Math 256 (Differential Equations), Winter 2015
Quasi Practice Final Solutions

March 13, 2015

1. Consider the spring equation $y'' + 3y' + 5y = \cos(wt)$, where $w \geq 0$.

   (a) Is it underdamped, overdamped, critically damped, or undamped? Is it forced or unforced?

   The discriminant $3^2 - 4 * 5$ is negative, so there are two complex roots. It is underdamped. It is forced (i.e. inhomogeneous).

   (b) Find a steady-state solution. (You need not compute the additional phase lag explicitly.)

   By the SRT, our steady-state solution is $\frac{1}{|p(wi)|}\cos(wt - \varphi)$ where φ is the argument of $p(wi)$. We have

   $$p(wi) = (5 - w^2) + (3w)i$$

   so that

   $$\frac{1}{|p(wi)|} = \frac{1}{\sqrt{(5-w^2)^2 + (3w)^2}} = \frac{1}{\sqrt{w^4 - w^2 + 25}}.$$

   (c) Find the number $w \geq 0$ which maximizes the amplitude of the steady-state solution.

   To maximize the amplitude, we can minimize the quantity under the square root. To minimize $w^4 - w^2 + 25$, we set its derivative to zero:

   $$4w^3 - 2w = 0 \implies w = 0, \pm\frac{1}{\sqrt{2}}.$$

   We also need its double derivative to be positive:

   $$12w^2 - 2 > 0 \implies w \neq 0.$$

   Thus the amplitude is maximized at $w = \frac{1}{\sqrt{2}}$.

   (d) Describe in a sentence or two what happens physically when w is the maximizing number you computed above. What is the name for this phenomenon?

   There is positive feedback between the forcing function $\cos(wt)$ and the homogeneous solutions. This is called practical resonance. (Not to be confused with true resonance, where the forcing function is actually a homogeneous solution.)

2. Consider the spring equation $y'' + 4y' + 4y = 0$.

   (a) Is it underdamped, overdamped, critically damped, or undamped? Is it forced or unforced?

   The discriminant $4^2 - 4 * 4$ is zero, so there is a repeated real root. It is critically damped. It is unforced (i.e. homogeneous).

   (b) Find the general solution.

   The repeated root is -2, so the general solution is

   $$y = c_1 e^{-2t} + c_2 t e^{-2t} = (c_1 + c_2 t) e^{-2t}.$$

   (c) Find the solution where $y(0) = 6$ and $y'(0) = -14$. Find t_0 such that $y(t_0) = 0$.

   $y(0) = c_1 = 6$ and $y'(0) = -2c_1 + c_2 = -14$ implies that $c_1 = 6$ and $c_2 = -2$. Therefore, if $y(t_0) = 0$ then

   $$(6 - 2t_0)e^{-2t} = 0 \implies 6 - 2t_0 = 0$$

   so that $t_0 = 3$.

   (d) Suppose that $y(0) = 6$ and $y'(0) = b$, and let t_0 be the time when $y(t_0) = 0$. For which b will it be true that $t_0 > 0$?

   By the same argument, $c_1 = 6$ and $c_2 = b + 12$, and $6 + (b + 12)t_0 = 0$. So $t_0 = \frac{-6}{b+12}$. When $b < -12$, this is positive. (Note that this agrees with the previous problem, which showed that $t_0 > 0$ when $b = -14$.)

3. Consider the 3×3 matrix
$$A = \begin{pmatrix} 4 & 1 & -1 \\ 0 & -6 & 6 \\ 1 & 1 & 2 \end{pmatrix}.$$

(a) Give the definitions of an *eigenvector* and an *eigenvalue* for the matrix A?

An *eigenvector* is a NONZERO vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$ for some scalar (i.e. number) λ. An eigenvalue is any scalar λ which appears this way, for some eigenvector.

(Note: we know that the eigenvalues are the roots of the characteristic polynomial of A, but this is not the definition! This is a theorem.)

