n^+.

Thm: $\mathrm{eval}_{n,\mathbb{C}} : \mathrm{Webs}_n^+ \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow \mathrm{Fund}\, GL_n$ is an equiv. of monoidal cats.

due to Cautis–Kamnitzer–Morrison '14.

That is:
- All morphisms are in the span of webs, and
- We found all the relations!

Various Remarks: 1) Proof uses Skew Howe Duality. We'll unravel a different proof in these talks; SHD won't work in other types/situations.

2) Both sides have a q-deformation: $\mathrm{Fund}\, U_q(\mathfrak{gl}_n)$

Replace $\binom{a}{b}$ with $\begin{bmatrix} a \\ b \end{bmatrix}_q$ and $(-1)^{\ell(S,S')}$ with $(-q)^{\ell(S,S')}$.

Not obvious what $\Lambda^k \mathbb{C}$ means... but same basis for L_i.

3) (E. '16 arXiv) gives meaning to Webs_n^+ before $- \otimes_{\mathbb{Z}} \mathbb{C}$, i.e. to the integral form of Webs. Matches $\mathrm{Tilt}\, GL_n$ in finite characteristic.

4) CKM technically do SL_n not GL_n, but it's basically the same story, GL_n has fewer technical details.

5) History: SL_2 1925 Rumer–Teller–Weyl
"Temperley–Lieb algebra"

SL_3 and other rank 2 groups: 1996 Kuperberg. Poses general question.

Morrison's thesis '07: relations, but no proof of equivalence.

CKM '14: Skew HD saves the day.

BGRT '21 arXiv: Type C, i.e. Rep Sp_{2n}.

other types: still open, but progress being made. Ask me.

Find GL_n vs. Rep GL_n

Let $\Lambda_{wt} = \mathbb{Z}^n$, the <u>weight lattice</u>. These parametrize the simult. eigenvalues for the abelian group $T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subset GL_n$ which appear in fd. GL_n reps.

We say $v \in X$ is a <u>weight vector</u> of weight $\underline{a} = (a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$ if $\forall$

$$t = \begin{pmatrix} t_1 & & 0 \\ & t_2 & \\ & & \ddots \\ 0 & & & t_n \end{pmatrix} \in T, \quad t \cdot v = t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n} v.$$

Ex: $e_i \in \mathbb{C}^n$ is a $\begin{array}{c} \, i \\ (000\,1\,0000) \end{array}$ weight vector $te_i = t_i e_i$.

Do $e_i \otimes e_j$ in $V \otimes V$.

$e_i \wedge e_j \in \Lambda^2 \mathbb{C}^n$ is $(0001\,000\,100)$

so is $e_i e_j \in S^2 \mathbb{C}^n$

$e_i^2 \in S^2 \mathbb{C}^n$ is $(000\,2\,000)$ as is $e_i \otimes e_i \in \mathbb{C}^n \otimes \mathbb{C}^n$.

$X[\underline{a}]$ is subspace of $\underline{a}$ weight vectors.

Big Thm: $$X = \bigoplus_{\underline{a} \in \Lambda_{wt}} X[\underline{a}]$$ for any fd. GL_n repn.

$\mathrm{wts}(X) := \{\underline{a} \in \Lambda_{wt} \mid X[\underline{a}] \neq 0\}$

Now the weights appearing in $(\mathbb{C}^n)^{\otimes d}$ or in L_i are all positive... they live in $\mathbb{N}^n \subset \mathbb{Z}^n$. $\boxed{\text{Fund} \subset \text{Rep}^+}$

<u>Def:</u> A weight $\underline{a}$ is <u>polynomial</u> if $a_i \geq 0 \; \forall i$. A repn is <u>poly</u> if all weights are poly. $\text{Rep}^+ GL_n$

(Why: the map $GL_n \to GL(X)$ is poly if all matrix entries in $GL(X)$ are polys in entries of GL_n. Poly repns have poly weights.)

<u>Ex:</u> $L_n = \text{Det}$ is 1D w/ weight $(1,1,1,\ldots,1)$.

Det^* is 1D w/ weight $(-1,-1,-1,\ldots,-1)$, i.e. g acts by $\det(g)^{-1}$.

NOT POLY. Rep^+ not closed under duals!

<u>Thm:</u> For any f.d. repn X, $X \otimes \text{Det}^{\otimes k}$ is poly for $k \gg 0$.

So basically $\text{Rep}^+ GL_n = \{\text{poly repns}\}$ understands all of $\text{Rep}\, GL_n$, just apply invertible functor $\otimes \text{Det}^{-1}$. We're "happy" if we understand the monoidal category $\text{Rep}^+ GL_n$.

but duals are nice....

<u>Thm:</u> Every irrep in $\text{Rep}^+ GL_n$ is a direct summand of some object in $\text{Fund}\, GL_n$.

$\rightsquigarrow \text{Rep}^+ GL_n \cong \text{Kar}(\text{Fund}\, GL_n)$. Pf in a bit.

The Karoubi envelope is a formal way to add direct summands to a category. We'll discuss soon.

<u>Cor:</u> $\text{Kar}(\text{Webs}_n^+) \cong \text{Rep}^+ GL_n$. We get it all.

Let's talk about ~~Irrep~~ of GL_n.

Def: A weight $\lambda = (a_1, a_2, \ldots, a_n)$ (let's switch to more common notation)

is <u>dominant</u> if $a_1 \geq a_2 \geq \cdots \geq a_n$.

$$\Lambda_{wt} \supset \Lambda^+_{wt} \supset \Lambda^{+,dom}_{wt} \qquad \Lambda_{wt} \supset \Lambda^{dom}_{wt} \supset \Lambda^{+,dom}_{wt}$$

$\Lambda^{+,dom}_{wt}$ ← corresponds to partition w/ $\leq n$ rows $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$.

Ex: $wts(L_1)$ $\quad \boxed{(1000)} \xleftarrow{+(1\,-1\,0\,0)} (0100) \xleftarrow{+(0\,1\,-1\,0)} (0010) \xleftarrow{+(0\,0\,1\,-1)} (0001)$

DO $wts(L_1 \otimes L_1)$ next. dom. weights add when you $\otimes$.

$wts(L_2)$ $\quad \boxed{(1100)} \leftarrow (1010) \leftarrow (1001) \quad (0110) \quad (0101) \quad (0011)$

$wts(S^2\mathbb{C}^n)$ $\quad \boxed{(2000)} \quad \boxed{(1100)} \quad (1010) \quad (0200) \quad \ldots$

Def: $\lambda \geq \mu$ if $\lambda - \mu$ is in $\mathbb{N}$-span of $(00\overset{i}{1}\overset{i+1}{-1}000)$ for $1 \leq i \leq n-1$

<u>dominance order</u>.

Thm: Any irrep L of GL_n has a ! weight λ (highest weight) which is maximal wrt $\geq$, and $\lambda \in \Lambda^{dom}_{wt}$. Moreover, $L \leftrightarrow \lambda$ is bijection

$$\{\text{Irrep } GL_n\} \leftrightarrow \{\Lambda^{dom}_{wt}\}$$
$$\cup \qquad\qquad \cup$$
$$\{\text{Irrep}^+ GL_n\} \leftrightarrow \{\Lambda^{+,dom}_{wt}\}$$

Moreover, $\dim L_\lambda[\lambda] = 1$. A basis vector in $L_\lambda[\lambda]$ is usually written v_+.

So $L_1 = L_{(1,000)}$ $\quad L_2 = L_{(1100)}$ $\quad S^2L_1 = L_{(2000)}$ etc.

it's very. p for paid. 15

Def: Let $\omega_k = (\overbrace{1111}^{k}\overbrace{0000}^{n-k}) \in \Lambda^{+,dom}_{wt}$.

Then $L_k = L_{\omega_k}$. $\{\omega_k\}_{k=1}^n$ is a basis for $\mathbb{Z}^n = \Lambda_{wt}$.

Moreover, $\Lambda^{+,dom}_{wt} = \mathbb{N}\cdot\{\omega_k\}_{k=1}^n$. That is,

any $\lambda = \sum^n c_i \omega_i$ and $\lambda \in \Lambda^{+,dom}_{wt} \Leftrightarrow c_i \geq 0 \; \forall i$.

Note: $\omega_n = (111111)$ and doesn't affect dominance.

$\lambda \in \Lambda^{dom}_{wt} \Leftrightarrow c_i \geq 0 \; \forall i < n$. (but maybe $c_n < 0$)

Finally: to prove thm: Let $\lambda \in \Lambda^{+,dom}_{wt}$, $\lambda = \sum^n c_i \omega_i$.

Consider $\underline{i} = (\underbrace{1,1,1,\ldots}_{c_1}, \underbrace{2,2,\ldots}_{c_2}, \ldots)$ and look at $L_{\underline{i}}$.

Then $v_+ \otimes v_+ \otimes \ldots \otimes v_+$ has weight λ, all other weights are $< \lambda$.

$L_{\underline{i}} = \bigoplus L_\mu^{\oplus m_\mu}$ and only way to get λ is if $m_\lambda = 1$.

(Also, $m_\mu = 0$ unless $\mu < \lambda$.)

So $L_\lambda \overset{\oplus}{\subset} L_{\underline{i}}$.

