28. Determine the a_n so that the equation

$$\sum_{n=1}^{\infty} na_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

is satisfied. Try to identify the function represented by the series $\sum_{n=0}^{\infty} a_n x^n$.

5.2 Series Solutions Near an Ordinary Point, Part I

In Chapter 3 we described methods of solving second order linear differential equations with constant coefficients. We now consider methods of solving second order linear equations when the coefficients are functions of the independent variable. In this chapter we will denote the independent variable by x. It is sufficient to consider the homogeneous equation

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0, \tag{1}$$

since the procedure for the corresponding nonhomogeneous equation is similar.

Many problems in mathematical physics lead to equations of the form (1) having polynomial coefficients; examples include the Bessel equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

where ν is a constant, and the Legendre equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

where α is a constant. For this reason, as well as to simplify the algebraic computations, we primarily consider the case in which the functions P, Q, and R are polynomials. However, as we will see, the method of solution is also applicable when P, Q, and R are general analytic functions.

For the present, then, suppose that P, Q, and R are polynomials and that there is no factor $(x - c)$ that is common to all three of them. If there is such a factor $(x - c)$, then divide it out before proceeding. Suppose also that we wish to solve Eq. (1) in the neighborhood of a point x_0. The solution of Eq. (1) in an interval containing x_0 is closely associated with the behavior of P in that interval.

A point x_0 such that $P(x_0) \neq 0$ is called an **ordinary point**. Since P is continuous, it follows that there is an interval about x_0 in which $P(x)$ is never zero. In that interval we can divide Eq. (1) by $P(x)$ to obtain

$$y'' + p(x)y' + q(x)y = 0, \tag{2}$$

where $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$ are continuous functions. Hence, according to the existence and uniqueness Theorem 3.2.1, there exists in that interval a unique solution of Eq. (1) that also satisfies the initial conditions $y(x_0) = y_0$, $y'(x_0) = y_0'$ for arbitrary values of y_0 and y_0'. In this and the following section, we discuss the solution of Eq. (1) in the neighborhood of an ordinary point.

On the other hand, if $P(x_0) = 0$, then x_0 is called a **singular point** of Eq. (1). In this case at least one of $Q(x_0)$ and $R(x_0)$ is not zero. Consequently, at least one of the coefficients p and q in Eq. (2) becomes unbounded as $x \to x_0$, and therefore Theorem 3.2.1 does not apply in this case. Sections 5.4 through 5.7 deal with finding solutions of Eq. (1) in the neighborhood of a singular point.

We now take up the problem of solving Eq. (1) in the neighborhood of an ordinary point x_0. We look for solutions of the form

$$y = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n + \cdots = \sum_{n=0}^{\infty} a_n(x - x_0)^n \tag{3}$$

and assume that the series converges in the interval $|x - x_0| < \rho$ for some $\rho > 0$. While at first sight it may appear unattractive to seek a solution in the form of a power series, this is actually a convenient and useful form for a solution. Within their intervals of convergence, power series behave very much like polynomials and are easy to manipulate both analytically and numerically. Indeed, even if we can obtain a solution in terms of elementary functions, such as exponential or trigonometric functions, we are likely to need a power series or some equivalent expression if we want to evaluate them numerically or to plot their graphs.

The most practical way to determine the coefficients a_n is to substitute the series (3) and its derivatives for y, y', and y'' in Eq. (1). The following examples illustrate this process. The operations, such as differentiation, that are involved in the procedure are justified so long as we stay within the interval of convergence. The differential equations in these examples are also of considerable importance in their own right.

EXAMPLE 1

Find a series solution of the equation

$$y'' + y = 0, \qquad -\infty < x < \infty. \tag{4}$$

As we know, $\sin x$ and $\cos x$ form a fundamental set of solutions of this equation, so series methods are not needed to solve it. However, this example illustrates the use of power series in a relatively simple case. For Eq. (4), $P(x) = 1$, $Q(x) = 0$, and $R(x) = 1$; hence every point is an ordinary point.

We look for a solution in the form of a power series about $x_0 = 0$

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots = \sum_{n=0}^{\infty} a_n x^n \tag{5}$$

and assume that the series converges in some interval $|x| < \rho$. Differentiating Eq. (5) term by term, we obtain

$$y' = a_1 + 2a_2 x + \cdots + n a_n x^{n-1} + \cdots = \sum_{n=1}^{\infty} n a_n x^{n-1} \tag{6}$$

and

$$y'' = 2a_2 + \cdots + n(n-1)a_n x^{n-2} + \cdots = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}. \tag{7}$$

Substituting the series (5) and (7) for y and y'' in Eq. (4) gives

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

To combine the two series, we need to rewrite at least one of them so that both series display the same generic term. Thus, in the first sum, we shift the index of summation by replacing n by $n+2$ and starting the sum at 0 rather than 2. We obtain

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}a_n x^n = 0$$

or

$$\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} + a_n]x^n = 0.$$

For this equation to be satisfied for all x, the coefficient of each power of x must be zero; hence we conclude that

$$(n+2)(n+1)a_{n+2} + a_n = 0, \qquad n = 0,1,2,3,\ldots. \tag{8}$$

Equation (8) is referred to as a **recurrence relation**. The successive coefficients can be evaluated one by one by writing the recurrence relation first for $n=0$, then for $n=1$, and so forth. In this example Eq. (8) relates each coefficient to the second one before it. Thus the even-numbered coefficients $(a_0, a_2, a_4, \ldots)$ and the odd-numbered ones $(a_1, a_3, a_5, \ldots)$ are determined separately. For the even-numbered coefficients we have

$$a_2 = -\frac{a_0}{2\cdot 1} = -\frac{a_0}{2!}, \qquad a_4 = -\frac{a_2}{4\cdot 3} = +\frac{a_0}{4!}, \qquad a_6 = -\frac{a_4}{6\cdot 5} = -\frac{a_0}{6!}, \ldots.$$

These results suggest that in general, if $n = 2k$, then

$$a_n = a_{2k} = \frac{(-1)^k}{(2k)!}a_0, \qquad k = 1,2,3,\ldots. \tag{9}$$

We can prove Eq. (9) by mathematical induction. First, observe that it is true for $k=1$. Next, assume that it is true for an arbitrary value of k and consider the case $k+1$. We have

$$a_{2k+2} = -\frac{a_{2k}}{(2k+2)(2k+1)} = -\frac{(-1)^k}{(2k+2)(2k+1)(2k)!}a_0 = \frac{(-1)^{k+1}}{(2k+2)!}a_0.$$

Hence Eq. (9) is also true for $k+1$, and consequently it is true for all positive integers k.

