1. Evaluate the following series exactly if they converge, or explain why they diverge.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{5^n}{3 \cdot 2^{3n+2}}$

   Geometric series. $r = \frac{5}{2^3} = \frac{5}{8} < 1$ *so it converges.* $a = \frac{5^0}{3 \cdot 2^2} = \frac{1}{12}$*. So the limit is*

   $$\frac{a}{1-r} = \frac{\frac{1}{12}}{1-\frac{5}{8}} = \frac{2}{9}.$$

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{5 \cdot 2^n}{7}$

   Geometric series. $r = 2$ *so diverges. (WARNING: Don't forget to make sure that* $r < 1$ *before blindly using the formula* $\frac{a}{1-r}$*.)*

   (c) $\displaystyle\sum_{n=4}^{\infty} \frac{1}{\ln(n^2)} - \frac{1}{\ln((n+1)^2)}$

   Telescoping sum.

   $$\lim_{n\to\infty} \frac{1}{\ln(n^2)} = 0$$

   and therefore the sum converges to

   $$\frac{1}{\ln(4^2)} - 0 = \frac{1}{\ln(16)}.$$

   (d) $\displaystyle\sum_{n=4}^{\infty} \ln(n^2) - \ln((n+1)^2)$

   Telescoping sum.

   $$\lim_{n\to\infty} \ln(n^2) = \infty$$

   and therefore the sum diverges.

(e) $4 + 12 + \frac{36}{2} + \frac{108}{6} + \frac{4 \cdot 81}{24} + \dots$

This is

$$4(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots)$$

with $x = 3$ *plugged in. Thus it is* $4e^3$.

(f) $7 - \frac{7^2}{2} + \frac{7^3}{3} - \frac{7^4}{4} + \dots$

This is

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

with $x = 7$ *plugged in. You might be fooled and say it is* $\ln(1+7) = \ln(8)$. *Unfortunately, 7 is not in the interval of convergence of this power series... it diverges, by the ratio test.*

(g) $\sum_{n=0}^{\infty} (-1)^n \frac{(13)^{2n}}{(2n)!} + (-1)^n \frac{9^{2n+1}}{(2n+1)!}$

The first sum is $\cos(13)$ *and the second sum is* $\sin(9)$, *thus the combo is* $\cos(13) + \sin(9)$.

2. Let $y = \sum_{n=0}^{\infty} a_n t^n$ be a power series centered at 0.

   (a) Write down power series centered at 0 for y', y'', and y'''.

$$y' = \sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n.$$

(Note: either one is ok, both answer the question. But the latter will be more useful later on.)

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n.$$

$$y''' = \sum_{n=3}^{\infty} n(n-1)(n-2)a_n t^{n-3} = \sum_{n=0}^{\infty}(n+3)(n+2)(n+1)a_{n+3}t^n.$$

(b) Suppose that $y(0) = 2$, $y'(0) = 1$, and $y''(0) = 6$. What does that say about the coefficients a_n?

$a_0 = y(0) = 2$, $a_1 = y'(0) = 1$ *and* $a_2 = \frac{y''(0)}{2!} = 3$.

(c) Suppose that y solves the differential equation $y''' - 2y'' + y' - 2y = 0$ with the initial conditions given above. Find a_n for $n \leq 4$.

(We add together the power series in the latter form above, so that they easily give us the coefficient of t^n.)

$$y'''-2y''+y'-2y = \sum_{n=0}^{\infty} \big((n+3)(n+2)(n+1)a_{n+3}-2(n+2)(n+1)a_{n+2}+(n+1)a_{n+1}-2a_n\big)t^n = 0$$

So each coefficient is zero. Solving for a_{n+3} we get

$$a_{n+3} = \frac{2(n+2)(n+1)a_{n+2} - (n+1)a_{n+1} + 2a_n}{(n+3)(n+2)(n+1)}$$

Plugging in $n = 0$ we get

$$a_3 = \frac{2(2)(1)a_2 - (1)a_1 + 2a_0}{(3)(2)(1)} = \frac{12 - 1 + 4}{6} = \frac{15}{6}.$$

Plugging in $n = 1$ we get

$$a_4 = \frac{(2(3)(2)a_3 - (2)a_2 + 2a_1}{(4)(3)(2)} = \frac{12 \cdot \frac{15}{6} - 6 + 2}{24} = \frac{26}{24}.$$

3. Let $y = \sum_{n=0}^{\infty} a_n (t-6)^n$ be a power series centered at 6.

   (a) Write down power series centered at 6 for y' and $(t+2)y$.

$$y' = \sum_{n=1}^{\infty} n a_n (t-6)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (t-6)^n$$

$$\begin{aligned}(t+2)y = ((t-6)+8)y = (t-6)y + 8y &= \sum_{n=0}^{\infty} a_n (t-6)^{n+1} + \sum_{n=0}^{\infty} 8a_n (t-6)^n \\ = \sum_{n=1}^{\infty} a_{n-1}(t-6)^n + \sum_{n=0}^{\infty} 8a_n (t-6)^n &= 8a_0 + \sum_{n=1}^{\infty} \big(8a_n + a_{n-1}\big)(t-6)^n\end{aligned}$$

(Remember, the steps here are: recenter the polynomial $(t+2)$ at 6, i.e. in the form $(t-6)+8$. Then find each term individually. Then, to add them together, reindex so the coefficient of $(t-6)^n$ is obvious. Then, when you add, keep track of small values of n separately; here, $n=0$ is special, because it is not in the first sum.)