$\boxed{\begin{array}{l}\text{Ex: } L_{\omega_1} \otimes L_{\omega_1} = L_{2\omega_1} \oplus L_{\omega_2} \\ \qquad \omega_2 < 2\omega_1.\end{array}}$

Def: For $\lambda \in \Lambda^{+,dom}_{wt}$ Let $P(\lambda) = \{\underline{i} = (i_1, \ldots, i_d) \mid \sum \omega_{i_k} = \lambda\}$. ← just reorder $\underline{i}$ above!!

Cor: For any $\underline{i} \in P(\lambda)$, $L_\lambda \underset{1}{\overset{\oplus}{\subset}} L_{\underline{i}}$ and all other summands are L_μ for $\mu < \lambda$.

mult. one

$***$

So, what's Kar and how do we study L_q using webs?

If $L \overset{\oplus}{\subset} X$ then [redacted] $\exists\ L \underset{p}{\overset{i}{\rightleftarrows}} X$ st. $p \circ i = \mathrm{Id}_L$.

$\Rightarrow$ $i \circ p = e \in \mathrm{End}(X)$ is idempotent.

For any Y, $\quad \mathrm{Hom}(X,Y) \underset{(-)\circ p}{\overset{(-)\circ i}{\rightleftarrows}} \mathrm{Hom}(L,Y)$, and

$$\mathrm{Hom}(L,Y) \cong \{\varphi \in \mathrm{Hom}(X,Y) \mid \varphi \circ e = \varphi\} \qquad \text{(Exercise if you're new.)}$$
$$= \mathrm{Hom}(X,Y)\circ e$$

Similarly, $\quad \mathrm{Hom}(Y,L) \cong e \circ \mathrm{Hom}(Y,X)$

<u>Def</u>: $\mathrm{Kar}(\mathcal{C})$ has Ob: pairs (X,e) w/ $e \in \mathrm{End}(X)$ idempotent

$$\mathrm{Hom}((X,e),(Y,f)) := f \circ \mathrm{Hom}(X,Y) \circ e$$

<u>Note</u>: $\mathcal{C} \hookrightarrow \mathrm{Kar}\,\mathcal{C}$ $\qquad$ Can also just add one summand at a time, partial idempotent completion.

$X \mapsto (X, \mathrm{id}_X)$

we still call it X.

Less standard: What if [redacted] something is a summand of <u>multiple</u> objects.

$$\begin{tikzcd} X \arrow[r, shift left, "{}_Y\varphi_X"] \arrow[dr, shift left, "p_X"] & Y \arrow[l, shift left] \arrow[r, shift left] \arrow[d, shift left] & Z \arrow[l, shift left] \arrow[dl, shift left] \\ & L \arrow[ul, shift left, "i_X"] \arrow[u, shift left] \arrow[ur, shift left] & \end{tikzcd}$$

${}_Y\varphi_X = i_Y \circ p_X$

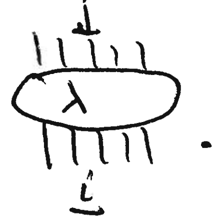
${}_X\varphi_Y \circ {}_Y\varphi_X = e_X = {}_X\varphi_X$ $\Leftarrow$ and implies idempotence

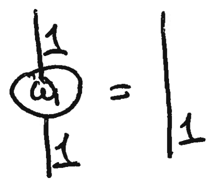
More generally, $\boxed{{}_Z\varphi_Y \circ {}_Y\varphi_X = {}_Z\varphi_X}$ $\circledast$

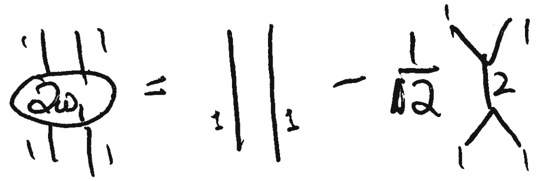
Only remembering idempotents $\{{}_X\varphi_X\}_X \rightsquigarrow$ a bunch of summands $(X, {}_X\varphi_X)$

but the whole family $\{{}_Y\varphi_X\}_{X,Y} \rightsquigarrow$ a <u>common</u> summand, i.e. a bunch of summands w/ fixed isoms between them!

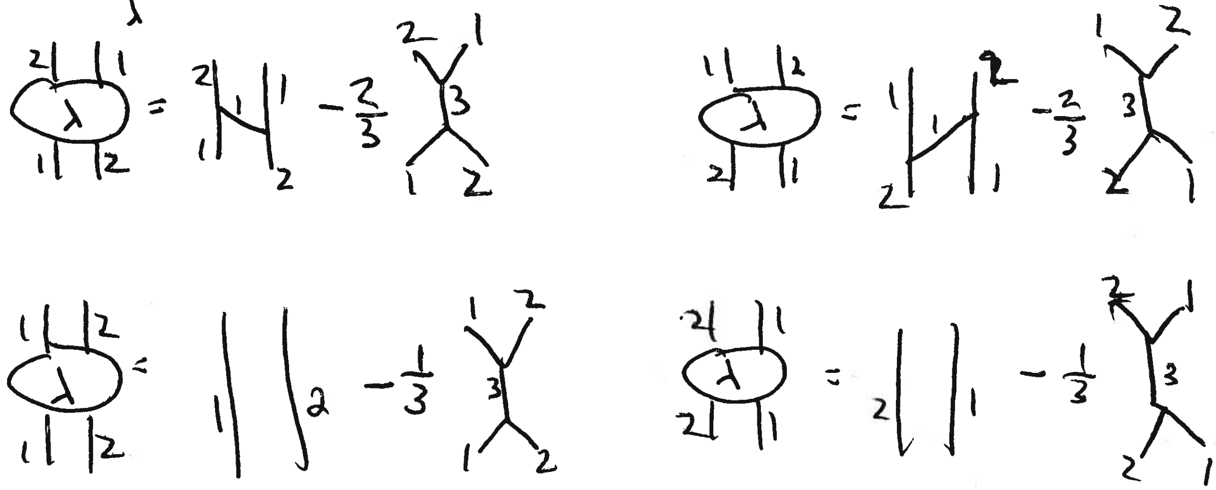
Def: For $\lambda \in \Lambda_{wt}^{+,dom}$, a <u>clasp</u> is ~~the~~ a family $\{{}_{\underline{j}}\varphi_{\underline{i}}\}_{\underline{i},\underline{j}\in P(\lambda)}$ satisfying ⊛, and $\underbrace{\text{picking out the summand } L_\lambda}_{\text{equiv. criteria soon.}}$.

We'll denote w/ oval: .

Ex:  $P(\omega_1)=\{(1)\}$.

Ex: $P(2\omega_1)=\{(1,1)\}$  ← why soon.

Ex: $P(\underset{\lambda}{\underbrace{\omega_1+\omega_2}})=\{(1,2),(2,1)\}$



do bottom first Ben!

Exercise: Check ⊛.

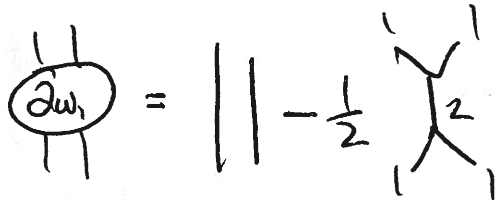
Moral 1: Kar is nice as a formal construction. But in practice, you need to find idempotents/clasps explicitly to use it!

Moral 2: Clasps are nasty linear combos of diagrams, and not easy to find!

Moral 3: Clasps have denominators – defined in $\mathrm{Webs}^+ \otimes_{\mathbb{Z}} \mathbb{Q}$
(but indep of n really!)
Don't exist in $\mathrm{Webs}^+ \otimes_{\mathbb{Z}} \mathbb{F}_p$ ... which is why rep theory in finite char is not semisimple and hard.

Moral 4: This is the reason to stick to Fund when presenting the category!
Fund is easy, but all reps are hard. If you wanted to present the category w/ objects $L_{\lambda_1} \otimes L_{\lambda_2} \otimes \cdots \otimes L_{\lambda_d}$, then i, p would be morphisms and $i \circ p = \text{clasp}$ would be a relation! Category would be defined over $\mathbb{Q}$, not $\mathbb{Z}$, and would need ∞-ly many nasty relations...

Why is



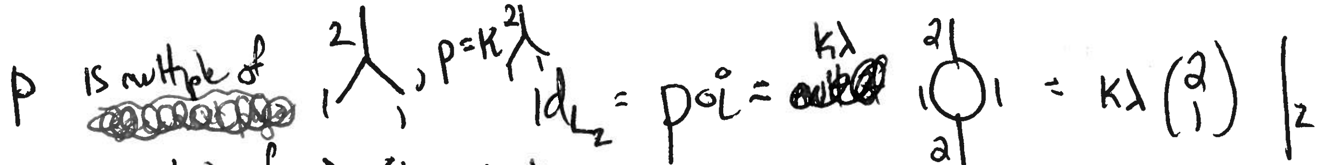
correct?