Similarly, for the odd-numbered coefficients

$$a_3 = -\frac{a_1}{2\cdot 3} = -\frac{a_1}{3!}, \qquad a_5 = -\frac{a_3}{5\cdot 4} = +\frac{a_1}{5!}, \qquad a_7 = -\frac{a_5}{7\cdot 6} = -\frac{a_1}{7!}, \ldots,$$

and in general, if $n = 2k+1$, then[2]

$$a_n = a_{2k+1} = \frac{(-1)^k}{(2k+1)!}a_1, \qquad k = 1,2,3,\ldots. \tag{10}$$

Substituting these coefficients into Eq. (5), we have

[2]The result given in Eq. (10) and other similar formulas in this chapter can be proved by an induction argument resembling the one just given for Eq. (9). We assume that the results are plausible and omit the inductive argument hereafter.

$$y = a_0 + a_1 x - \frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5$$

$$+ \cdots + \frac{(-1)^n a_0}{(2n)!}x^{2n} + \frac{(-1)^n a_1}{(2n+1)!}x^{2n+1} + \cdots$$

$$= a_0\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots\right]$$

$$+ a_1\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + \cdots\right]$$

$$= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}. \quad (11)$$

Now that we have formally obtained two series solutions of Eq. (4), we can test them for convergence. Using the ratio test, we can show that each of the series in Eq. (11) converges for all x, and this justifies retroactively all the steps used in obtaining the solutions. Indeed, we recognize that the first series in Eq. (11) is exactly the Taylor series for $\cos x$ about $x = 0$ and that the second is the Taylor series for $\sin x$ about $x = 0$. Thus, as expected, we obtain the solution $y = a_0 \cos x + a_1 \sin x$.

Notice that no conditions are imposed on a_0 and a_1; hence they are arbitrary. From Eqs. (5) and (6) we see that y and y' evaluated at $x = 0$ are a_0 and a_1, respectively. Since the initial conditions $y(0)$ and $y'(0)$ can be chosen arbitrarily, it follows that a_0 and a_1 should be arbitrary until specific initial conditions are stated.

Figures 5.2.1 and 5.2.2 show how the partial sums of the series in Eq. (11) approximate $\cos x$ and $\sin x$. As the number of terms increases, the interval over which the approximation is satisfactory becomes longer, and for each x in this interval the accuracy of the approximation improves. However, you should always remember that a truncated power series provides only a local approximation of the solution in a neighborhood of the initial point $x = 0$; it cannot adequately represent the solution for large $|x|$.

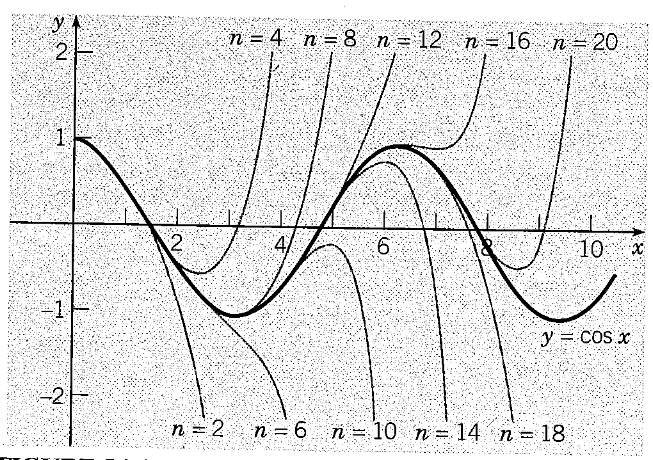

FIGURE 5.2.1 Polynomial approximations to $\cos x$. The value of n is the degree of the approximating polynomial.

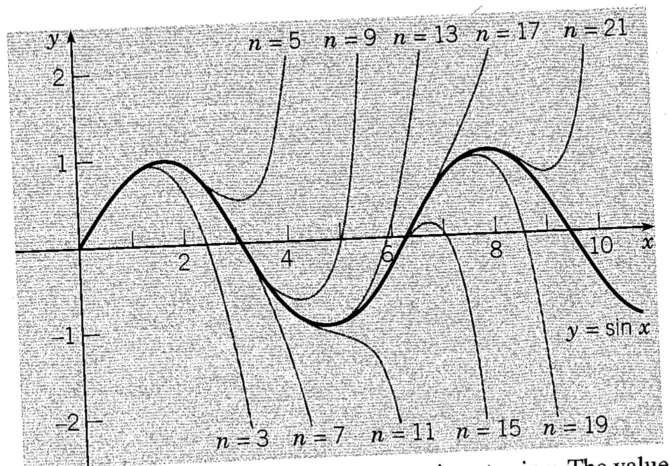

FIGURE 5.2.2 Polynomial approximations to $\sin x$. The value of n is the degree of the approximating polynomial.

In Example 1 we knew from the start that $\sin x$ and $\cos x$ form a fundamental set of solutions of Eq. (4). However, if we had not known this and had simply solved Eq. (4) using series methods, we would still have obtained the solution (11). In recognition of the fact that the differential equation (4) often occurs in applications, we might decide to give the two solutions of Eq. (11) special names, perhaps

$$C(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \qquad S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}. \tag{12}$$

Then we might ask what properties these functions have. For instance, can we be sure that $C(x)$ and $S(x)$ form a fundamental set of solutions? It follows at once from the series expansions that $C(0) = 1$ and $S(0) = 0$. By differentiating the series for $C(x)$ and $S(x)$ term by term, we find that

$$S'(x) = C(x), \qquad C'(x) = -S(x). \tag{13}$$

Thus, at $x = 0$ we have $S'(0) = 1$ and $C'(0) = 0$. Consequently, the Wronskian of C and S at $x = 0$ is

$$W(C, S)(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \tag{14}$$

so these functions do indeed form a fundamental set of solutions. By substituting $-x$ for x in each of Eqs. (12), we obtain $C(-x) = C(x)$ and $S(-x) = -S(x)$. Moreover, by calculating with the infinite series,[3] we can show that the functions $C(x)$ and $S(x)$ have all the usual analytical and algebraic properties of the cosine and sine functions, respectively.

---

[3] Such an analysis is given in Section 24 of K. Knopp, *Theory and Applications of Infinite Series* (New York: Hafner, 1951).

