Besse 2018, Lecture II

Main Exercise 1. This exercise gives practice with plethysm paths and light ladders.

(a) Compute all the plethysm paths for $\underline{i} = (2,3,2)$ when $n = 17$. (Hint: There are 13, with one being different than all the others.)

(b) For each of the plethysm paths, draw the clasped light ladder. (You need not compute the clasps, but you should draw the elementary light ladders and neutral ladders.)

(c) What is the dimension of $\mathrm{End}(L_{(2,3,2)})$? Draw all the clasped double ladders. (You can stop when you feel like it becomes busywork.)

Main Exercise 2. This exercise tries to motivate why the neutral ladders and elementary light ladders function as they should, using the subset state model to evaluate webs. Recall that this model counts consistent labelings of strands by subsets of $\{1, 2, \ldots, n\}$.

Definition 1. The *highest label* that one can put on the strands in a sequence $\underline{i} = (a, b, \ldots, c)$ is $(\{1, \ldots, a\}, \{1, \ldots, b\}, \ldots, \{1, \ldots, c\})$.

(a) Let N be a neutral ladder, and label the output strands with the highest label. Prove that there are no consistent labelings if the source is not labeled with the highest label, and there is exactly one consistent labeling if it is. (Starting example: the neutral rung from $(5,7)$ to $(7,5)$.)

(b) Let E be an elementary light ladder which models the projection $L_\lambda \otimes L_k \to L_{\lambda+\nu}$. Label the output strands with the highest label. Label the source strands with the highest label, except for the final input strand k, which is labeled with a subset S. Prove that there is a unique consistent labeling, and that the unique set S which has this consistent labeling corresponds to ν in some way. (Starting examples: the elementary ladder for $\nu = (0,1,0,0,1,0)$, and then for $\nu = (0,1,1,0,1,0)$.)

Note: a supplementary exercise will tease out why these computations imply that neutral and elementary light ladders descend to the maps they should.

Besse 2018, Lecture II supplementary exercises

Exercise 1. Draw the basis of clasped double ladders for $\mathrm{Hom}(L_{(2,3,2)}, L_{(1,1,5)})$.

Exercise 2. Continue the ideas of Main Exercise 2b. Let E be an elementary light ladder, and label the input strands with the highest label, except for the final input strand k, labeled with S. (Do not assume the output has the highest label.) Show that whenever S corrresponds to a weight μ which is higher than ν in the dominance order, then there are no consistent labels. (Starting example: the elementary ladder for $\nu = (0, 1, 1, 0, 1, 0)$.)

Exercise 3. Suppose that $\underline{i} \in P(\lambda)$, so that $L_\lambda \overset{\oplus}{\subset} L_{\underline{i}}$. Let Y be an arbitrary representation of $\mathfrak{gl}_n$. We claim that a morphism $f \in \mathrm{Hom}(L_{\underline{i}}, Y)$ descends to a nonzero morphism in $\mathrm{Hom}(L_\lambda, Y)$ if and only if $f(e_+) \neq 0$. Prove it, and a similar statement about $\mathrm{Hom}(Y, L_{\underline{i}})$ and $\mathrm{Hom}(Y, L_\lambda)$. Why does this imply that neutral ladders and elementary light ladders descend to the appropriate morphisms?