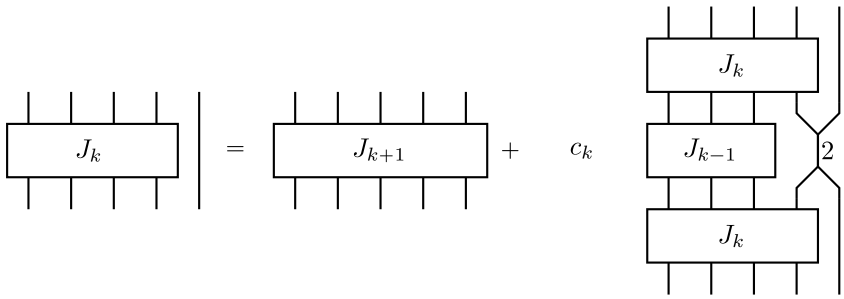
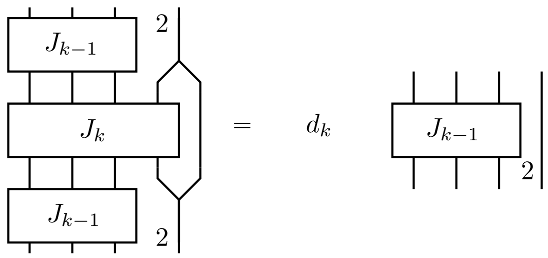
Besse 2018, Lecture I

Main Exercise 1. The goal of this exercise is to explore the double-recursion that allows one to compute the Jones-Wenzl projectors, which are the clasps for $\mathfrak{gl}_2$. We work in the category Webs_2^+. We will inductively define endomorphisms J_k of $L_1^{\otimes k}$ by the following recursive formula, whose coefficients c_k are not yet determined. All unmarked strands are labeled 1.



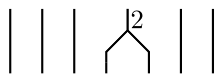
(1)

The following formula will define certain scalars d_k, when $k > 1$.



(2)

(a) This diagram is the merging of two strands:



(3)

(Sanity check: there are $k-1$ such mergings we can apply to $L_1^{\otimes k}$.) Assume that J_k is killed by postcomposition with any merging. Find the relationship between c_k and d_k which makes it true for J_{k+1} as well.

(b) Assuming J_k is an idempotent, prove that J_{k+1} is an idempotent.

(c) Use (1) for $k-1$ to compute d_k in terms of c_{k-1}.

(d) You now have a double recursion, which you can use to solve for both c_k and d_k. The base case is $c_0 = 0$.

(e) What is the representation-theoretic meaning of the formula (1)? (Hint: What is $L_{k\omega_1} \otimes L_1$?)

Besse 2018, Lecture I supplementary exercises

Exercise 1. Explain what the square flop relation really says, in terms of the "state model" evaluation using subsets of $\{1, \dots, n\}$. (You can assume $q = 1$ for this exercise.)

Exercise 2. Using webs, prove that the determinant representation L_n tensor-commutes with L_k for any $0 \leqslant k \leqslant n$. That is, find isomorphisms $L_n \otimes L_k \to L_k \otimes L_n$ and their inverses for each k. If you're feeling ambitious, you can prove that these isomorphisms satisfy some naturality properties with respect to morphisms.

Here is the q-deformed formula for multiplication and comultiplication:

$$m(e_S \otimes e_{S'}) = \begin{cases} (-q)^{\ell(S,S')} e_{S \cup S'} & \text{if } S \text{ and } S' \text{ are disjoint} \\ 0 & \text{else,} \end{cases} \tag{4}$$

$$\Delta(e_T) = \sum_{S \cup S' = T} (-1)^{\ell(S,S')} q^{-\ell(S',S)} e_S \otimes e_{S'}. \tag{5}$$

In these formulas S is forced to have size i, S' to have size j, and T to have size $i + j$, when the map relates $L_i \otimes L_j$ and L_{i+j}.

Exercise 3. Verify that m is associative.

Exercise 4. Verify that $m \circ \Delta = \binom{i+j}{i}_q \mathrm{id}$. We did this in class when $q = 1$.

Exercise 5. Return to the setting $n = 2$ of the main exercise. Inside $\mathrm{End}(L_1^{\otimes k})$, let T denote the space of maps which are killed by postcomposition with any merging (3), and B the space of maps which are killed by precomposition with any splitting. For a web $x \in \mathrm{End}(L_1^{\otimes k})$, let $\overline{x}$ denote x flipped upside-down. For example, $x \in T$ if and only if $\overline{x} \in B$.

We assume a fact which will follow from the double ladders basis later in the course: the endomorphism ring of $L_1^{\otimes k}$ has a basis of diagrams, where one diagram is the identity, and the remaining diagrams have both a merging on bottom and a split on top. When we talk about the coefficient of the identity, we refer to this basis. If a morphism ever factors through an object with fewer than k tensor factors, its coefficient of the identity will be zero.

(a) We now make the assumption (*): there exists some $f \in T$ for which the coefficient of the identity diagram is invertible. Why is this equivalent to the analogous assumption for B?

(b) Let $f \in T$, with invertible coefficient c for the identity diagram. Let $g \in B$, with invertible coefficient d for the identity diagram. Compute the composition gf in two ways, and deduce that f and g are colinear.

(c) Assuming (*) deduce that $T = B$, and the space is one-dimensional, and that $f = \overline{f}$ for $f \in T$.

(d) Thus, assuming (*), there is a unique element $J_k \in T$ whose identity coefficient is 1. Prove that J_k is idempotent.

(e) In the main exercise you constructed a morphism J_k which satisfied T. Prove using the recursive formula (1) that its identity coefficient is 1.

(f) The recursive formula (1) only works when the coefficients c_k are in the base ring. What is the smallest extension of $\mathbb{Z}[q, q^{-1}]$ (or $\mathbb{Z}$, when $q = 1$) where (*) will hold for all k?