Although you probably first saw the sine and cosine functions defined in a more elementary manner in terms of right triangles, it is interesting that these functions can be defined as solutions of a certain simple second order linear differential equation. To be precise, the function $\sin x$ can be defined as the unique solution of the initial value problem $y'' + y = 0$, $y(0) = 0$, $y'(0) = 1$; similarly, $\cos x$ can be defined as the unique solution of the initial value problem $y'' + y = 0$, $y(0) = 1$, $y'(0) = 0$. Many other functions that are important in mathematical physics are also defined as solutions of certain initial value problems. For most of these functions there is no simpler or more elementary way to approach them.

EXAMPLE 2

Find a series solution in powers of x of Airy's[4] equation

$$y'' - xy = 0, \qquad -\infty < x < \infty. \tag{15}$$

For this equation $P(x) = 1$, $Q(x) = 0$, and $R(x) = -x$; hence every point is an ordinary point. We assume that

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{16}$$

and that the series converges in some interval $|x| < \rho$. The series for y'' is given by Eq. (7); as explained in the preceding example, we can rewrite it as

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n. \tag{17}$$

Substituting the series (16) and (17) for y and y'' in Eq. (15), we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n = x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1}. \tag{18}$$

Next, we shift the index of summation in the series on the right side of Eq. (18) by replacing n by $n - 1$ and starting the summation at 1 rather than zero. Thus we have

$$2 \cdot 1 a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}x^n = \sum_{n=1}^{\infty} a_{n-1}x^n.$$

Again, for this equation to be satisfied for all x in some interval, the coefficients of like powers of x must be equal; hence $a_2 = 0$, and we obtain the recurrence relation

$$(n+2)(n+1)a_{n+2} = a_{n-1} \qquad \text{for} \quad n = 1, 2, 3, \ldots. \tag{19}$$

Since a_{n+2} is given in terms of a_{n-1}, the a's are determined in steps of three. Thus a_0 determines a_3, which in turn determines $a_6, \ldots$; a_1 determines a_4, which in turn determines $a_7, \ldots$; and a_2 determines a_5, which in turn determines $a_8, \ldots$. Since $a_2 = 0$, we immediately conclude that $a_5 = a_8 = a_{11} = \cdots = 0$.

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[4] Sir George Biddell Airy (1801–1892), an English astronomer and mathematician, was director of the Greenwich Observatory from 1835 to 1881. He studied the equation named for him in an 1838 paper on optics. One reason why Airy's equation is of interest is that for x negative the solutions are similar to trigonometric functions, and for x positive they are similar to hyperbolic functions. Can you explain why it is reasonable to expect such behavior?

For the sequence $a_0, a_3, a_6, a_9, \ldots$ we set $n = 1, 4, 7, 10, \ldots$ in the recurrence relation:

$$a_3 = \frac{a_0}{2 \cdot 3}, \qquad a_6 = \frac{a_3}{5 \cdot 6} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6}, \qquad a_9 = \frac{a_6}{8 \cdot 9} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}, \ldots.$$

These results suggest the general formula

$$a_{3n} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-1)(3n)}, \qquad n \ge 4.$$

For the sequence $a_1, a_4, a_7, a_{10}, \ldots$, we set $n = 2, 5, 8, 11, \ldots$ in the recurrence relation:

$$a_4 = \frac{a_1}{3 \cdot 4}, \qquad a_7 = \frac{a_4}{6 \cdot 7} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7}, \qquad a_{10} = \frac{a_7}{9 \cdot 10} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10}, \ldots.$$

In general, we have

$$a_{3n+1} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3n)(3n+1)}, \qquad n \ge 4.$$

Thus the general solution of Airy's equation is

$$y = a_0 \left[1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \cdots + \frac{x^{3n}}{2 \cdot 3 \cdots (3n-1)(3n)} + \cdots \right]$$
$$+ a_1 \left[x + \frac{x^4}{3 \cdot 4} + \frac{x^7}{3 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{x^{3n+1}}{3 \cdot 4 \cdots (3n)(3n+1)} + \cdots \right]. \qquad (20)$$

Having obtained these two series solutions, we can now investigate their convergence. Because of the rapid growth of the denominators of the terms in the series (20), we might expect these series to have a large radius of convergence. Indeed, it is easy to use the ratio test to show that both of these series converge for all x; see Problem 20.

Assuming for the moment that the series do converge for all x, let y_1 and y_2 denote the functions defined by the expressions in the first and second sets of brackets, respectively, in Eq. (20). Then, by choosing first $a_0 = 1, a_1 = 0$ and then $a_0 = 0, a_1 = 1$, it follows that y_1 and y_2 are individually solutions of Eq. (15). Notice that y_1 satisfies the initial conditions $y_1(0) = 1$, $y_1'(0) = 0$ and that y_2 satisfies the initial conditions $y_2(0) = 0$, $y_2'(0) = 1$. Thus $W(y_1, y_2)(0) = 1 \neq 0$, and consequently y_1 and y_2 are a fundamental set of solutions. Hence the general solution of Airy's equation is

$$y = a_0 y_1(x) + a_1 y_2(x), \qquad -\infty < x < \infty.$$

In Figures 5.2.3 and 5.2.4, respectively, we show the graphs of the solutions y_1 and y_2 of Airy's equation, as well as graphs of several partial sums of the two series in Eq. (20). Again, the partial sums provide local approximations to the solutions in a neighborhood of the origin. Although the quality of the approximation improves as the number of terms increases, no polynomial can adequately represent y_1 and y_2 for large $|x|$. A practical way to estimate the interval in which a given partial sum is reasonably accurate is to compare the graphs of that partial sum and the next one, obtained by including one more term. As soon as the graphs begin to separate noticeably, we can be confident that the original partial sum is no longer accurate. For example, in Figure 5.2.3 the graphs for $n = 24$ and $n = 27$ begin to separate at about $x = -9/2$. Thus, beyond this point, the partial sum of degree 24 is worthless as an approximation to the solution.

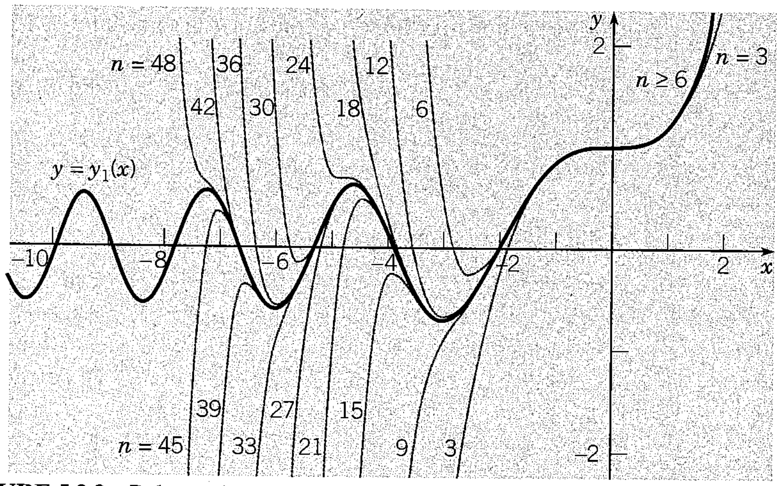

FIGURE 5.2.3 Polynomial approximations to the solution $y_1(x)$ of Airy's equation. The value of n is the degree of the approximating polynomial.

