1. (a) Rewrite $-\sqrt{3}+i$ in the form $Ae^{i\theta}$.

   The answer is $2e^{i\frac{5\pi}{6}}$. This is because $A=\sqrt{(-\sqrt{3})^2+(1)^2}$, and $\frac{5\pi}{6}$ is the argument (i.e. angle) of the corresponding point.

   Note that if you wrote the angle as $\arctan(\frac{-1}{\sqrt{3}})$, you would be wrong. This gives a point in the lower right quadrant, not the upper left quadrant. The difference between these two angles is π.

   (b) Rewrite $5e^{i\arctan(\frac{3}{4})}$ in the form $a+bi$.

   The answer is $4+3i$.

   One has $a=5\cos(\arctan(\frac{3}{4}))$ and $b=5\sin(\arctan(\frac{3}{4}))$. By drawing a right triangle with edge lengths $3,4,5$ you see that $\cos(\arctan(\frac{3}{4}))$ is actually $\frac{4}{5}$, etcetera. I checked that I am correctly in the upper right quadrant.

   (c) Rewrite $-\cos(3t)-3\sin(3t)$ in the form $C\cos(\omega t-\varphi)$. What is the amplitude of this sinusoidal function?

   The answer is $\sqrt{10}\cos(3t-\pi-\arctan(3))$. The amplitude is $\sqrt{10}$.

   Writing $(-1,-3)$ in polar coordinates, it has magnitude $\sqrt{10}$ and angle $\arctan(3)+\pi$. (The $+\pi$ is because we're in the lower left quadrant.)

   (d) Compute $\frac{2-3i}{3-4i}$ in either rectangular or polar form.

   The answer is $\frac{18}{25}+\frac{-1}{25}i$.

   We clear denominators by multiplying by $\frac{3+4i}{3+4i}$. It is way easier to do this in rectangular form; DON'T use polar form.

   (e) Compute $\frac{\sqrt{5}e^{\frac{3}{4}\pi i}}{4e^{-\frac{5}{8}\pi i}}$ in either rectangular or polar form.

   The answer is $\frac{\sqrt{5}}{4}e^{\frac{11}{8}\pi i}$.

   This is because the magnitudes divide and the arguments add/subtract (in this case, the argument is $\frac{3\pi}{4}-\frac{-5\pi}{8}$). It is way easier in polar form; DON'T use rectangular form.

2. Consider the differential equation $y'' - 4y' + 29y = 5e^t - 3e^{-2t}$.

   (a) What kind of differential equation is this?
   It is a 2LODEwCC, inhomogeneous.

   (b) Find the general solution.
   First find the general homogeneous solution. Let $p(x) = x^2 - 4x + 29$. By the quadratic formula, the roots are $\frac{4 \pm \sqrt{16 - 4(29)}}{2} = 2 \pm \sqrt{4 - 29} = 2 \pm 5i$. Thus the general homogeneous solution is $h = c_1 e^{2t} \cos(5t) + c_2 e^{2t} \sin(5t)$.
   Now find y_1 with $p(D)y_1 = 5e^t$. Since 1 is not a root, we use the easy form of the ERF. So $y_1 = \frac{5}{p(1)} e^t = \frac{5}{26} e^t$.
   Now find y_2 with $p(D)y_2 = -3e^{-2t}$. Since -2 is not a root, we use the easy form of the ERF. So $y_2 = \frac{-3}{p(-2)} e^{-2t} = \frac{-3}{41} e^{-2t}$.
   Thus the general solution is

$$y = y_1 + y_2 + h = \frac{5}{26} e^t - \frac{3}{41} e^{-2t} + c_1 e^{2t} \cos(5t) + c_2 e^{2t} \sin(5t).$$

