1. What does the **ratio test** say about the following series?

   (a) $\sum_{n=1}^{\infty}(-1)^n \frac{2^n}{n!}$

   $|\frac{a_{n+1}}{a_n}| = \frac{2}{n+1}$ *and the limit as* $n \to \infty$ *is* 0. *The series absolutely converges.*

   (b) $\sum_{n=1}^{\infty} \frac{n^2+3}{n^3+2}$

   $|\frac{a_{n+1}}{a_n}| = \frac{((n+1)^2+3)(n^3+2)}{((n+1)^3+2)(n^2+3)} = \frac{n^5+\dots}{n^5+\dots}$ *and the limit as* $n \to \infty$ *is* 1. *The ratio test is inconclusive.*

2. Find the interval of convergence of the following power series.

   (a) $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n3^n}$

   Ratio test: $|\frac{a_{n+1}}{a_n}| = \frac{|x-2|}{3}\frac{n}{n+1}$ *and the limit is* $\frac{|x-2|}{3}$. *This converges when* $|x-2| < 3$, *so the radius is* 3, *centered at* 2.
   Checking the boundary: When $x = 5$ *we have* $\sum \frac{1}{n}$ *which diverges by* p*-test,* $p = 1$. *When* $x = -1$ *we have* $\sum \frac{(-1)^n}{n}$ *which converges by AST. So the interval of convergence is* $[-1, 5)$.

   (b) $\sum_{n=0}^{\infty} \frac{4^n(x+9)^n}{n^3+1}$

   Ratio test: $|\frac{a_{n+1}}{a_n}| = 4|x+9|\frac{(n+1)^3+1}{n^3+1}$ *and the limit is* $4|x+9|$. *This converges when* $|x+9| < \frac{1}{4}$, *so the radius is* $\frac{1}{4}$, *centered at* -9.
   Checking the boundary: When $x = -8.75$ *we have* $\sum \frac{1}{n^3+1}$ *which converges by (limit) comparison test to* p*-test,* $p = 3$. *When* $x = -9.25$ *we have* $\sum \frac{(-1)^n}{n^3+1}$ *which converges by AST (or because it absolutely converges). So the interval of convergence is* $[-9.25, 8.75]$.

3. Find a power series centered at zero for the following functions. (Note: I could also ask for the radius of convergence.)

   (a) $\dfrac{1}{4-3x}$

   $\dfrac{1}{4-3x} = \dfrac{1}{4}\dfrac{1}{(1-\frac{3}{4}x)} = \dfrac{1}{4}\sum_{n=0}^{\infty}(\dfrac{3}{4})^n x^n$. *The radius of convergence is* $\frac{4}{3}$ *(easy ratio test).*

   (b) $\displaystyle\int_0^x \frac{1}{1+t^3}dt$

   $\dfrac{1}{1+t^3} = \sum_{n=0}^{\infty}(-1)^n t^{3n}$. *So the integral is* $C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{3n+1}}{3n+1}$. *Clearly* $C = 0$ *since the integral begins at* 0. *The radius of convergence is* 1 *(easy ratio test). (Remember: radius of convergence doesn't change when you integrate or derive! However, interval of convergence may change - stuff can happen at the boundary!)*

   (c) The derivative of $\displaystyle\sum_{n=0}^{\infty}\frac{2^n(n!)x^n}{(3n)!}$.

   $\displaystyle\sum_{n=0}^{\infty}\frac{2^n(n!)nx^{n-1}}{(3n)!}$. *The radius of convergence is* ∞ *(harder ratio test).*

4. Compute $\int_0^{1/10}\frac{1}{1+t^3}dt$ to within 10^{-9}.

   We've already seen $\displaystyle\int_0^x \frac{1}{1+t^3}dt = \sum_{n=0}^{\infty}(-1)^n\frac{x^{3n+1}}{3n+1}$. *Plugging in* $x = 10^{-1}$, *we get an alternating series, so we are interested in when the* $(k+1)$*-st term has absolute value less than* 10^{-9}. *Clearly the* $n = 3$ *term is less than* 10^{-9}, *so one can take* $\sum_{n=0}^{2}(-1)^n\frac{10^{-(3n+1)}}{3n+1}$ *as our estimate.*

5. Find a power series centered at zero for the following functions. Write out the first three nonzero terms explicitly. (Note: I could also ask for the radius of convergence.)

   (a) e^{x^3}

   Just plug in x^3 to the formula for e^x.

   $1 + x^3 + \frac{1}{2}x^6 + \ldots = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}.$

   The radius of convergence is ∞. After all, the radius of convergence of e^x is ∞, and plugging in x^3 takes the cube root of that. Or you could do a ratio test.

   (b) $\frac{1}{(1+2x)^{3.5}}$

   Binomial expansion, plug in $2x$, and $k = -3.5$.

   $\sum_{n=0}^{\infty} \binom{-3.5}{n} (2x)^n = 1 + \frac{-3.5}{1}(2x) + \frac{(-3.5)(-4.5)}{(2)(1)} 2^2 x^2 + \ldots$

   The radius of convergence is $\frac{1}{2}$. After all, the radius of convergence of $(1+x)^k$ is 1, and plugging in $2x$ cuts that in half. Or you could do a ratio test.

   (c) $\ln(1 - x^3)$

   $\sum_{n=1}^{\infty} (-1)^{n-1}(-1)^n \frac{x^{3n}}{n} = -x^3 - \frac{x^6}{2} - \frac{x^9}{3} - \ldots$

   The radius of convergence is 1. After all, the radius of convergence of $\ln(1+x)$ is 1, and plugging in x^3 takes the cube root of that. Or you could do a ratio test.