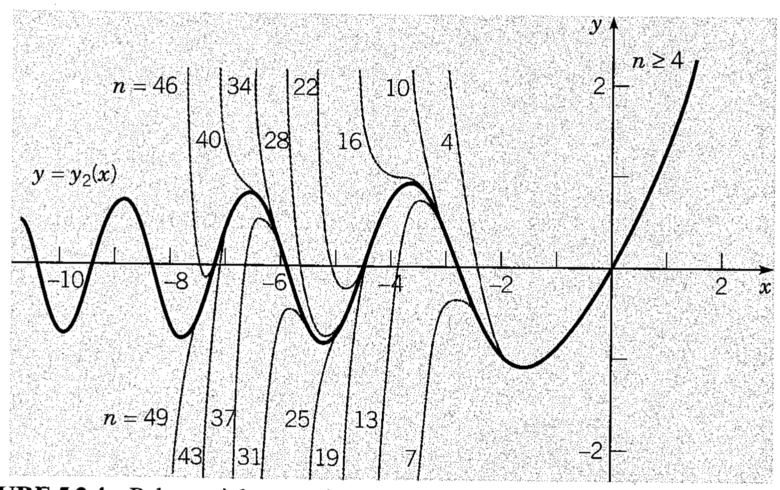

FIGURE 5.2.4 Polynomial approximations to the solution $y_2(x)$ of Airy's equation. The value of n is the degree of the approximating polynomial.

Observe that both y_1 and y_2 are monotone for $x > 0$ and oscillatory for $x < 0$. You can also see from the figures that the oscillations are not uniform but, rather, decay in amplitude and increase in frequency as the distance from the origin increases. In contrast to Example 1, the solutions y_1 and y_2 of Airy's equation are not elementary functions that you have already encountered in calculus. However, because of their importance in some physical applications, these functions have been extensively studied, and their properties are well known to applied mathematicians and scientists.

EXAMPLE 3

Find a solution of Airy's equation in powers of $x - 1$.

The point $x = 1$ is an ordinary point of Eq. (15), and thus we look for a solution of the form

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n,$$

where we assume that the series converges in some interval $|x-1| < \rho$. Then

$$y' = \sum_{n=1}^{\infty} na_n(x-1)^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n,$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n.$$

Substituting for y and y'' in Eq. (12), we obtain

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n = x\sum_{n=0}^{\infty} a_n(x-1)^n. \tag{21}$$

Now to equate the coefficients of like powers of $(x-1)$, we must express x, the coefficient of y in Eq. (15), in powers of $x-1$; that is, we write $x = 1 + (x-1)$. Note that this is precisely the Taylor series for x about $x = 1$. Then Eq. (21) takes the form

$$\begin{aligned}\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n &= [1+(x-1)]\sum_{n=0}^{\infty} a_n(x-1)^n\\ &= \sum_{n=0}^{\infty} a_n(x-1)^n + \sum_{n=0}^{\infty} a_n(x-1)^{n+1}.\end{aligned}$$

Shifting the index of summation in the second series on the right gives

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n = \sum_{n=0}^{\infty} a_n(x-1)^n + \sum_{n=1}^{\infty} a_{n-1}(x-1)^n.$$

Equating coefficients of like powers of $x-1$, we obtain

$$\begin{aligned}2a_2 &= a_0,\\ (3\cdot 2)a_3 &= a_1 + a_0,\\ (4\cdot 3)a_4 &= a_2 + a_1,\\ (5\cdot 4)a_5 &= a_3 + a_2,\\ &\ \ \vdots\end{aligned}$$

The general recurrence relation is

$$(n+2)(n+1)a_{n+2} = a_n + a_{n-1} \qquad \text{for} \quad n \geq 1. \tag{22}$$

Solving for the first few coefficients a_n in terms of a_0 and a_1, we find that

$$a_2 = \frac{a_0}{2}, \qquad a_3 = \frac{a_1}{6} + \frac{a_0}{6}, \qquad a_4 = \frac{a_2}{12} + \frac{a_1}{12} = \frac{a_0}{24} + \frac{a_1}{12}, \qquad a_5 = \frac{a_3}{20} + \frac{a_2}{20} = \frac{a_0}{30} + \frac{a_1}{120}.$$

Hence

$$\begin{aligned} y = a_0&\left[1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \frac{(x-1)^5}{30} + \cdots\right]\\ &+ a_1\left[(x-1) + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \frac{(x-1)^5}{120} + \cdots\right]. \end{aligned} \tag{23}$$

In general, when the recurrence relation has more than two terms, as in Eq. (22), the determination of a formula for a_n in terms a_0 and a_1 will be fairly complicated, if not impossible. In this example such a formula is not readily apparent. Lacking such a formula, we cannot test the two series in Eq. (23) for convergence by direct methods such as the ratio test. However, we shall see in Section 5.3 that even without knowing the formula for a_n, it is possible to establish that the two series in Eq. (23) converge for all x. Further, they define functions y_3 and y_4 that are a fundamental set of solutions of the Airy equation (15). Thus

$$y = a_0 y_3(x) + a_1 y_4(x)$$

is the general solution of Airy's equation for $-\infty < x < \infty$.

It is worth emphasizing, as we saw in Example 3, that if we look for a solution of Eq. (1) of the form $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$, then the coefficients $P(x)$, $Q(x)$, and $R(x)$ in Eq. (1) must also be expressed in powers of $x - x_0$. Alternatively, we can make the change of variable $x - x_0 = t$, obtaining a new differential equation for y as a function of t, and then look for solutions of this new equation of the form $\sum_{n=0}^{\infty} a_n t^n$. When we have finished the calculations, we replace t by $x - x_0$ (see Problem 19).

In Examples 2 and 3 we have found two sets of solutions of Airy's equation. The functions y_1 and y_2 defined by the series in Eq. (20) are a fundamental set of solutions of Eq. (15) for all x, and this is also true for the functions y_3 and y_4 defined by the series in Eq. (23). According to the general theory of second order linear equations, each of the first two functions can be expressed as a linear combination of the latter two functions, and vice versa—a result that is certainly not obvious from an examination of the series alone.