(b) Is $\mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ an eigenvector for A? If so, what is its eigenvalue?

$$A\mathbf{v} = \begin{pmatrix} 4*2+1*0+(-1)*(-1) \\ 0*2+(-6)*0+6*(-1) \\ 1*2+1*0+2*(-1) \end{pmatrix} = \begin{pmatrix} 9 \\ -6 \\ 0 \end{pmatrix}.$$

Since $A\mathbf{v}$ is not a multiple of $\mathbf{v}$, $\mathbf{v}$ is not an eigenvector.

(c) Is $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ an eigenvector for A? If so, what is its eigenvalue?

$$A\mathbf{v} = \begin{pmatrix} 4*1+1*2+(-1)*3 \\ 0*1+(-6)*2+6*3 \\ 1*1+1*2+2*3 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}.$$

Since $A\mathbf{v} = 3\mathbf{v}$, $\mathbf{v}$ is an eigenvector with eigenvalue 3.

(d) Compute A^2.

$$A^2 = \begin{pmatrix} 15 & -3 & 0 \\ 6 & 42 & -24 \\ 6 & -3 & 9 \end{pmatrix}.$$

(e) Compute $\det A$.

$$\det A = 4*(-6)*2+1*6*1+(-1)*0*1-(-1)*(-6)*1-1*0*2-4*6*1 = -72.$$

4. Consider the differential equation

$$\begin{aligned} x_1' &= 2x_1 + \frac{1}{t-1}x_2 - x_3 \\ x_2' &= \frac{1}{t-2}x_1 + x_2 + x_3 + t \\ x_3' &= 3x_2 - \frac{1}{t-3}x_3 \end{aligned}$$

(a) What kind of differential equation is it?

A first order linear system with three functions, inhomogeneous. (Not with constant coefficients). Shorthand: inhomog. 1LSys, 3 funct.

(b) Write the equation in matrix-vector form.

$$D\mathbf{x} = A\mathbf{x} + \mathbf{g}$$

where

$$A = \begin{pmatrix} 2 & \frac{1}{t-1} & -1 \\ \frac{1}{t-2} & 1 & 1 \\ 0 & 3 & \frac{-1}{t-3} \end{pmatrix}, \qquad \mathbf{g} = \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix}.$$

(c) Write down an initial value for this equation, at time $t_0 = 1.5$. Your IVP will be used in the rest of the problem.

$x_1(1.5) = 1$, $x_2(1.5) = 2$, $x_3(1.5) = 3$. (Note: if you want to make it easier, why not let them all be zero?)

(d) Find the solution to your IVP.

This is a trick question. We don't know how to solve 1LSys without constant coefficients.

(e) What is the domain of definition for the solution to your IVP?

Even without the solution, we know this! The differential equation is continuous and differentiable except at $t = 1, 2, 3$. By the existence and uniqueness theorem for linear systems, our solution is defined on the interval $(1, 2)$, the biggest interval containing 1.5 where the equation is continuous and differentiable.

(f) Use the Euler method with step size 1 to compute the value of your solution at time 2.5. What do you think about your estimate?

$$x_1'(1.5) = 2+\frac{2}{.5}-3 = 3, \quad x_2'(1.5) = \frac{1}{-.5}+2+3+1.5 = 4.5, \quad x_3'(1.5) = 6-\frac{3}{-1.5} = 8$$

so

$$x_1(2.5) \approx 1 + 3 = 4, \quad x_2(2.5) \approx 2 + 4.5 = 6.5, \quad x_3(2.5) \approx 3 + 8 = 11.$$

On the other hand, the solution has unknown and mysterious behavior around $t = 2$, and need not even extend uniquely beyond the time interval $(1, 2)$, so any approximation of what happens at time 2.5 is meaningless!!

5. Consider the differential equation

$$\begin{aligned} x_1' &= 8x_1 - 3x_2 \\ x_2' &= 6x_1 - x_2 \end{aligned}$$

(a) What kind of differential equation is it?

A first order linear system with constant coefficients, homogeneous, with 2 functions. 1LSyswCC, 2 functions, homog.

(b) Find the general solution.

The trace is 7. The determinant is 10. The characteristic polynomial is $\lambda^2 - 7\lambda + 10 = (\lambda - 2)(\lambda - 5)$, which has roots 2 and 5.

