# Math 256 (Differential Equations), Winter 2015
# Ben's problems for HW 6

February 27, 2015

1. Consider the differential equation

$$y''' - 2y'' + 3y' - 4y = 3e^{2t}\cos(3t) - 4e^{2t}\sin(3t).$$

   (a) Rewrite the forcing function in the form $Ae^{at}\cos(bt - \varphi)$.
   The complex number $3 - 4i$ has polar coordinates $5e^{i\varphi}$ where $\varphi = \arctan(\frac{-4}{3})$. So the forcing function equals $5e^{2t}\cos(3t - \varphi)$.

   (b) Find a particular solution to the inhomogeneous equation. What is the amplitude? What is the gain? What is the additional phase lag?
   Letting $p(x) = x^3 - 2x^2 + 3x - 4$, and $r = 2 + 3i$, we have $p(r) = 34 - 6i$ (I did this on wolfram alpha). Since $p(r) \neq 0$, the forcing function is not a homogeneous solution, and we can use the sinusoidal response formula. Thus our particular solution is

$$\frac{5}{\sqrt{34^2 + 6^2}}e^{2t}\cos(3t - \varphi - \arctan(\frac{-6}{34})).$$

   The amplitude is $\frac{5}{\sqrt{34^2+6^2}}$, the gain is $\frac{1}{\sqrt{34^2+6^2}}$, and the additional phase lag is $\arctan(\frac{-6}{34})$.

2. Consider the differential equation $y'' + 2y' + 3y = 10\cos(wt)$ for some positive numbers $A$ and $w$.

   (a) Is this overdamped, underdamped, or critically damped? Is it forced or unforced?
   By the quadratic formula, the roots are $\frac{-2\pm\sqrt{4-12}}{2} = -1 \pm \sqrt{2}i$. Since these are complex with negative real part, it is underdamped. It is forced, since it is not homogeneous.

   (b) Find the general solution.
   Since $\cos(wt)$ is never a homogeneous solution, the sinusoidal response formula gives a particular solution. $p(wi) = 3 - w^2 + 2wi$, so the SRF yields

$$\frac{10}{\sqrt{(3-w^2)^2 + (2w)^2}}\cos(wt - \varphi)$$

where $\varphi$ is the argument of $p(wi)$. Meanwhile the general homogeneous solution is $c_1e^{-t}\cos(\sqrt{2}t) + c_2e^{-t}\sin(\sqrt{2}t)$. Thus the general solution is

$$\frac{10}{\sqrt{(3-w^2)^2+(2w)^2}}\cos(wt-\varphi) + c_1e^{-t}\cos(\sqrt{2}t) + c_2e^{-t}\sin(\sqrt{2}t).$$

(c) Find a formula for the amplitude of the steady state solution.

The steady state solution is the (periodic) particular solution, which is given by the SRF. It has amplitude $\frac{10}{\sqrt{(3-w^2)^2+(2w)^2}}$.

(d) Find $w > 0$ which maximizes the amplitude of the steady state solution.

To maximize the amplitude we could take the derivate with respect to $w$ of the above formula, and find the zeroes. However, it is easier to just minimize the part under the square root, which is $w^4 - 2w^2 + 9$. Taking the derivative of that we get $4w^3 - 4w$, which has a zero at $w = -1, 0, 1$. Taking the double derivative, it is a minimum only at $w = -1, +1$. Only $w > 0$ makes sense for frequencies. So the answer is $w = 1$.

3. (a) Find a particular solution to $y'' + 3y' + 2y = 5e^{-t}\cos(t)$.

$p(x) = x^2 + 3x + 2$ and $p(-1+i) = -1+i$. This is nonzero, so we can use the SRF. The answer is

$$\frac{5}{\sqrt{2}}e^{-t}\cos(t - \frac{3\pi}{4}).$$

(b) Find a particular solution to $y'' + 2y' + 2y = 5e^{-t}\cos(t)$.

$p(x) = x^2 + 2x + 2$ and $p(-1+i) = 0$, so we can not use the SRF. There is a fancier version of the SRF that would work, but we'll just do it by hand. We guess something of the form

$$y = Ate^{-t}\cos(t) + Bte^{-t}\sin(t).$$

Computing derivatives, the left hand side is $2Be^{-t}\cos(t) - 2Ae^{-t}\sin(t)$. For this to equal $5e^{-t}\cos(t)$ we set $A = 0$ and $B = 2.5$.

4. Consider the differential equation $y'' + by' + 4y = 0$.

(a) For which values of $b \geq 0$ is it overdamped? Underdamped? Critically damped? Undamped?

The question is whether $b^2 - 16$ is positive, zero, or negative. For $b > 4$ it is overdamped. For $b = 4$ it is critically damped. For $0 < b < 4$ it is underdamped. For $b = 0$ it is undamped.

(b) Suppose that it is underdamped. Write down a formula for the quasi-period of a solution, in terms of $b$. As $b$ gets smaller, what happens to the quasi-period?

When $0 < b < 4$ the roots are $\frac{-b\pm\sqrt{b^2-16}}{2} = \frac{-b}{2} \pm \frac{\sqrt{16-b^2}}{2}i$, and a homogeneous solution has the form $Ce^{-\frac{b}{2}t}\cos(\frac{\sqrt{16-b^2}}{2}t - \varphi)$. Thus the quasi-period is $\frac{4\pi}{\sqrt{16-b^2}}$. As $b$ approaches zero, the denominator gets larger until it approaches $4$, so the quasi-period gets smaller until it approaches $\pi$.

(c) Suppose that it is critically damped. Find the general solution. How many times will a solution satisfy $y(t) = 0$?

So $b = 4$, and the repeated root is $-2$. The general solution is $(c_1 + c_2 t)e^{-2t}$. This satisfies $y(t) = 0$ exactly once at time $t_0$, when $c_1 + t_0 c_2 = 0$.

(d) Continue to assume that it is critically damped. Suppose that $y(0) = 5$ and $y'(0) = 3$. At what time(s) will the solution satisfy $y(t) = 0$?

$y(0) = c_1 = 5$ and $y'(0) = -2c_1 + c_2 = 3$ so $c_2 = 13$. Therefore, $t_0 = \frac{-c_1}{c_2} = \frac{-5}{13}$.

(e) Suppose that $b = 5$. Suppose that $y(0) = 5$. For while values of $y'(0)$ will the solution NEVER satisfy $y(t) = 0$?

The roots are $-1$ and $-4$, so the solution is $c_1e^{-t} + c_2e^{-4t}$. If the solution satisfies $y(t_0) = 0$ for a particular time $t_0$ then $c_1e^{-t_0} = -c_2e^{-4t_0}$, meaning that $\frac{c_1}{c_2} = -e^{-3t_0}$ is a negative number. Any negative number can be obtained this way for some $t_0$, so the solution will never satisfy $y(t) = 0$ if and only if $\frac{c_1}{c_2}$ is positive.

Now $y'(0) = -c_1 - 4c_2$ and $y(0) = c_1 + c_2 = 5$. Thus $-3c_2 = 5 + y'(0)$ and $3c_1 = 20 + y'(0)$. In particular, $c_1$ and $c_2$ have the same sign when $20 + y'(0)$ and $5 + y'(0)$ have opposite signs. This will happen precisely for $-20 < y'(0) < -5$.