Finally, we emphasize that it is not particularly important if, as in Example 3, we are unable to determine the general coefficient a_n in terms of a_0 and a_1. What is essential is that we can determine *as many coefficients as we want.* Thus we can find as many terms in the two series solutions as we want, even if we cannot determine the general term. While the task of calculating several coefficients in a power series solution is not difficult, it can be tedious. A symbolic manipulation package can be very helpful here; some are able to find a specified number of terms in a power series solution in response to a single command. With a suitable graphics package we can also produce plots such as those shown in the figures in this section.

PROBLEMS

In each of Problems 1 through 14:

(a) Seek power series solutions of the given differential equation about the given point x_0; find the recurrence relation.

(b) Find the first four terms in each of two solutions y_1 and y_2 (unless the series terminates sooner).

(c) By evaluating the Wronskian $W(y_1, y_2)(x_0)$, show that y_1 and y_2 form a fundamental set of solutions.

(d) If possible, find the general term in each solution.

1. $y'' - y = 0, \quad x_0 = 0$
2. $y'' - xy' - y = 0, \quad x_0 = 0$
3. $y'' - xy' - y = 0, \quad x_0 = 1$
4. $y'' + k^2x^2y = 0, \quad x_0 = 0$, k a constant
5. $(1 - x)y'' + y = 0, \quad x_0 = 0$
6. $(2 + x^2)y'' - xy' + 4y = 0, \quad x_0 = 0$
7. $y'' + xy' + 2y = 0, \quad x_0 = 0$
8. $xy'' + y' + xy = 0, \quad x_0 = 1$
9. $(1 + x^2)y'' - 4xy' + 6y = 0, \quad x_0 = 0$
10. $(4 - x^2)y'' + 2y = 0, \quad x_0 = 0$

11. $(3-x^2)y''-3xy'-y=0, \qquad x_0=0$
12. $(1-x)y''+xy'-y=0, \qquad x_0=0$
13. $2y''+xy'+3y=0, \qquad x_0=0$
14. $2y''+(x+1)y'+3y=0, \qquad x_0=2$

In each of Problems 15 through 18:
(a) Find the first five nonzero terms in the solution of the given initial value problem.
(b) Plot the four-term and the five-term approximations to the solution on the same axes.
(c) From the plot in part (b) estimate the interval in which the four-term approximation is reasonably accurate.

15. $y''-xy'-y=0, \qquad y(0)=2, \quad y'(0)=1;$ see Problem 2
16. $(2+x^2)y''-xy'+4y=0, \qquad y(0)=-1, \quad y'(0)=3;$ see Problem 6
17. $y''+xy'+2y=0, \qquad y(0)=4, \quad y'(0)=-1;$ see Problem 7
18. $(1-x)y''+xy'-y=0, \qquad y(0)=-3, \quad y'(0)=2;$ see Problem 12
19. (a) By making the change of variable $x-1=t$ and assuming that y has a Taylor series in powers of t, find two series solutions of

$$y''+(x-1)^2y'+(x^2-1)y=0$$

in powers of $x-1$.
(b) Show that you obtain the same result by assuming that y has a Taylor series in powers of $x-1$ and also expressing the coefficient x^2-1 in powers of $x-1$.
20. Show directly, using the ratio test, that the two series solutions of Airy's equation about $x=0$ converge for all x; see Eq. (20) of the text.
21. **The Hermite Equation.** The equation

$$y''-2xy'+\lambda y=0, \qquad -\infty<x<\infty,$$

where λ is a constant, is known as the Hermite[5] equation. It is an important equation in mathematical physics.
(a) Find the first four terms in each of two solutions about $x=0$ and show that they form a fundamental set of solutions.
(b) Observe that if λ is a nonnegative even integer, then one or the other of the series solutions terminates and becomes a polynomial. Find the polynomial solutions for $\lambda=0$, 2, 4, 6, 8, and 10. Note that each polynomial is determined only up to a multiplicative constant.
(c) The Hermite polynomial $H_n(x)$ is defined as the polynomial solution of the Hermite equation with $\lambda=2n$ for which the coefficient of x^n is 2^n. Find $H_0(x),\ldots,H_5(x)$.
22. Consider the initial value problem $y'=\sqrt{1-y^2}, y(0)=0$.
(a) Show that $y=\sin x$ is the solution of this initial value problem.
(b) Look for a solution of the initial value problem in the form of a power series about $x=0$. Find the coefficients up to the term in x^3 in this series.

[5]Charles Hermite (1822–1901) was an influential French analyst and algebraist. An inspiring teacher, he was professor at the École Polytechnique and the Sorbonne. He introduced the Hermite functions in 1864 and showed in 1873 that e is a transcendental number (that is, e is not a root of any polynomial equation with rational coefficients). His name is also associated with Hermitian matrices (see Section 7.3), some of whose properties he discovered.

In each of Problems 23 through 28, plot several partial sums in a series solution of the given initial value problem about $x = 0$, thereby obtaining graphs analogous to those in Figures 5.2.1 through 5.2.4.

23. $y'' - xy' - y = 0, \quad y(0) = 1, \quad y'(0) = 0;$ see Problem 2
24. $(2 + x^2)y'' - xy' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0;$ see Problem 6
25. $y'' + xy' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 1;$ see Problem 7
26. $(4 - x^2)y'' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 1;$ see Problem 10
27. $y'' + x^2y = 0, \quad y(0) = 1, \quad y'(0) = 0;$ see Problem 4
28. $(1 - x)y'' + xy' - 2y = 0, \quad y(0) = 0, \quad y'(0) = 1$

5.3 Series Solutions Near an Ordinary Point, Part II

In the preceding section we considered the problem of finding solutions of

$$P(x)y'' + Q(x)y' + R(x)y = 0, \tag{1}$$

where P, Q, and R are polynomials, in the neighborhood of an ordinary point x_0. Assuming that Eq. (1) does have a solution $y = \phi(x)$ and that ϕ has a Taylor series

$$y = \phi(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n \tag{2}$$

that converges for $|x - x_0| < \rho$, where $\rho > 0$, we found that the a_n can be determined by directly substituting the series (2) for y in Eq. (1).

Let us now consider how we might justify the statement that if x_0 is an ordinary point of Eq. (1), then there exist solutions of the form (2). We also consider the question of the radius of convergence of such a series. In doing this, we are led to a generalization of the definition of an ordinary point.