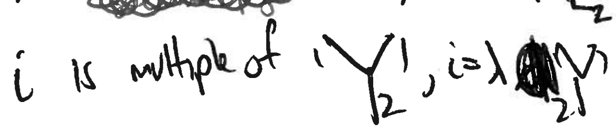
Reason 1:
$L_1 \otimes L_1 \cong L_{2\omega_1} \oplus L_{\omega_2}$, i.e. $V \otimes V \cong S^2 V \oplus \Lambda^2 V$.

not in Fund ↗ $L_{2\omega_1}$; ↖ already in Fund! L_{ω_2}

Schur ⟹ $\dim \mathrm{Hom}(L_1 \otimes L_1, L_2) = 1$ so unique maps up to scalar.

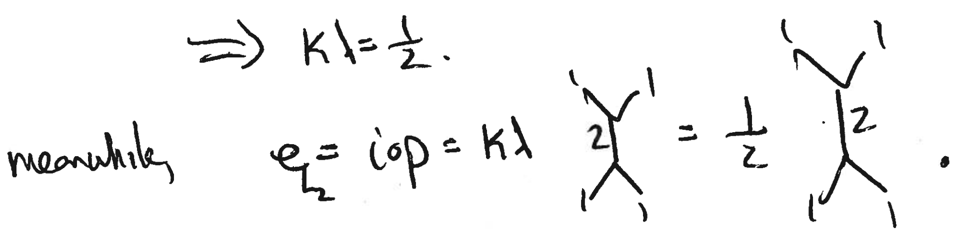
We can already compute $L_1 \otimes L_1 \underset{i}{\overset{p}{\rightleftarrows}} L_2$ for the other summand



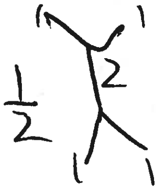


$= 2k\lambda \, \mathrm{id}_{L_2}$

$\Rightarrow k\lambda = \frac{1}{2}$.



Now $\mathrm{id}_{L_1 \otimes L_1} = e_{L_{2\omega_1}} + e_{L_2}$ ← orthog. idempotents

$\Longrightarrow e_{L_{2\omega_1}} = \mathrm{id} - e_{L_2} = \;||\; - \frac{1}{2}$  .

For second example, $L_1 \otimes L_2 = L_{\omega_1+\omega_2} \oplus L_{\omega_3}$.

<u>Reason 2</u>: Spose you've checked $= e \in \mathrm{End}(L_{\underline{i}})$ is an idempotent.

What is $\mathrm{Im}\, e$? Some (not necessary indecomposable) summand of $L_{\underline{i}}$.

How would you know it was L_λ? Well, remember ~~xxx~~

$\mathrm{Im}\, e \cong L_\lambda \iff \mathrm{Im}\, e \neq 0$ and $\mathrm{Hom}(\mathrm{Im}\, e, L_\mu) = 0 \;\; \forall \mu < \lambda$.

$\iff e \neq 0$ and $\mathrm{Hom}(L_{\underline{i}}, L_\mu)\, e = 0 \;\; \forall \mu < \lambda$

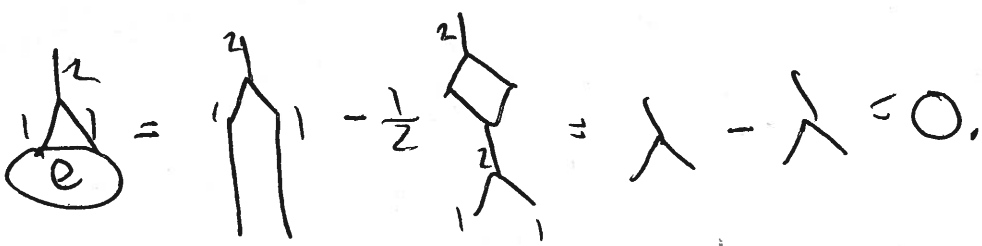
$\iff e \neq 0$ and $\mathrm{Hom}(L_{\underline{i}}, L_{\underline{j}})\, e = 0 \;\; \forall \underline{j} \in P(\mu) \; \forall \mu < \lambda$.

(b/c all summands of $L_{\underline{j}}$ are L_ν for $\nu \leq \mu < \lambda$.)

(Exercise $\iff e \neq 0$ and $\mathrm{Hom}_{<\lambda}(L_{\underline{i}}, L_{\underline{i}})\, e = 0$.)

<u>Ex</u>: What's $\leq 2\omega_1$? Just ω_2. $\mathrm{Hom}(L_1 \otimes L_1, L_2)$ spanned by 

(in $\Lambda^{+,\mathrm{dom}}_{\mathrm{wt}}$) (it's 1D by Schur lemma)

and 

The clasp is <u>orthogonal to lower terms</u>. How to check? Need to know a basis of maps to lower terms ⇝ soon.

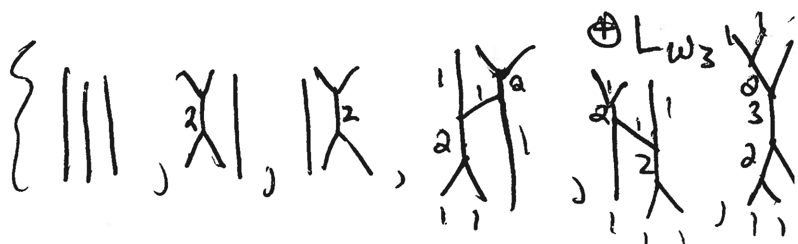
Exercise: Show that ${}_{\underline{\lambda}}\varphi_{\underline{\lambda}} \in \mathrm{End}(L_{\underline{i}})$ is equivalent to id modulo $\mathrm{Hom}_{<\lambda}(L_{\underline{i}}, L_{\underline{i}})$. Use this to deduce that ${}_{x}\varphi_{x}$ is idempotent. (given orthogonality to lower terms). Is

1) ${}_{\underline{\lambda}}\varphi_{\underline{\lambda}}$ is idemp w/ image L_λ

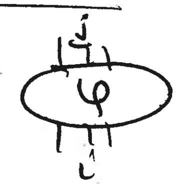
equivalent to

2) ${}_{\underline{\lambda}}\varphi_{\underline{\lambda}} \equiv \mathrm{id}$ modulo $\mathrm{Hom}_{<\lambda}$ and ${}_{\underline{i}}\varphi_{\underline{i}}$ is orthog. to lower terms?

Exercise: Compute . You can use $L_1 \otimes L_1 \otimes L_1 \cong L_{3\omega_1} \oplus L_{\omega_1+\omega_2}^{\oplus 2} \oplus L_{\omega_3}$ and that $\mathrm{End}(L_1 \otimes L_1 \otimes L_1)$ has basis

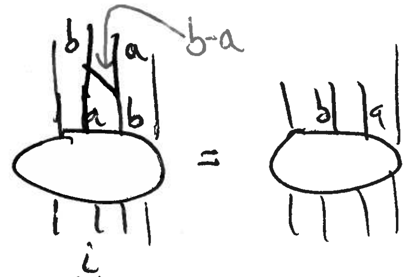


Exercise: To compute the clasp, you need only compute a single



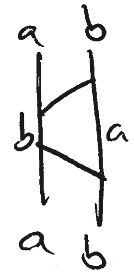
because

Thm: $\boxed{b>a}$



and sim. for $a<b$

to prove, look at



verify

$= \text{(Square flop)}$  $+ \mathrm{Hom}_{<\omega_a+\omega_b}$

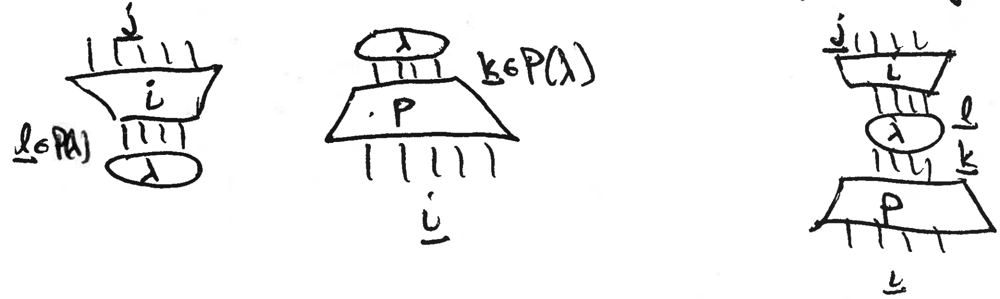
and deduce the result.

Exercise: Compute  knowing $L_2\otimes L_2 \cong L_{2\omega_2}\oplus L_{\omega_1+\omega_3}\oplus L_{\omega_4}$

To really work w/ webs we need a **basis** for Hom spaces. * TO p21.5

Recall: $\mathrm{Hom}(L_{\underline{i}}, L_{\underline{j}}) = \bigoplus_\lambda \mathrm{Hom}_\lambda(L_{\underline{i}}, L_{\underline{j}})$ span of maps $L_{\underline{i}} \to L_\lambda \to L_{\underline{j}}$

and $\mathrm{Hom}(L_\lambda, L_{\underline{j}})\otimes \mathrm{Hom}(L_{\underline{i}}, L_\lambda) \xrightarrow[\sim]{\text{compose}} \mathrm{Hom}_\lambda(L_{\underline{i}}, L_{\underline{j}})$ by Schur's lemma.



Qn: How to find a basis for $\mathrm{Hom}(L_{\underline{i}}, L_\lambda)$? Dim could be very high!!

Motto: We hate linear algebra!

