Math 253 (Calc III), Winter 2015
Practice Midterm 1 Solutions

1. Does the sequence converge or diverge? WHY? If it converges, what is the limit.

   (a)
   $$a_n = \sqrt{\frac{4n^2 - 2n + 3}{2n^2 + 17}}$$
   Inside converges to $\frac{4}{2} = 2$ by leading terms (or by L'Hop). f of limit is limit of f. Converges to $\sqrt{2}$.

   (b)
   $$b_n = \frac{n^n}{n!}$$
   $$b_n = \frac{n \cdot (n \cdot \ldots \cdot n)}{1 \cdot (2 \cdot \ldots \cdot n)} > n$$
   so it diverges, goes to ∞.

   (c)
   $$c_n = (\frac{1}{2}, \frac{-1}{2}, \frac{1}{3}, \frac{-2}{3}, \frac{1}{4}, \frac{-3}{4}, \frac{1}{5}, \frac{-4}{5}, \ldots)$$
   Half the terms go to 0, other half to -1, but doesn't stay close to one limit. Diverges.

   (d)
   $$d_n = 8000(\frac{99}{100})^n$$
   Geometric sequence, $r = \frac{99}{100} < 1$. Converges to zero.

   (e)
   $$e_n = \frac{\ln n}{\ln(\ln n)}$$
   Use L'Hop. $\lim e_n = \lim \frac{\frac{1}{n}}{\frac{1}{n \ln n}} = \lim \ln n = \infty$. Diverges.

2. Consider the sequence $a_n = \frac{n^2-2}{n^2}$.
   (a) How many terms of the sequence are needed before it gets within $.01$ of the limit? Within $.001$ of the limit?
   (b) Using the mathematical definition of limit, prove that the sequence converges.

$a_n = 1 - \frac{2}{n^2}$, so $\lim a_n = 1$. $|a_n - 1| = \frac{2}{n^2}$.

For $\frac{2}{n^2} < .01 = \frac{1}{100}$, need $n^2 > 200$. First happens for $n = 15$. (Good enough to say $n > \sqrt{200}$.)

For $\frac{2}{n^2} < .001 = \frac{1}{1000}$, need $n^2 > 2000$. First happens for $n > \sqrt{2000}$.

For $\frac{2}{n^2} < \varepsilon$, need $n^2 > \frac{2}{\varepsilon}$. Happens when $n > \sqrt{\frac{2}{\varepsilon}}$. So let $N = \sqrt{\frac{2}{\varepsilon}}$. Can find N for any $\varepsilon > 0$. This proves that the sequence converges to 1.

3. Consider the sequence $a_1 = 10$ and $a_{n+1} = \frac{1}{4}a_n + 3$.
   (a) Show that the sequence is monotone and bounded, and therefore converges.
   (b) What is the limit?

Let L be a potential limit. Then $L = \frac{1}{4}L + 3$ so $L = 4$. $a_1 = 10 > 4$ so we expect decreasing, bounded below by 4.

Suppose $a_n \geq 4$. Then $a_{n+1} = \frac{1}{4}a_n + 3 \geq \frac{1}{4}4 + 3 = 4$. Since $a_1 > 4$, induction says $a_k \geq 4$ for all k.

Then $a_n - a_{n+1} = \frac{3}{4}a_n - 3 \geq \frac{3}{4}4 - 3 = 0$. So the sequence is decreasing.

4. Does the series converge or diverge? WHY?

(a)

$$\sum_{n=3}^{\infty} \frac{1}{n^{1.05}}$$

p-test, $p = 1.05 > 1$ so converges.

(b)

$$\sum_{n=1}^{\infty} \frac{1}{n^{1.05}} + \frac{1}{n^{.95}}$$

For each term, p-test. $p = .95 < 1$ diverges, first half converges. Diverge + converge = Diverge. So diverges.

(c)

$$\sum_{n=1}^{\infty} \frac{n10^n}{3^{3n+2}}$$

$\frac{n10^n}{3^{3n+2}} = \frac{n10^n}{9*27^n} = \frac{n}{9}(\frac{10}{27})^n$. *Now $n10^n$ is eventually less than 11^n. Thus $\frac{n}{9}(\frac{10}{27})^n$ is eventually less than $\frac{1}{9}(\frac{11}{27})^n$. This converges by geom series, $r = \frac{11}{27} < 1$. So by comparison test, it converges.*

(d)

$$\sum_{n=1}^{\infty} \sin(n) - \sin(n+1)$$

Telescoping sum. The partial sum s_n is $\sin(1) - \sin(n+1)$, and the potential limit is $\sin(1)$. The difference $\sin(n+1)$ does not go to zero. Diverges.

(e)

$$\sum_{n=0}^{\infty} \frac{n^2+2}{n^3+3}$$

By limit comparison test, we instead look at $\sum \frac{n^2}{n^3} = \sum \frac{1}{n}$. By p-test, $p = 1$, this diverges.

(f)

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

The antiderivative of $\frac{1}{n \ln n}$ is $\ln(\ln t)$. This goes to infinity, so $\int_N^\infty \frac{1}{t \ln t} dt$ diverges for any N. Integral test says our series diverges.

(g)

$$-1 - \frac{1}{4} + \frac{1}{9} + \frac{1}{16} - \frac{1}{25} - \frac{1}{36} + \frac{1}{49} + \dots$$

This is $\sum \pm \frac{1}{n^2}$. *Its absolute value sequence* $\sum \frac{1}{n^2}$ *converges by* p*-test,* $p = 2$. *Absolute convergence implies convergence.*

5. Consider the series

$$\frac{1}{4} - \frac{1}{8} + \frac{1}{12} - \frac{1}{16} + \dots$$

(a) Is the series convergent? WHY?
Alternating, decreasing to zero. Alternating series test says converges.

(b) How many terms of the sum must one take in order to be within .1 of the limit?
Error after N *terms is bounded by* $|a_{N+1}|$. *Need* $|a_{N+1}| < \frac{1}{10}$. *But* $\frac{1}{12} < \frac{1}{10}$ *so* $N = 2$ *works.*

(c) Is the series absolutely convergent? WHY?
First observe our series is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4n}.$$

$\sum \frac{1}{4n}$ *diverges by* p*-test,* $p = 1$. *No, not absolutely convergent.*

6. What is the sum of the following series?

$$\sum_{n=6}^{\infty}(\frac{6}{7})^n$$

Geom series. First term is $(\frac{6}{7})^6$*. Ratio* $r = \frac{6}{7}$*. Limit is*

$$\frac{(\frac{6}{7})^6}{1-\frac{6}{7}}.$$

7. How many terms of the series $\sum_{n=1}^{\infty} \frac{4}{n^5}$ must you take to approximate the sum of the series to within 10^{-8}?

Bound error with integral test.

$$\int_N^{\infty} \frac{4}{t^5} dt = \frac{-1}{t^4}\Big|_N^{\infty} = \frac{1}{N^4}.$$

So

$$\sum_{n=N+1}^{\infty} \frac{4}{n^5} < \frac{1}{N^4}.$$

If we want $\frac{1}{N^4} \leq 10^{-8}$ *then we need* $N \geq 10^2 = 100$*. 100 terms will work.*