Suppose, then, that there is a solution of Eq. (1) of the form (2). By differentiating Eq. (2) m times and setting x equal to x_0, we obtain

$$m!a_m = \phi^{(m)}(x_0).$$

Hence, to compute a_n in the series (2), we must show that we can determine $\phi^{(n)}(x_0)$ for $n = 0, 1, 2, \ldots$ from the differential equation (1).

Suppose that $y = \phi(x)$ is a solution of Eq. (1) satisfying the initial conditions $y(x_0) = y_0, y'(x_0) = y_0'$. Then $a_0 = y_0$ and $a_1 = y_0'$. If we are solely interested in finding a solution of Eq. (1) without specifying any initial conditions, then a_0 and a_1 remain arbitrary. To determine $\phi^{(n)}(x_0)$ and the corresponding a_n for $n = 2, 3, \ldots$, we turn to Eq. (1). Since ϕ is a solution of Eq. (1), we have

$$P(x)\phi''(x) + Q(x)\phi'(x) + R(x)\phi(x) = 0.$$

For the interval about x_0 for which P is nonzero, we can write this equation in the form

$$\phi''(x) = -p(x)\phi'(x) - q(x)\phi(x), \tag{3}$$

where $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$. Setting x equal to x_0 in Eq. (3) gives

$$\phi''(x_0) = -p(x_0)\phi'(x_0) - q(x_0)\phi(x_0).$$

Hence a_2 is given by

$$2!a_2 = \phi''(x_0) = -p(x_0)a_1 - q(x_0)a_0. \tag{4}$$

To determine a_3, we differentiate Eq. (3) and then set x equal to x_0, obtaining

$$\begin{aligned} 3!a_3 = \phi'''(x_0) &= -[p\phi'' + (p' + q)\phi' + q'\phi]\Big|_{x=x_0} \\ &= -2!p(x_0)a_2 - [p'(x_0) + q(x_0)]a_1 - q'(x_0)a_0. \end{aligned} \tag{5}$$

Substituting for a_2 from Eq. (4) gives a_3 in terms of a_1 and a_0. Since P, Q, and R are polynomials and $P(x_0) \neq 0$, all the derivatives of p and q exist at x_0. Hence, we can continue to differentiate Eq. (3) indefinitely, determining after each differentiation the successive coefficients $a_4, a_5, \ldots$ by setting x equal to x_0.

Notice that the important property that we used in determining the a_n was that we could compute infinitely many derivatives of the functions p and q. It might seem reasonable to relax our assumption that the functions p and q are ratios of polynomials and simply require that they be infinitely differentiable in the neighborhood of x_0. Unfortunately, this condition is too weak to ensure that we can prove the convergence of the resulting series expansion for $y = \phi(x)$. What is needed is to assume that the functions p and q are *analytic* at x_0; that is, they have Taylor series expansions that converge to them in some interval about the point x_0:

$$p(x) = p_0 + p_1(x - x_0) + \cdots + p_n(x - x_0)^n + \cdots = \sum_{n=0}^{\infty} p_n(x - x_0)^n, \tag{6}$$

$$q(x) = q_0 + q_1(x - x_0) + \cdots + q_n(x - x_0)^n + \cdots = \sum_{n=0}^{\infty} q_n(x - x_0)^n. \tag{7}$$

With this idea in mind, we can generalize the definitions of an ordinary point and a singular point of Eq. (1) as follows: if the functions $p = Q/P$ and $q = R/P$ are analytic at x_0, then the point x_0 is said to be an **ordinary point** of the differential equation (1); otherwise, it is a **singular point**.

Now let us turn to the question of the interval of convergence of the series solution. One possibility is actually to compute the series solution for each problem and then to apply one of the tests for convergence of an infinite series to determine its radius of convergence. Unfortunately, these tests require us to obtain an expression for the general coefficient a_n as a function of n, and this task is often quite difficult, if not impossible; recall Example 3 in Section 5.2. However, the question can be answered at once for a wide class of problems by the following theorem.

Theorem 5.3.1 If x_0 is an ordinary point of the differential equation (1)

$$P(x)y'' + Q(x)y' + R(x)y = 0,$$

that is, if $p = Q/P$ and $q = R/P$ are analytic at x_0, then the general solution of Eq. (1) is

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0y_1(x) + a_1y_2(x), \tag{8}$$

where a_0 and a_1 are arbitrary, and y_1 and y_2 are two power series solutions that are analytic at x_0. The solutions y_1 and y_2 form a fundamental set of solutions. Further, the radius of convergence for each of the series solutions y_1 and y_2 is at least as large as the minimum of the radii of convergence of the series for p and q.

To see that y_1 and y_2 are a fundamental set of solutions, note that they have the form $y_1(x) = 1 + b_2(x - x_0)^2 + \cdots$ and $y_2(x) = (x - x_0) + c_2(x - x_0)^2 + \cdots$, where $b_2 + c_2 = a_2$. Hence y_1 satisfies the initial conditions $y_1(x_0) = 1, y_1'(x_0) = 0$, and y_2 satisfies the initial conditions $y_2(x_0) = 0, y_2'(x_0) = 1$. Thus $W(y_1, y_2)(x_0) = 1$.

Also note that although calculating the coefficients by successively differentiating the differential equation is excellent in theory, it is usually not a practical computational procedure. Rather, you should substitute the series (2) for y in the differential equation (1) and determine the coefficients so that the differential equation is satisfied, as in the examples in the preceding section.

We will not prove this theorem, which in a slightly more general form was established by Fuchs.[6] What is important for our purposes is that there is a series solution of the form (2) and that the radius of convergence of the series solution cannot be less than the smaller of the radii of convergence of the series for p and q; hence we need only determine these.

This can be done in either of two ways. Again, one possibility is simply to compute the power series for p and q and then to determine the radii of convergence by using one of the convergence tests for infinite series. However, there is an easier way when P, Q, and R are polynomials. It is shown in the theory of functions of a complex variable that the ratio of two polynomials, say, Q/P, has a convergent power series expansion about a point $x = x_0$ if $P(x_0) \neq 0$. Further, if we assume that any factors common to Q and P have been canceled, then the radius of convergence of the power series for Q/P about the point x_0 is precisely the distance from x_0 to the nearest zero of P. In determining this distance, we must remember that $P(x) = 0$ may have complex roots, and these must also be considered.

EXAMPLE 1

What is the radius of convergence of the Taylor series for $(1 + x^2)^{-1}$ about $x = 0$?

One way to proceed is to find the Taylor series in question, namely,

$$\frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots.$$

Then it can be verified by the ratio test that $\rho = 1$. Another approach is to note that the zeros of $1 + x^2$ are $x = \pm i$. Since the distance in the complex plane from 0 to i or to $-i$ is 1, the radius of convergence of the power series about $x = 0$ is 1.