Glorious Fact, one of the real reasons we love Fund!: multiplicity-free branching

$\forall \lambda \in \Lambda^{+,\mathrm{dom}}_{\mathrm{wt}}$ $\quad L_\lambda \otimes L_i \cong \bigoplus L_\nu^{\oplus c_\nu}$ and $c_\nu \in \{0,1\}$

$\forall i \geq 0$

so $\mathrm{Hom}(L_\lambda\otimes L_i, L_\nu)$ is at most 1D. Basis is unique up to rescaling!!

NO LINEAR ALG.

We'll bootstrap into a canonical (up to rescaling) basis of $\mathrm{Hom}(L_{\underline{i}}, L_\lambda)$ adapted to monoidal structure.

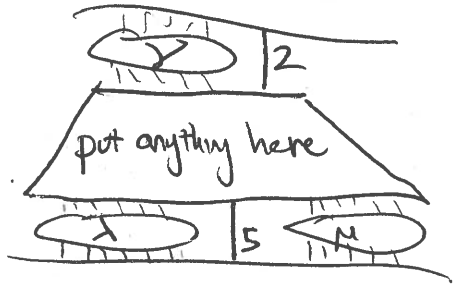
First, let's just look at $L_\lambda \otimes L_i$.

For pedagogical reasons I will develop the theory of bases assuming we can compute all the clasps! Which seems ridiculous...

But later I'll argue you can remove all the clasps and still get a basis.

So just pretend right now that clasps aren't awful linear combos but are instead nice happy black boxes... er, ovals.

$\mathrm{Hom}(L_\lambda \otimes L_5 \otimes L_\mu, L_\nu \otimes L_2)$ = span of "diagrams" of the form



← pick $\underline{i} \in P(\mu)$ but choice is irrelevant.

Steinberg $\otimes$ formula:
$$L_\lambda \otimes L_\mu = \bigoplus_{\nu \in \text{wts}(L_\mu)} (\dim L_\mu[\nu]) \cdot \text{"}L_{\lambda+\nu}\text{"}$$

Ex: $n=2$. $\text{wts}(S^2\mathbb{C}^2 = L_{2\omega_1}) = \{(2,0) \quad (1,1) \quad (0,2)\}$

so $L_{(8,4)} \otimes L_{2\omega_1} = L_{(10,4)} \oplus L_{(9,5)} \oplus L_{(8,6)}$.

Problem: $\lambda+\nu$ might not be dominant! What is $L_{\lambda+\nu}$ then?

In general, some "signed" irrep which might cancel out another summand!!

BUT — recall $\lambda = \sum c_i \omega_i$ is dominant $\iff$ $c_i \geq 0$. $\forall i$

If $c_i = -1$ for some i we say λ is "on the wall."

If λ is on the wall, then "L_λ" $= 0$.

Ex: $L_{(1,0)} \otimes L_{2\omega_1} = L_{(3,0)} \oplus L_{(2,1)} \oplus \cancel{\text{"}L_{(1,2)}\text{"}}$ 0 $\leftarrow 2\omega_2 - \omega_1$

Ex: $\lambda = (a_1, a_2, \ldots, a_n)$ $\quad a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$.

$\boxed{\lambda = (a_1-a_2)\omega_1 + (a_2-a_3)\omega_2 + \cdots + a_n\omega_n}$

$(n=7)$ $\nu = (0,1,1,0,0,1,0)$

$\lambda+\nu = (a_1, a_2+1, a_3+1, a_4, a_5, a_6+1, a_7)$ where $a_1 \overset{?}{\geq} a_2+1 \geq a_3+1 \geq a_4 \geq a_5 \overset{?}{\geq} a_6+1 \geq a_7$

if $a_1 = a_2$ then $a_1 \not\geq a_2+1$ but coeff of ω_1 is -1 on the wall!

if $a_1 > a_2$ then $a_1 \geq a_2+1$

Sim, if $a_5 = a_6$ then $\lambda+\nu$ on the wall, else $a_5 \geq a_6+1$. So $\lambda+\nu$ is dominant or on the wall.

Said another way: $\gamma = -\omega_1+\omega_3-\omega_5+\omega_6$

$$\lambda = \sum c_i\omega_i$$

so $\lambda+\gamma = (c_1-1)\omega_1 + c_2\omega_2 + (c_3+1)\omega_3 + \omega_4 + (c_5-1)\omega_5 + (c_6+1)\omega_6 + c_7\omega_7$

all coeffs ≥ -1, and if ≥ 0 then dominant.

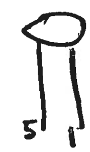
When NOT dominant? When $c_1=0$ (i.e. $a_1=a_2$) or $c_5=0$ (i.e. $a_5=a_6$).

Now, $\text{wts}(L_k) = \left\{ \underset{e_3\wedge e_5\wedge e_6\wedge e_8\wedge\ldots}{(0010110110101011\ldots)} \right.$ $\left. \begin{matrix} k \text{ ones} \\ n-k \text{ zeros} \end{matrix} \right\}$, all of mult. 1.

Ex: $L_5\otimes L_1 = L_{(11111\,00000)+(10000)} \oplus L_{(11111\,000)+(01000)} \oplus \cdots$

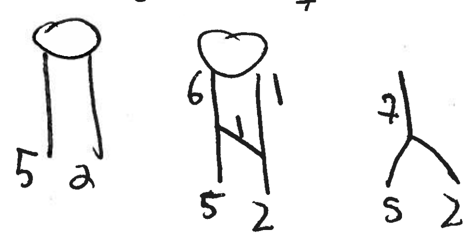
$= L_{(21111000)} \oplus \cancel{L_{(12111\,000)}} \oplus \cdots \oplus L_{(111111\,000)} \oplus \cancel{L_{(11110100)}} \oplus$

γ $\qquad 0$

$= L_{\omega_5+\omega_1} \oplus L_{\omega_6}$. Proj maps up to scalar:  and 

Ex: $L_5\otimes L_2 = L_{(22111\,000)} \oplus L_{(21111\,1\,0000)} \oplus L_{(11111\,11\,000)}$

$= L_{\omega_2+\omega_5} \oplus L_{\omega_1+\omega_6} \oplus L_{\omega_7}$.

Proj maps up to scalar: 

these are non-zero maps living in 1D hom space.

state sum: show map is nonzero on hw vector.

$\nu = (0,1,1,0,0,1,0) \in \mathrm{wts}(L_3)$, $\nu = -\omega_1 + \omega_3 - \omega_5 + \omega_6$

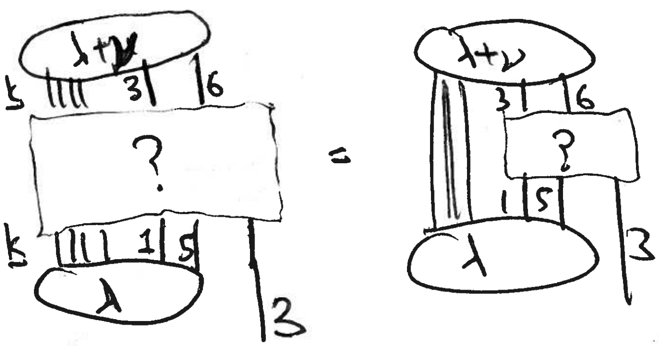
Ex: When does $L_\lambda \otimes L_3$ have summand $L_{\lambda+\nu}$? When $\lambda+\nu$ dominant

$\iff$ when $\lambda = \sum c_i \omega_i$ with $c_1, c_5 > 0$.

So $\exists\, \underline{i} \in P(\lambda)$ of the form $(\underline{k}, 1, 5)$

Let's construct map $L_\lambda \otimes L_3 \to L_{\lambda+\nu}$. FOR ALL λ!!! Note: $(\underline{k}, 3, 6) \in P(\lambda+\nu)$

remove 1,5 add 3,6.

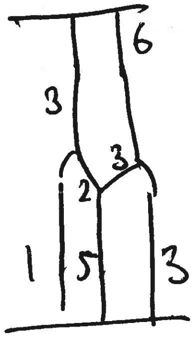


wlog, solve problem for minimal example $\lambda = \omega_1 + \omega_5$

$\lambda + \nu = \omega_3 + \omega_6$.

This is why the problem is tractable!

∞-ly many λ, but only one minimal example!!

? = 

hooray.

(To prove it works, just need to find a vector $X \in L_3$ s.t.

$v_+ \otimes X \longmapsto$ nonzero mult of v_+

Hint: $X \in L_3[\nu]$.)

EXERCISE

I call this an elementary light ladder.

You've just experienced the phenomenon of branching patterns! The pattern "$+\nu$" matched a class of λ, and the projection maps for this pattern were all constructed en masse from the minimal match.

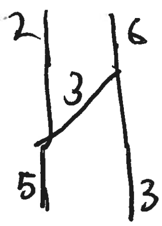
Exercise: Find all elementary light ladders for $\mathrm{wts}(L_2)$ when $n=4$.

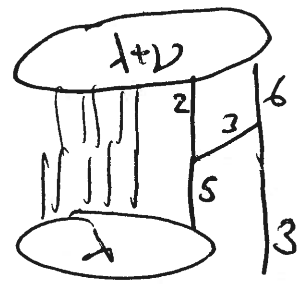
Exercise: Find all elementary light ladders for $\mathrm{wts}(L_2)$ when $n=6$.