$A - 2I$ has first row $(6, -3)$ so $\mathbf{v}_2 = \begin{pmatrix} -3 \\ -6 \end{pmatrix}$ is an eigenvector with eigenvalue 2.

One can also use $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ or any other multiple of $\mathbf{v}_2$.

$A - 5I$ has first row $(3, -3)$ so $\mathbf{v}_5 = \begin{pmatrix} -3 \\ -3 \end{pmatrix}$ is an eigenvector with eigenvalue 5.

Thus the general solution is $\mathbf{x} = c_1 e^{2t}\mathbf{v}_2 + c_2 e^{5t}\mathbf{v}_5$.

(c) Is it a node, a spiral, or a saddle? Is it stable or unstable?

Two unequal positive eigenvalues is an unstable node.

(d) Draw the phase portrait. Include the trajectories which go through $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively.

To be scanned.

6. Consider the differential equation

$$\begin{aligned} x_1' &= 3x_1 - 8x_2 \\ x_2' &= 4x_1 - 5x_2 \end{aligned}$$

(a) What kind of differential equation is it?
1LSyswCC, 2 functions, homog.

(b) Find the general solution.
Trace is -2. Determinant is 17. Characteristic polynomial is $\lambda^2 + 2\lambda + 17$. Roots are $\frac{-2\pm\sqrt{4(1-17)}}{2} = -1 \pm 4i$.
$A-(-1+4i)I$ has first row $(4-4i, -8)$ so that $\mathbf{v}_1 = \begin{pmatrix} -8 \\ -4+4i \end{pmatrix}$ is an eigenvector with eigenvalue $-1+4i$. To make my life easier, I will instead use $\mathbf{v} = \begin{pmatrix} 2 \\ 1-i \end{pmatrix}$ which is $\frac{1}{-4}\mathbf{v}_1$. Thus a complex valued solution is $\mathbf{z} = e^{(-1+4i)t}\mathbf{v}$.
More concretely,

$$\mathbf{z} = e^{-t}\begin{pmatrix} 2*(\cos(4t)+i\sin(4t)) \\ (1-i)*(\cos(4t)+i\sin(4t)) \end{pmatrix} = \begin{pmatrix} 2\cos(4t)+2\sin(4t)i \\ (\cos(4t)+\sin(4t))+(\sin(4t)-\cos(4t))i \end{pmatrix}.$$

To get a real solution, take linear combinations of the real and imaginary parts of the complex solution. Thus our solution is

$$\mathbf{x} = c_1 e^{-t}\begin{pmatrix} 2\cos(4t) \\ \cos(4t)+\sin(4t) \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 2\sin(4t) \\ \sin(4t)-\cos(4t) \end{pmatrix}.$$

(c) Is it a node, a spiral, or a saddle? Is it stable or unstable?
Two complex eigenvalues with negative real part is a stable spiral.

(d) Draw the phase portrait. Include the trajectories which go through $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively.
To be scanned.

7. For each of the following 2×2 matrices, classify the corresponding homogeneous first order linear system with constant coefficients as either

   - a node,
   - a spiral,
   - a saddle, or
   - none of the above.

   Also classify the equilibrium solution at the origin as either

   - stable,
   - unstable, or
   - none of the above.

   (a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

   Trace > 0, determinant < 0. Saddle, unstable.

   (b) $\begin{pmatrix} 1 & 2 \\ 3 & -100 \end{pmatrix}$

   Whoops, this wasn't the problem I meant to assign. Oh well. Trace < 0, determinant < 0, saddle, unstable.

   I meant to have the upper left corner be -1. Then the trace < 0, determinant > 0, and Tr^2 is much larger than $4\det$. Thus we have 2 real negative roots. Stable node.

   (c) $\begin{pmatrix} 4 & 1 \\ -3 & 2 \end{pmatrix}$

   Trace > 0, det > 0, and Tr$^2 - 4\det < 0$. Unstable spiral.

   (d) $\begin{pmatrix} -5 & 2 \\ -2 & -5 \end{pmatrix}$

   Trace < 0, det > 0, and Tr$^2 - 4\det < 0$. Stable spiral.