[6] Lazarus Immanuel Fuchs (1833–1902), a German mathematician, was a student and later a professor at the University of Berlin. He proved the result of Theorem 5.3.1 in 1866. His most important research was on singular points of linear differential equations. He recognized the significance of regular singular points (Section 5.4), and equations whose only singularities, including the point at infinity, are regular singular points are known as Fuchsian equations.

EXAMPLE 2

What is the radius of convergence of the Taylor series for $(x^2 - 2x + 2)^{-1}$ about $x = 0$? about $x = 1$?

First notice that

$$x^2 - 2x + 2 = 0$$

has solutions $x = 1 \pm i$. The distance in the complex plane from $x = 0$ to either $x = 1 + i$ or $x = 1 - i$ is $\sqrt{2}$; hence the radius of convergence of the Taylor series expansion $\sum_{n=0}^{\infty} a_n x^n$ about $x = 0$ is $\sqrt{2}$.

The distance in the complex plane from $x = 1$ to either $x = 1 + i$ or $x = 1 - i$ is 1; hence the radius of convergence of the Taylor series expansion $\sum_{n=0}^{\infty} b_n(x - 1)^n$ about $x = 1$ is 1.

According to Theorem 5.3.1, the series solutions of the Airy equation in Examples 2 and 3 of the preceding section converge for all values of x and $x - 1$, respectively, since in each problem $P(x) = 1$ and hence is never zero.

A series solution may converge for a wider range of x than indicated by Theorem 5.3.1, so the theorem actually gives only a lower bound on the radius of convergence of the series solution. This is illustrated by the Legendre polynomial solution of the Legendre equation given in the next example.

EXAMPLE 3

Determine a lower bound for the radius of convergence of series solutions about $x = 0$ for the Legendre equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

where α is a constant.

Note that $P(x) = 1 - x^2$, $Q(x) = -2x$, and $R(x) = \alpha(\alpha + 1)$ are polynomials, and that the zeros of P, namely, $x = \pm 1$, are a distance 1 from $x = 0$. Hence a series solution of the form $\sum_{n=0}^{\infty} a_n x^n$ converges at least for $|x| < 1$, and possibly for larger values of x. Indeed, it can be shown that if α is a positive integer, one of the series solutions terminates after a finite number of terms and hence converges not just for $|x| < 1$ but for all x. For example, if $\alpha = 1$, the polynomial solution is $y = x$. See Problems 22 through 29 at the end of this section for a further discussion of the Legendre equation.

EXAMPLE 4

Determine a lower bound for the radius of convergence of series solutions of the differential equation

$$(1 + x^2)y'' + 2xy' + 4x^2y = 0 \tag{9}$$

about the point $x = 0$; about the point $x = -\frac{1}{2}$.

Again P, Q, and R are polynomials, and P has zeros at $x = \pm i$. The distance in the complex plane from 0 to $\pm i$ is 1, and from $-\frac{1}{2}$ to $\pm i$ is $\sqrt{1 + \frac{1}{4}} = \sqrt{5}/2$. Hence in the first case the series $\sum_{n=0}^{\infty} a_n x^n$ converges at least for $|x| < 1$, and in the second case the series $\sum_{n=0}^{\infty} b_n \left(x + \frac{1}{2}\right)^n$ converges at least for $|x + \frac{1}{2}| < \sqrt{5}/2$.

An interesting observation that we can make about Eq. (9) follows from Theorems 3.2.1 and 5.3.1. Suppose that initial conditions $y(0) = y_0$ and $y'(0) = y_0'$ are given. Since $1 + x^2 \neq 0$ for all x, we know from Theorem 3.2.1 that there exists a unique solution of the initial value problem on $-\infty < x < \infty$. On the other hand, Theorem 5.3.1 only guarantees a series solution of the form $\sum_{n=0}^{\infty} a_n x^n$ (with $a_0 = y_0, a_1 = y_0'$) for $-1 < x < 1$. The unique solution on the interval $-\infty < x < \infty$ may not have a power series about $x = 0$ that converges for all x.

EXAMPLE 5

Can we determine a series solution about $x = 0$ for the differential equation

$$y'' + (\sin x)y' + (1 + x^2)y = 0,$$

and if so, what is the radius of convergence?

For this differential equation, $p(x) = \sin x$ and $q(x) = 1 + x^2$. Recall from calculus that $\sin x$ has a Taylor series expansion about $x = 0$ that converges for all x. Further, q also has a Taylor series expansion about $x = 0$, namely, $q(x) = 1 + x^2$, that converges for all x. Thus there is a series solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$ with a_0 and a_1 arbitrary, and the series converges for all x.

PROBLEMS

In each of Problems 1 through 4, determine $\phi''(x_0)$, $\phi'''(x_0)$, and $\phi^{(4)}(x_0)$ for the given point x_0 if $y = \phi(x)$ is a solution of the given initial value problem.

1. $y'' + xy' + y = 0; \quad y(0) = 1, \quad y'(0) = 0$
2. $y'' + (\sin x)y' + (\cos x)y = 0; \quad y(0) = 0, \quad y'(0) = 1$
3. $x^2y'' + (1 + x)y' + 3(\ln x)y = 0; \quad y(1) = 2, \quad y'(1) = 0$
4. $y'' + x^2y' + (\sin x)y = 0; \quad y(0) = a_0, \quad y'(0) = a_1$

In each of Problems 5 through 8, determine a lower bound for the radius of convergence of series solutions about each given point x_0 for the given differential equation.

5. $y'' + 4y' + 6xy = 0; \quad x_0 = 0, \quad x_0 = 4$
6. $(x^2 - 2x - 3)y'' + xy' + 4y = 0; \quad x_0 = 4, \quad x_0 = -4, \quad x_0 = 0$
7. $(1 + x^3)y'' + 4xy' + y = 0; \quad x_0 = 0, \quad x_0 = 2$
8. $xy'' + y = 0; \quad x_0 = 1$
9. Determine a lower bound for the radius of convergence of series solutions about the given x_0 for each of the differential equations in Problems 1 through 14 of Section 5.2.
10. **The Chebyshev Equation.** The Chebyshev[7] differential equation is

$$(1 - x^2)y'' - xy' + \alpha^2 y = 0,$$

where α is a constant.

(a) Determine two solutions in powers of x for $|x| < 1$, and show that they form a fundamental set of solutions.

(b) Show that if α is a nonnegative integer n, then there is a polynomial solution of degree n. These polynomials, when properly normalized, are called the Chebyshev polynomials. They are very useful in problems that require a polynomial approximation to a function defined on $-1 \leq x \leq 1$.