Exercise: —————— " —————— $\mathrm{wts}(L_3)$ when $n=6$.

Ex: $\nu = (1100100) = \omega_2 - \omega_5 + \omega_6 \in \mathrm{wts}(L_3)$

minimal match: $\lambda = \omega_5$ $\quad \lambda+\nu = \omega_2+\omega_6$

ELL = 

general projection 

Ok, so we know how to project from $L_\lambda \otimes L_i$ to any of its summands. Now what?

Let's decompose $L_2 \otimes L_2 \otimes L_1$ into its direct summands $(n \gg 0)$ and find projection maps

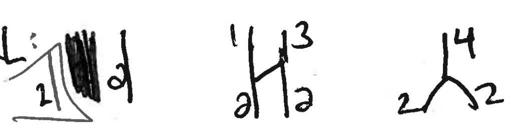
$\quad \shortparallel$

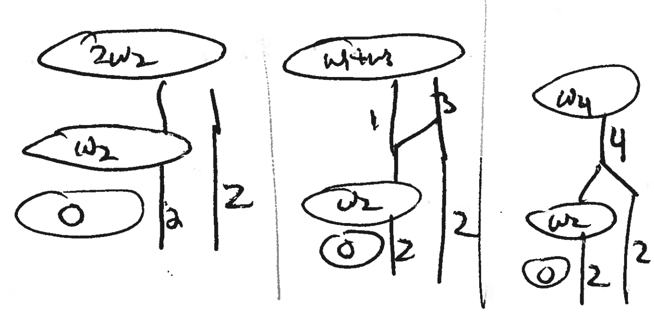
$\mathbb{1} \otimes L_2 \otimes L_2 \otimes L_1$ do it one $\otimes$ factor at a time

$\mathbb{1} \otimes L_2 \longrightarrow L_2 = L_{\omega_2}$ only summand. $\quad (\mathbb{1} = L_0$ $\quad 0+\nu$ dominant $\Leftrightarrow \nu = \omega_2$, $\nu \in \mathrm{wts}(L_2)$)

proj = 

$L_2 \otimes L_2 \cong L_{2\omega_2} \oplus L_{\omega_1+\omega_3} \oplus L_{\omega_4}$

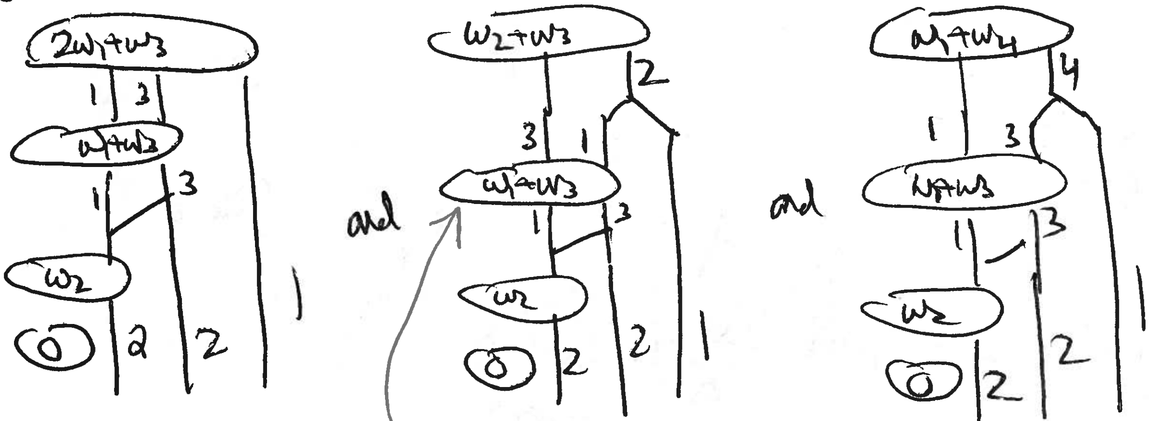
ELL: 

proj 

Now $\otimes$ w/ L_1. Let's analyze $\underset{(21100)}{L_{\omega_1+\omega_3}} \otimes L_1 = L_{(31100)} \oplus L_{(22100)} \oplus L_{(21110)}$

$+\omega_1$ $\quad$ $+\omega_2-\omega_1$ $\quad$ $+\omega_4-\omega_3$

so these lead to



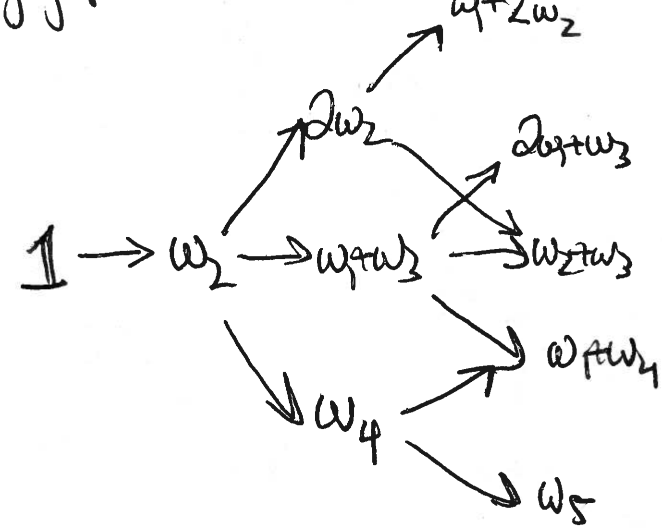
we switched the expression in $P(\omega_1+\omega_3)$

this flexibility really helps!!

Hooray, clasps > idempotents.

If we analyze $L_{2\omega_2}\otimes L_1$ and $L_{\omega_4}\otimes L_1$ similarly, we construct projections from $L_2\otimes L_2\otimes L_1$ to all summands.

Branching graph:

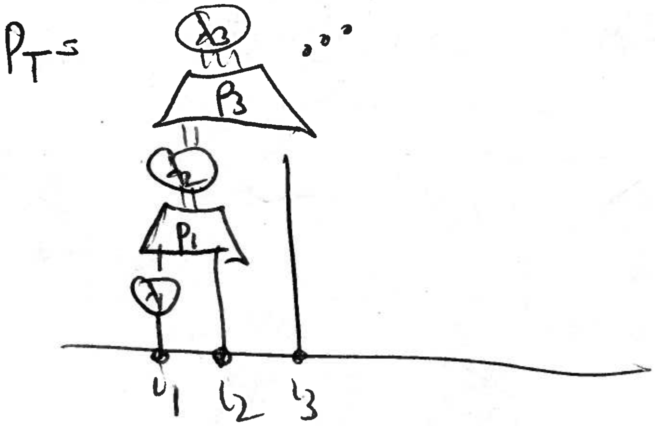


each path gave a summand of $L_2\otimes L_2\otimes L_1$ with a unique up to rescaling projection map factoring thru the intermediate summands!

So, to summarize: to each path T in the branching graph of $L_{\underline{i}}$ we have a summand $L_\lambda \overset{\oplus}{\subset} L_{\underline{i}}$ and a projection $p_T : L_{\underline{i}} \to L_\lambda$

$T = (\mathbb{1}, \lambda_1, t_2, \dots, \lambda_d = \lambda)$

$p_T =$

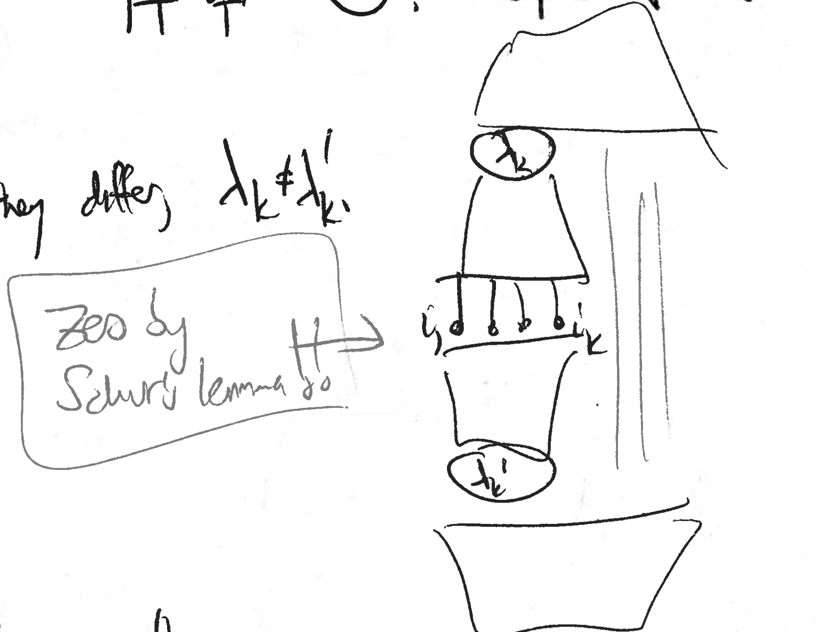


(where each p_i is $\underset{\mathrm{id}}{||||} \otimes$ ELL.)

Flipping upside-down is a symmetry of webs, gives $i_T : L_\lambda \to L_{\underline{i}}$.