6. Find $\cos(.5)$ to within $\frac{1}{500}$. Use the Taylor Inequality Estimate to justify your answer.

   We use the usual Taylor series of $f(x) = \cos x$ centered at 0. Note that $|f^{(k)}(x)|$ is bounded above by $M = 1$ for any k. Thus the TIE says that

   $$|R_k(.5)| \leq \frac{(.5)^{k+1}}{(k+1)!} = \frac{1}{2^{k+1}(k+1)!}$$

   When $k = 4$, we have $|R_k(.5)| \leq \frac{1}{500}$. So our estimate is $T_4(.5)$, which is $1 - \frac{(.5)^2}{2!} + \frac{(.5)^4}{4!}$.

7. Using any method, find the first few terms of the Taylor series, up to the cubic term (i.e. the x^3 term).

   (a) $e^x \cos x$ centered at 0.

   Method 1:
   $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$ and $\cos x = 1 - \frac{x^2}{2} + \ldots$ *so when we multiply we get*
   $1 + x + x^2(\frac{1}{2} - \frac{1}{2}) + x^3(\frac{1}{6} - \frac{1}{2}) + \ldots = 1 + x - \frac{1}{3}x^3 + \ldots$

   Method 2:
   $f(x) = e^x \cos x$ so $f(0) = 1$. $f'(x) = e^x(\cos x - \sin x)$ so $f'(0) = 1$. $f''(x) = -2e^x \sin x$ so $f''(0) = 0$. $f'''(x) = -2e^x(\cos x + \sin x)$ so $f'''(0) = -2$. Thus
   $f(x) = 1 + 1x + \frac{0}{2!}x^2 + \frac{-2}{3!}x^3 + \ldots = 1 + x - \frac{1}{3}x^3 \ldots$

   (b) $\sqrt{x - 3}$ centered at 2.

   This function is not defined at 2. *Looks like a trick question to me.*
   Just in case you want some practice, I'll do a non-trick question: $\sqrt{x-2}$ *centered at* 3.

   Method 1:
   $f(3) = 1$, $f'(x) = (.5)(x-2)^{-.5}$ so $f'(3) = .5$, $f''(x) = (.5)(-.5)(x-2)^{-1.5}$ so $f''(3) = (.5)(-.5)$, $f'''(x) = (.5)(-.5)(-1.5)(x-2)^{-2.5}$ so $f'''(3) = (.5)(-.5)(-1.5)$.
   $f(x) = 1 + .5(x-3) + \frac{(.5)(-.5)}{2}(x-3)^2 + \frac{(.5)(-.5)(-1.5)}{3!}(x-3)^3 + \ldots$

   Method 2: if you're clever, you can use a binomial expansion. This is $\sqrt{1 + (x-3)}$.

   (c) e^{3x} centered at -5.

   $f^{(k)}(x) = 3^k e^{3x}$. *So we have*
   $f(x) = e^{-15} + 3e^{-15}(x+5) + \frac{3^2 e^{-15}}{2}(x+5)^2 + \frac{3^3 e^{-15}}{3!}(x+5)^3 + \ldots$

8. Find the third-order approximation to $\frac{1}{1-x}$ at 5. Bound the error on the interval $(4, 6)$.

   One has $f^{(k)}(x) = \frac{k!}{(1-x)^{k+1}}$. *Thus* $T_3(x) = \frac{1}{-4} + \frac{1}{(-4)^2}(x-5) + \frac{1}{(-4)^3}(x-5)^2 + \frac{1}{(-4)^4}(x-5)^3$.

   On the interval $(4, 6)$, *the 4th derivative is decreasing in absolute value, so the maximum is obtained at* 4. *Thus* $M = \frac{4!}{(3)^5}$ *is a bound for the absolute value of* $f^{(4)}$ *on this interval. The radius for the interval is* $d = 1$. *Hence* $|R_3(x)| \leq \frac{4!}{4! \cdot 3^5} = \frac{1}{3^5}$.

9. Find the fifth-order approximation to $3\sin x$ at 0. Use the Taylor Inequality Estimate to find the radius d such that the error is less than $.2$ for x in the interval $(-d, d)$.

   Using the standard Taylor series at 0, we have $T_5(x) = 3x - 3\frac{x^3}{3!} + 3\frac{x^5}{5!}$.

   Every derivative of $3\sin x$ *is bounded in absolute value by* $M = 3$. *So on a radius of interval* d *we have* $|R_5(x)| \leq \frac{3d^6}{6!}$. *Now we solve* $\frac{3d^6}{6!} = .2$ *and get* $d^6 = (.2)(6!)/3 = \kappa$ *so* $d = \kappa^{\frac{1}{6}}$.

10. Prove that the Taylor series of e^x centered at 2 will converge to the function e^x everywhere.

    For $f(x) = e^x$ *one has* $f^{(k)} = e^x$ *is increasing and positive. So for any* d*, the maximum absolute value of* $f^{(k+1)}(x)$ *on the interval* $(2-d, 2+d)$ *is* e^{2+d}. *Hence one has* $|R_k(x)| \leq \frac{e^{2+d}d^{k+1}}{(k+1)!}$. *For any given* d*, this goes to* 0 *as* $k \to \infty$. *So* $T(x)$ *converges to* $f(x)$ *on the interval* $(2-d, 2+d)$. *But this is true for all* d*, so* $T(x)$ *converges to* $f(x)$ *everywhere.*