(c) Find a polynomial solution for each of the cases $\alpha = n = 0, 1, 2, 3$.

[7]Pafnuty L. Chebyshev (1821–1894), the most influential nineteenth-century Russian mathematician, was for 35 years professor at the University of St. Petersburg, which produced a long line of distinguished mathematicians. His study of Chebyshev polynomials began in about 1854 as part of an investigation of the approximation of functions by polynomials. Chebyshev is also known for his work in number theory and probability.

For each of the differential equations in Problems 11 through 14, find the first four nonzero terms in each of two power series solutions about the origin. Show that they form a fundamental set of solutions. What do you expect the radius of convergence to be for each solution?

11. $y'' + (\sin x)y = 0$
12. $e^x y'' + xy = 0$
13. $(\cos x)y'' + xy' - 2y = 0$
14. $e^{-x}y'' + \ln(1+x)y' - xy = 0$
15. Let x and x^2 be solutions of a differential equation $P(x)y'' + Q(x)y' + R(x)y = 0$. Can you say whether the point $x = 0$ is an ordinary point or a singular point? Prove your answer.

First Order Equations. The series methods discussed in this section are directly applicable to the first order linear differential equation $P(x)y' + Q(x)y = 0$ at a point x_0, if the function $p = Q/P$ has a Taylor series expansion about that point. Such a point is called an ordinary point, and further, the radius of convergence of the series $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ is at least as large as the radius of convergence of the series for Q/P. In each of Problems 16 through 21, solve the given differential equation by a series in powers of x and verify that a_0 is arbitrary in each case. Problems 20 and 21 involve nonhomogeneous differential equations to which series methods can be easily extended. Where possible, compare the series solution with the solution obtained by using the methods of Chapter 2.

16. $y' - y = 0$
17. $y' - xy = 0$
18. $y' = e^{x^2} y$, three terms only
19. $(1 - x)y' = y$
20. $y' - y = x^2$
21. $y' + xy = 1 + x$

The Legendre Equation. Problems 22 through 29 deal with the Legendre[8] equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0.$$

As indicated in Example 3, the point $x = 0$ is an ordinary point of this equation, and the distance from the origin to the nearest zero of $P(x) = 1 - x^2$ is 1. Hence the radius of convergence of series solutions about $x = 0$ is at least 1. Also notice that we need to consider only $\alpha > -1$ because if $\alpha \leq -1$, then the substitution $\alpha = -(1 + \gamma)$, where $\gamma \geq 0$, leads to the Legendre equation $(1 - x^2)y'' - 2xy' + \gamma(\gamma + 1)y = 0$.

22. Show that two solutions of the Legendre equation for $|x| < 1$ are

$$y_1(x) = 1 - \frac{\alpha(\alpha+1)}{2!}x^2 + \frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!}x^4$$
$$+ \sum_{m=3}^{\infty}(-1)^m \frac{\alpha\cdots(\alpha-2m+2)(\alpha+1)\cdots(\alpha+2m-1)}{(2m)!}x^{2m},$$

$$y_2(x) = x - \frac{(\alpha-1)(\alpha+2)}{3!}x^3 + \frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{5!}x^5$$
$$+ \sum_{m=3}^{\infty}(-1)^m \frac{(\alpha-1)\cdots(\alpha-2m+1)(\alpha+2)\cdots(\alpha+2m)}{(2m+1)!}x^{2m+1}.$$

[8] Adrien-Marie Legendre (1752–1833) held various positions in the French Académie des Sciences from 1783 onward. His primary work was in the fields of elliptic functions and number theory. The Legendre functions, solutions of Legendre's equation, first appeared in 1784 in his study of the attraction of spheroids.

23. Show that if α is zero or a positive even integer $2n$, the series solution y_1 reduces to a polynomial of degree $2n$ containing only even powers of x. Find the polynomials corresponding to $\alpha = 0$, 2, and 4. Show that if α is a positive odd integer $2n+1$, the series solution y_2 reduces to a polynomial of degree $2n+1$ containing only odd powers of x. Find the polynomials corresponding to $\alpha = 1, 3,$ and 5.

24. The Legendre polynomial $P_n(x)$ is defined as the polynomial solution of the Legendre equation with $\alpha = n$ that also satisfies the condition $P_n(1) = 1$.
    (a) Using the results of Problem 23, find the Legendre polynomials $P_0(x), \ldots, P_5(x)$.
    (b) Plot the graphs of $P_0(x), \ldots, P_5(x)$ for $-1 \le x \le 1$.
    (c) Find the zeros of $P_0(x), \ldots, P_5(x)$.

25. It can be shown that the general formula for $P_n(x)$ is

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k},$$

where $\lfloor n/2 \rfloor$ denotes the greatest integer less than or equal to $n/2$. By observing the form of $P_n(x)$ for n even and n odd, show that $P_n(-1) = (-1)^n$.

26. The Legendre polynomials play an important role in mathematical physics. For example, in solving Laplace's equation (the potential equation) in spherical coordinates, we encounter the equation

$$\frac{d^2F(\varphi)}{d\varphi^2} + \cot\varphi \frac{dF(\varphi)}{d\varphi} + n(n+1)F(\varphi) = 0, \qquad 0 < \varphi < \pi,$$

where n is a positive integer. Show that the change of variable $x = \cos\varphi$ leads to the Legendre equation with $\alpha = n$ for $y = f(x) = F(\arccos x)$.

27. Show that for $n = 0, 1, 2, 3$, the corresponding Legendre polynomial is given by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n.$$

This formula, known as Rodrigues's formula,[9] is true for all positive integers n.

28. Show that the Legendre equation can also be written as

$$[(1 - x^2)y']' = -\alpha(\alpha + 1)y.$$

Then it follows that

$$[(1 - x^2)P_n'(x)]' = -n(n+1)P_n(x) \quad \text{and} \quad [(1 - x^2)P_m'(x)]' = -m(m+1)P_m(x).$$

By multiplying the first equation by $P_m(x)$ and the second equation by $P_n(x)$, integrating by parts, and then subtracting one equation from the other, show that

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0 \qquad \text{if} \quad n \ne m.$$

This property of the Legendre polynomials is known as the orthogonality property. If $m = n$, it can be shown that the value of the preceding integral is $2/(2n+1)$.

[9] Benjamin Olinde Rodrigues (1795–1851) published this result as part of his doctoral thesis from the University of Paris in 1815. He then became a banker and social reformer but retained an interest in mathematics. Unfortunately, his later papers were not appreciated until the late twentieth century.