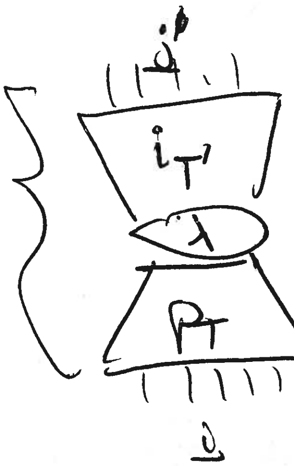
Thm: If $T \neq T'$ then $p_T \circ i_{T'} = 0$.

Pf: Look at first place they differ, $\lambda_k \neq \lambda_k'$.



This kind of structure makes sense for many other semisimple monoidal cats! We'll summarize our assumptions later.

Conclusion: A basis for $\mathrm{Hom}(L_{\underline{i}}, L_{\underline{j}})$ is



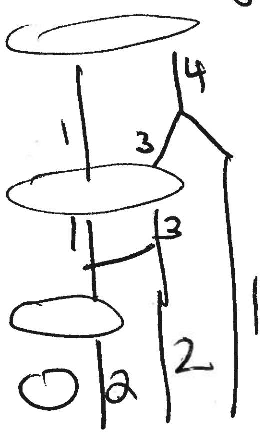
$\Bigg\{$ for $\lambda \in \Lambda^{+,\mathrm{dom}}_{wt}$, T a branching path for $\underline{i}$ ending in λ, T' — " — $\underline{j}$ — " — $\Bigg\}$

and this is an Artin-Wedderburn basis!! (by previous theorem)

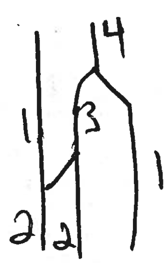
The problem: All those awful clasps. This basis is defined over $\mathbb{Q}$ not $\mathbb{Z}$, and every element is a linear combination of diagrams, impossible to compute...

The fix: what if we just ignore the clasps?

replace

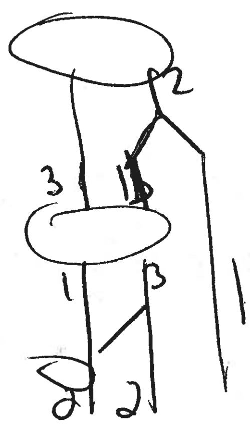


with

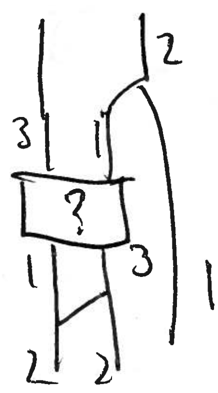


. Call it LL^{λ}_{T} instead of p_T, Π^{λ}_{T} instead of i_T.

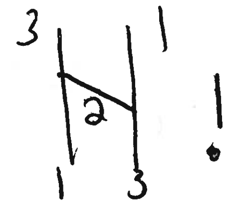
replace



with



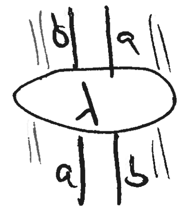
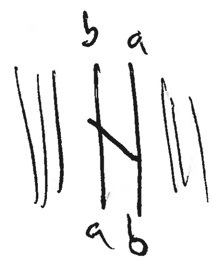
Use



!

neutral rung

What are we doing? (29)

Replacing  with $\mathrm{id}_{\underline{i}}$ or  with 

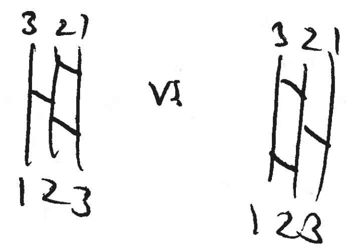
They agree modulo $\mathrm{Hom}_{<\lambda}$!!

By a terribly convoluted induction argument:

$\{\Gamma\Gamma_{T'}^{\lambda} \circ LL_T^{\lambda}\}$ is a basis of $\mathrm{Hom}(L_{\underline{i}}, L_{\underline{j}})$

w/ upper triangular c.o.b. matrix to $\{\iota_{T'}^{\lambda} \circ p_T^{\lambda}\}$

But $\Gamma\Gamma_{T'} \circ LL_T$ is just a diagram, not a linear combo of diagrams!

(not canonical though, make some choices. E.g. )

Look ma, no clasps!

Don't get an A-W basis, but don't expect one over $\mathbb{Z}$ anyway!!

<u>Moral</u>: In Webs, we don't have a direct handle on L_λ without computing clasps.

Can't access Hom_λ. But we can access $\mathrm{Hom}_{<\lambda}$ because it agrees with maps factoring thru $L_{\underline{i}}$ for $\underline{i} \in P(\mu)$ for $\mu < \lambda$.

Don't have clasp, but have id or "neutral ladder" which agrees modulo $\mathrm{Hom}_{<\lambda}$.

Good enough to crank the abstract machinery!

What really makes this tick: (assumptions) For semisimple $\otimes$ cat $\mathcal{C}$:

1) Monoidal unitriangularity: $\forall \lambda \in \mathrm{Irr}\,\mathcal{C}$ $\exists$ nonempty set $P(\lambda)$ of objects $\underline{i}$ s.th. $L_\lambda \overset{\oplus}{\underset{1}{\subset}} L_{\underline{i}}$ and all other summands $L_\mu \overset{\oplus}{\subset} L_{\underline{i}}$ satisfy $\mu < \lambda$ (for some p.o. on $\mathrm{Irr}\,\mathcal{C}$)

This fails for other popular approaches to rep theory, e.g. the full subcat $\{V^{\otimes k}\}_{k \geq 0}$, which does contain every irrep as a summand.

M.U. allows us to study filtration $\mathrm{Hom}_{\leq \lambda}$ on $\mathcal{C}$ using objects in $\mathcal{C}$, and thus to understand Hom_λ as $\mathrm{Hom}_{\leq \lambda} / \mathrm{Hom}_{< \lambda}$.

Here $\mathrm{Hom}_{\leq \lambda} = \left\{ \begin{array}{l} \text{Span of maps factoring thru } L_{\underline{i}} \text{ for } \underline{i} \in P(\mu) \\ \text{for } \mu \leq \lambda \end{array} \right\}$.

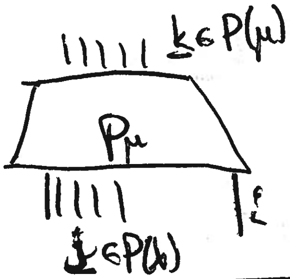
2) Multiplicity-free branching: this is OPTIONAL.
It removes linear algebra from the picture, but won't ~~work~~ hold outside type A.
E.g. Fund rep ~~[illegible]~~ might have 2D weight space... a little lin. alg. not a lot.

3) Lws vectors: A technical tool for proving correctness of elementary projections, a nice testing spot – but there are alternatives.

4) Branching patterns: Surprisingly ubiquitous! Finite work $\rightsquigarrow$ infinite pleasure

Summary: • Within the integral form $\mathrm{Webs}^+_{\mathbb{Z}}$ (not $\mathrm{Webs}^+ \otimes_{\mathbb{Z}} \mathbb{Q}$) 31

for each λ and $\underline{i}$ and summand $L_\mu \subseteq L_\lambda \otimes L_{\underline{i}}$ we have

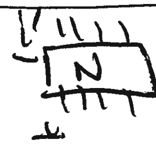


which descends in $\mathrm{Webs}_{\mathbb{Q}}$ (where clasps exist) to

a nonzero map

.

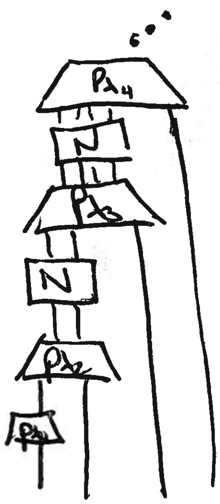
(*) • For $\underline{i}, \underline{i}' \in P(\lambda)$ have neutral map ${}_{\underline{i}'}N_{\underline{i}}$:



• Using this, for each branching path $T = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ subordinate to $\underline{j}$ we have

Write $T \in E(\underline{j}, \lambda_d)$

$P_T =$



.

From $\underline{j}$ to $\underline{k} \in P(\lambda_d)$.

Similarly $i_T : \underline{k} \in P(\lambda_d)$ to $\underline{j}$.

• If $T \in E(\underline{j}, \lambda)$ then $S \in E(\underline{j}', \lambda)$

$\mathbb{LL}^\lambda_{S,T} := i_S \circ P_T : \underline{j} \to \underline{j}'$

and $\{\mathbb{LL}^\lambda_{S,T}\}$ is basis for $\mathrm{Hom}(\underline{j}, \underline{j}')$, upper tri cob to an A-W basis

• Now $\mathrm{Webs}^+_{\mathbb{Z}}$ is an <u>multi-object adapted cellular category</u>: lots of free machinery to study its rep theory.

$\Rightarrow$ which descends in $\mathrm{Webs}_{\mathbb{Q}}$ to nonzero map

.

This helps you present your category too!

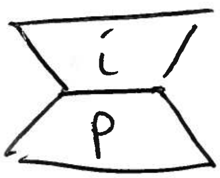
<u>Obj generators</u>: enough to get monoidal unitriangularity, but keep the branching graph simple w/ low multiplicities

<u>Morph generators</u>: enough to have all P_T and all N.