   (b) Suppose that y solves the differential equation $y' - (t+2)y = 15$. Write down a recursive formula for the coefficients a_n.

$$y' - (t+2)y = (a_1 - 8a_0 - 15)(t-6)^0 + \sum_{n=1}^{\infty} \big((n+1)a_{n+1} - 8a_n - a_{n-1}\big)(t-6)^n = 0$$

Each coefficient is zero. So our recursive formula is special at the start:

$$a_1 = 8a_0 + 15,$$

and then for $n \geq 1$ is:

$$a_{n+1} = \frac{8a_n + a_{n-1}}{n+1}.$$

(Note: You will make fewer mistakes if you keep track of for which n the formula is valid. This formula right here is valid only for $n \geq 1$, not for $n = 0$, where you have to use the formula $a_1 = 8a_0 + 15$ instead.)

(c) Find the first three non-zero terms of the general solution centered at 6.

For the general solution, we just let a_0 be an independent variable, and solve everything in terms of it.

$$a_1 = 8a_0 + 15$$

$$a_2 = \frac{8a_1 + a_0}{2} = \frac{64a_0 + a_0 + 120}{2} = \frac{65}{2}a_0 + 60$$

(Warning: It is not good enough to leave it at $a_2 = \frac{8a_1+a_0}{2}$. You have to solve in terms of a_0. The point is that a_0 determines the initial conditions, from which everything else should be determined.)
So the first three terms are $a_0 + (8a_0 + 15)(t-6) + (\frac{65}{2}a_0 + 60)(t-6)^2$.

4. Suppose that y solves the differential equation $y'' + ty' + y = 0$ with initial conditions $y(0) = 1$ and $y'(0) = 0$.

(a) Suppose that $y = \sum a_n t^n$. Find a recursive formula for the coefficients a_n.

$$y'' = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n$$

$$ty' = \sum_{n=0}^{\infty}(n+1)a_{n+1}t^{n+1} = \sum_{n=1}^{\infty} na_n t^n$$

$$y'' + ty' + y = \big((2)(1)a_2 + a_0\big)t^0 + \sum_{n=1}^{\infty}\big((n+2)(n+1)a_{n+2} + na_n + a_n\big)t^n$$

so

$$a_2 = \frac{-a_0}{2}$$

and

$$a_{n+2} = \frac{-(n+1)a_n}{(n+2)(n+1)} = \frac{-a_n}{n+2}$$

for $n \geq 1$.
(Secretly, though, this formula also holds for $n = 0$*. After all, looking at* ty'*, it is the case that* $\sum_{n=1}^{\infty} na_n t^n = \sum_{n=0}^{\infty} na_n t^n$*. But there are cases where trying to simplify like this will just screw you up or confuse you unless you're a careful person...)*

(b) Find the values of a_n for $n \leq 8$.

$a_0 = y(0) = 1$ *and* $a_1 = y'(0) = 0$.
$a_2 = \frac{-1}{2}$ *and* $a_3 = 0$
$a_4 = \frac{-a_2}{4} = \frac{1}{2 \cdot 4}$ *and* $a_5 = 0$.
$a_6 = \frac{-a_4}{6} = \frac{-1}{2 \cdot 4 \cdot 6}$ *and* $a_7 = 0$.
$a_8 = \frac{1}{2 \cdot 4 \cdot 6 \cdot 8}$.
(The pattern is clear if you needed to find a_n *for all* n*.)*

(c) Approximate the value of $y(0.1)$ to within 10^{-6}. Justify your answer.
By the above, we have that $y(t)$ *is approximately equal to*

$$1 - \frac{1}{2}t^2 + \frac{1}{2 \cdot 4}t^4 - \frac{1}{2 \cdot 4 \cdot 6}t^6 + \frac{1}{2 \cdot 4 \cdot 6 \cdot 8}t^8 - \ldots$$

Plugging in $t = 0.1 = 10^{-1}$*, we get an alternating series. Since* $\frac{1}{2 \cdot 4 \cdot 6} 10^{-6} < 10^{-6}$ *while* $\frac{1}{2 \cdot 4} 10^{-4} > 10^{-6}$*, we see that the fourth term is the first one which is smaller than* 10^{-6}*. So we should stop after the third term, which gives the approximation*

$$y(0.1) \approx 1 - \frac{1}{2}10^{-2} + \frac{1}{2 \cdot 4}10^{-4}.$$