   (c) Find a solution for which $y(0) = 1$ and $y'(0) = 2$.
   Plugging in we get

$$y(0) = \frac{5}{26} - \frac{3}{41} + c_1 = 1$$

   and

$$y'(0) = \frac{5}{26} + \frac{6}{41} + 2c_1 + 5c_2 = 2.$$

   Therefore,

$$c_1 = 1 - \frac{5}{26} + \frac{3}{41}$$

   and

$$c_2 = \frac{1}{5}(2 - 2c_1 - \frac{5}{26} - \frac{6}{41}) = \frac{1}{5}(\frac{5}{26} - \frac{12}{41}).$$

3. Let $\mathcal{L}$ denote the operator (on functions in a variable t) defined by

$$\mathcal{L}[y] = y'' + (1000 - \sin(3t))y' - 2y.$$

(a) Is $\mathcal{L}$ a linear operator?

Yes. (Taking the derivative and multiplying by a function of t are linear operators, and so is their composition and sum.)

(b) What kind of differential equation is $\mathcal{L}[y] = 0$?

A homogeneous 2LODE. (Not with constant coefficients.)

(c) Let $f = \mathcal{L}[e^{2t}]$. Find f.

Plugging in, we get

$$f = 4e^{2t} + (1000 - \sin(3t))2e^{2t} - 2e^{2t} = 2002e^{2t} - 2e^{2t}\sin(3t).$$

(d) Find a solution to $\mathcal{L}[y] = f$.

We know that $\mathcal{L}[e^{2t}] = f$, so e^{2t} is a solution (duh!). You should get this right even if you get the previous problem wrong!

(e) Find the general solution to $\mathcal{L}[y] = f$.

We don't know how to do this. Trick question!

(f) Suppose that z is a solution to $\mathcal{L}[y] = f$ satisfying $z(5) = 6$ and $z'(5) = -3$. Does z exist? Where is z defined? Why?

It exists and is defined everywhere. This is by the existence and uniqueness theorem for LINEAR ODEs, and the fact that the coefficient functions and f are all defined and continuous and differentiable everywhere.

(g) Find a linear operator $\mathcal{L}_2$ for which the solutions to $\mathcal{L}_2[y] = f$ approximate the solutions to $\mathcal{L}[y] = f$, and for which you know how to solve $\mathcal{L}_2[y] = f$. You don't need to find the solutions.

$\mathcal{L}_2[y] = y'' + 1000y' - 2y$ is a 2LODEwCC, and we know how to solve it. It is a good approximation because $1000 + \sin(3t)$ is always close to 1000, proportionally speaking.

4. Let $\mathcal{L} = p(D)$ for the polynomial $p(x) = x^2(x-1)(x+2)^3(x^2-2x+2)^2(x^2+5x+1)$. Here, D represents the derivative operator.

   (a) What kind of differential equation is $\mathcal{L}[y] = e^{5t}$?
   An inhomogeneous 12LODEwCC.

   (b) Find the general homogeneous solution.
   The roots are: 0 twice, 1 once, -2 thrice, $-1 \pm i$ twice each, and $\frac{-5\pm\sqrt{21}}{2}$ each once. This leads to twelve basic solutions.

$$c_1 + c_2 t + c_3 e^t + c_4 e^{-2t} + c_5 t e^{-2t} + c_6 t^2 e^{-2t} +$$
$$c_7 e^{-t}\cos(t) + c_8 e^{-t}\sin(t) + c_9 t e^{-t}\cos(t) + c_{10} t e^{-t}\sin(t) +$$
$$c_{11} e^{\frac{-5+\sqrt{21}}{2}t} + c_{12} e^{\frac{-5-\sqrt{21}}{t}}$$

   (c) Write down a guess for a particular solution to $\mathcal{L}[y] = (3t-5)e^{5t}$. You need not determine any coefficients.
   Try $y = (At + b)e^{5t}$.

   (d) Write down a guess for a particular solution to $\mathcal{L}[y] = 16e^{-2t}$. You need not determine any coefficients.
   Try $y = At^3 e^{-2t}$.

   (e) Write down a guess for a particular solution to $\mathcal{L}[y] = \cos(t)$. You need not determine any coefficients.
   Try $y = A\cos(t) + B\sin(t)$.