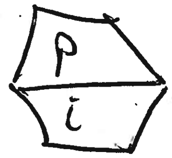
<u>Relations</u>: Enough to make $\{LL_{S,T}\}$ span!

goes back up →

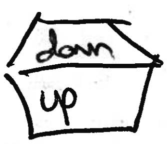
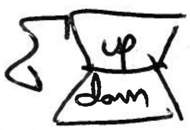
goes down in dominance order →

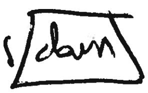
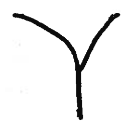


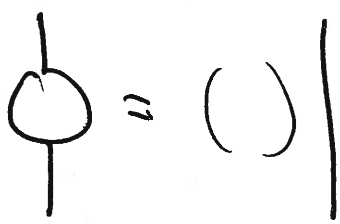
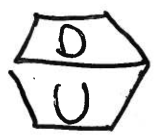
good

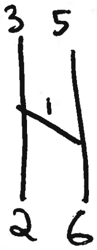
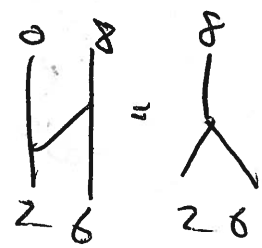


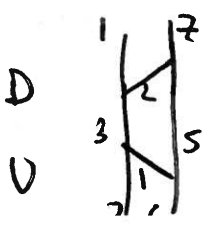
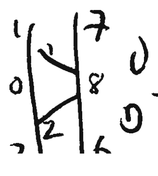
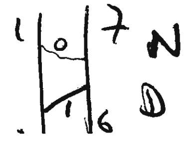
bad

any  should go to $\sum$ .

<u>Ex</u>: $\omega_{i+j} < \omega_i + \omega_j$ so  is   is 

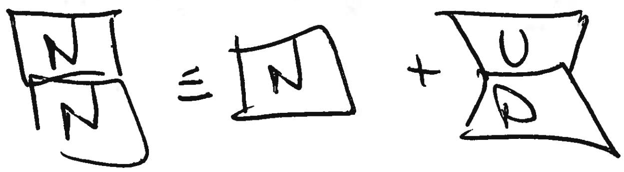
 replace  with $\boxed{N}$. Yay.

<u>Ex</u>:  is up,  is down (think  is D)

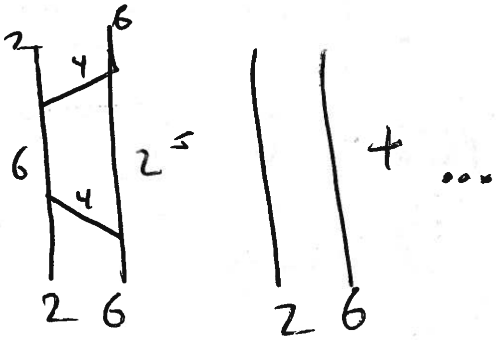
 $= c_1$  $+ c_2$  Yay.

Additional Considerations: $\underline{i},\underline{i}' \in P(A)$, $\mathrm{Hom}(\underline{i},\underline{i}') / \mathrm{Hom}_{<}(\underline{i},\underline{i}')$ is 1D.

so neutral maps live in 1D space m.l.t.



Ex:



There are a few more ingredients but there is almost a general proof of correctness of presentation requiring relations of a certain form...

Clasps | **Def:** Fix $\lambda \in \Lambda$ w/ reduced expressions $P(\lambda)$. The clasp, if it exists, is a family of morphisms ${}_{\underline{j}}\varphi_{\underline{i}} : \underline{i} \to \underline{j}$ for $\underline{i}, \underline{j} \in P(\lambda)$ satisfying

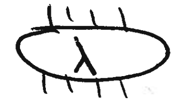
1. ${}_{\underline{k}}\varphi_{\underline{j}} \circ {}_{\underline{j}}\varphi_{\underline{i}} = {}_{\underline{k}}\varphi_{\underline{i}}$
2. ${}_{\underline{i}}\varphi_{\underline{i}} = id_{\underline{i}}$ modulo $I_{<\lambda}$
3. ${}_{\underline{j}}\varphi_{\underline{i}}\, a = b\, {}_{\underline{j}}\varphi_{\underline{i}} = 0$ for any $a, b \in I_{<\lambda}$

(This way of formulating it obviates the need for discussing the object L_λ.)

Exercise: The clasp is unique if it exists.

Finding a closed formula for the clasp as a linear combo of webs seems out of reach. It was done by Morrison for gl_3 and it is complicated! But inductive formulas are philosophically important and practical too. Triple Clasp expansion -

Suppose we have computed all the clasps less than $\lambda + \varpi_k$, including λ.

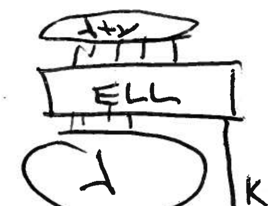


ovals

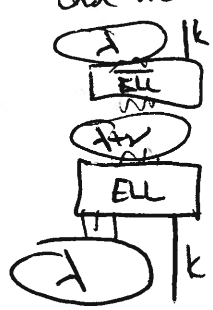
We know $L_\lambda \otimes L_{\varpi_k} = \bigoplus L_{\lambda+\nu}$ for $\nu \in wts(L_{\varpi_k})$ w/ $\lambda+\nu$ dominant.

One summand is $L_{\lambda+\varpi_k}$, we want this idempotent so we want to subtract off all the others!

dim $Hom(L_\lambda \otimes L_k, L_{\lambda+\nu}) = 1$ and we have a basis for it:

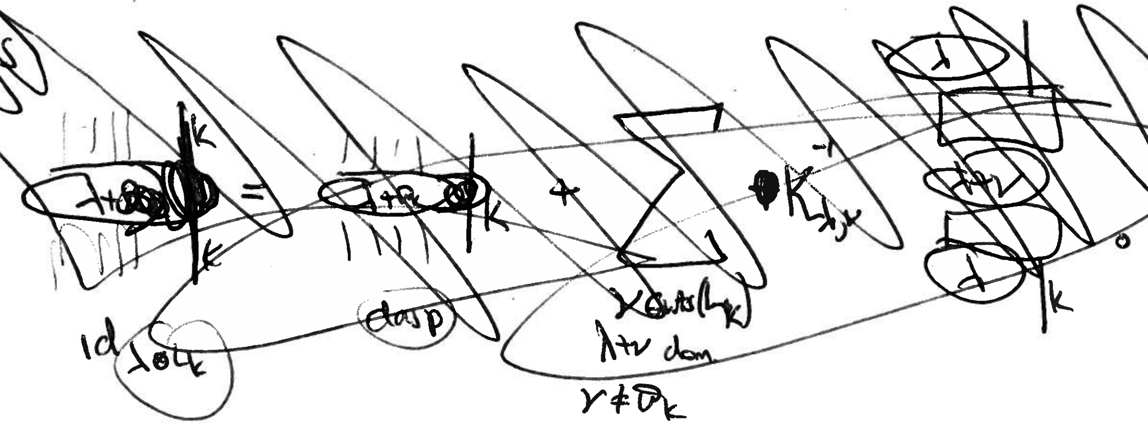


So the idempotent is $K^{-1}_{\lambda,\nu}$



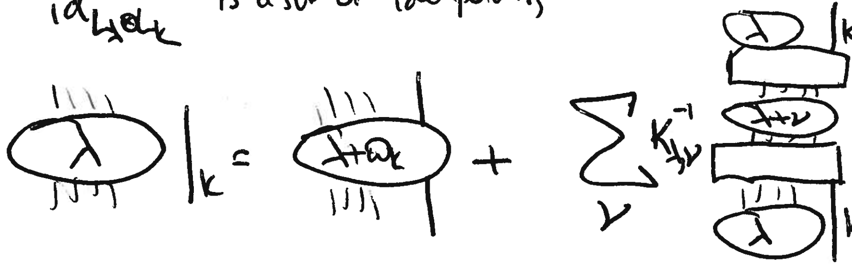
for some scalar $K^{-1}_{\lambda,\nu}$. reason for inverse coming soon

~~Thus~~



~~Great, completely formal argument~~

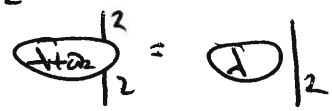
Thus $\mathrm{id}_{L_\lambda \otimes L_k}$ is a sum of idempotents, or

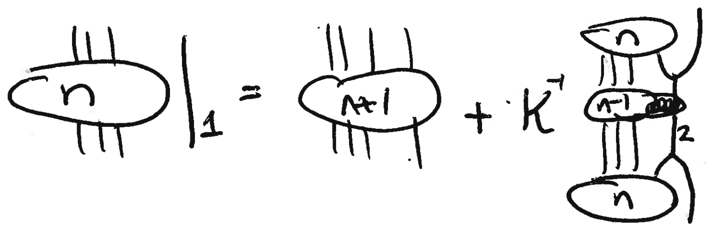


A formal argument gives this recursive formula!

Ex: $\mathfrak{gl}_2$, write (n) for $n\omega_1 + x\omega_2$ where x determined from context –

ω_2 is determinant rep and



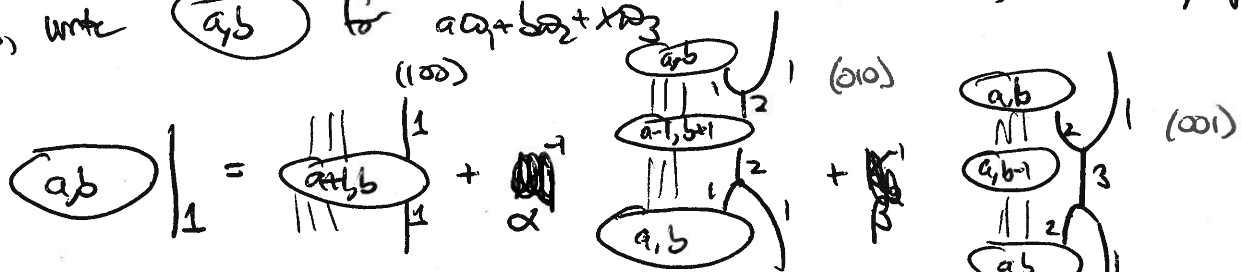


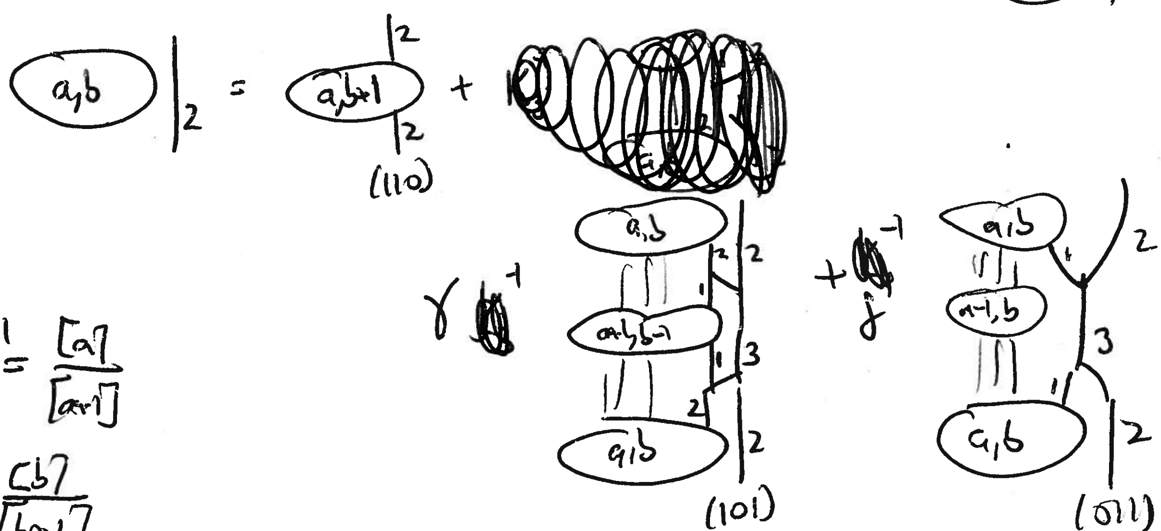
and $K^{-1} = \frac{[n]}{[n+1]}$

$\boxed{\text{Note: 3 clasps have a name.}}$

term exists only when $n > 0$! But when $n = 0$, $K^{-1} = 0$ anyway!

Ex: $\mathfrak{gl}_3$, write (a,b) for $a\omega_1 + b\omega_2 + x\omega_3$





$\alpha^{-1} = \frac{[a]}{[a+1]}$

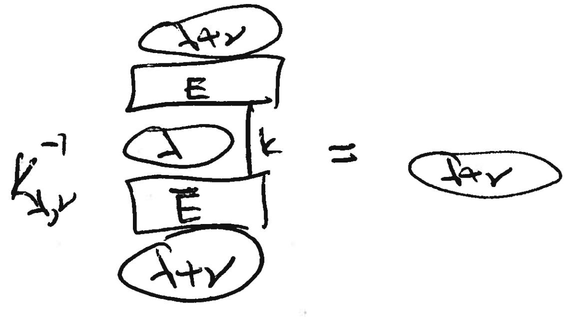
$\gamma^{-1} = \frac{[b]}{[b+1]}$

$\beta^{-1} = \frac{[b][a+b+1]}{[b+1][a+b+2]}$

$\gamma^{-1} = \frac{[a][a+b+1]}{[a+1][a+b+2]}$

How to find coeffs? ☹

Well, it's supposed to be $ip = e$, with $pi = \mathrm{id}_{L_{\lambda+\nu}}$. So want

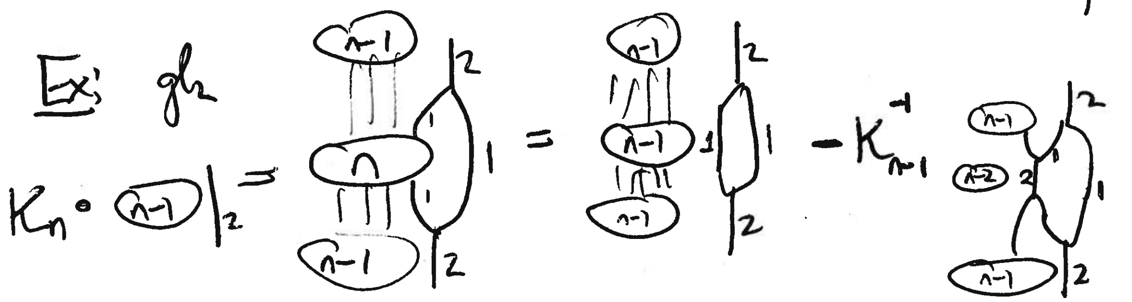


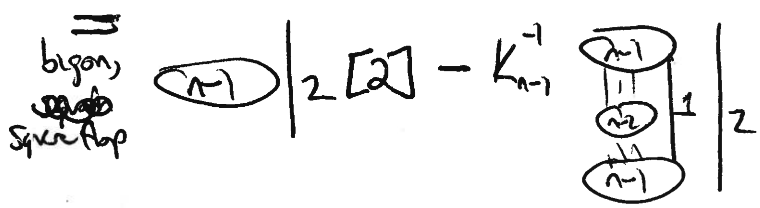
, i.e. $K_{\lambda,\nu}$ is coeff of id, i.e.

$K_{\lambda,\nu}$ = cellular pairing of E with itself (a 1×1 matrix)
hooray for multiplicity freeness.

We can just compute this

<u>Ex:</u> gl_2





So $K_n = [2] - K^{-1}_{n-1}$.

As noted $K_0^{-1} = 0$ since this term does not exist. Can solve, get $K_n = \frac{[n+1]}{[n]}$

Using the formal recursion for clasp, get recursion for coeffs in that formal recursion.

<u>Ex:</u> gl_3 get $\alpha_{a,b} = [2] - \alpha^{-1}_{a-1,b+1}$ and $\alpha^{-1}_{0,b} = 0$

$\rightsquigarrow \alpha_{a,b} = \frac{[a+1]}{[a]}$

get $\beta_{a,b} = [3] - \frac{\alpha_{a+1,b-2}}{\gamma_{a,b-1}} - \frac{1}{\delta_{a,b-1}}$ and similar for δ...

quite complicated !!

<u>Rmk:</u> (Reverse) dominance order on ν controls which coeffs can appear in recursive formula for which others.

Morally it's clear what to do! Compute various $K_{\lambda,\nu}$ by finding recursive formulas + solving them. Finding + Solving each take lots of work, but it's still tractable thanks to branching patterns. C4

Thm (E conjecture '16, Martin-Spacek '22)
$$K_{\lambda,\nu} = \prod_{\substack{\text{pos roots } \alpha \\ \text{inverted in } \nu}} \frac{\langle \lambda+\rho, \alpha\rangle}{\langle \lambda+\rho, \alpha\rangle - 1}$$

Ex: $\nu = (0101100)$ — this set is $\varepsilon_1-\varepsilon_2, \varepsilon_1-\varepsilon_4, \varepsilon_1-\varepsilon_5$, $\varepsilon_3-\varepsilon_4, \varepsilon_3-\varepsilon_5$

so if $\lambda = c_1\omega_1 + \dots + c_n\omega_n$ then
$$\lambda+\rho = (c_1+1)\omega_1 + \dots + (c_{n-1}+1)\omega_{n-1} + c_n\omega_n$$
$$\langle \lambda+\rho, \varepsilon_1-\varepsilon_2\rangle = c_1+1$$
$$\langle \lambda+\rho, \varepsilon_1-\varepsilon_4\rangle = (c_1+1)+(c_2+1)+(c_3+1)$$

$\langle \nu, \varepsilon_1-\varepsilon_2\rangle = -1$
$\langle \nu, \varepsilon_1-\varepsilon_4\rangle = -1$ etc

so denom is also $\langle \lambda+\rho+\nu, \alpha\rangle$

$$K_{\lambda,\nu} = \frac{(c_1+1)(c_1+c_2+c_3+3)(c_1+c_2+c_3+c_4+4)(c_3+1)(c_3+c_4+2)}{(c_1)(c_1+c_2+c_3+2)(c_1+c_2+c_3+c_4+3)(c_3)(c_3+c_4+1)}$$

After reconfiguring M-S relate to work of Tolstoy '90, but I'm still not sure there is a good combinatorial or philosophical explanation.